Genus-2 Lefschetz Fibrations

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October 16, 2024
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History of Lefschetz Fibrations

- Solomon Lefschetz (1884-1972)
- Worked on applications of algebraic topology in algebraic geometry

"La théorie des surfaces est de beaucoup la plus riche et la mieux perfectionnée."

Géométrie sur les surfaces et les variétés algébriques, 1929

Image from wikipedia.com

Why Lefschetz Fibrations?

Image from
breakthroughprize.org

symplectic manifolds have the structure of a Lefschetz fibration

Lefschetz fibrations in symplectic geometry, 1998

Lefschetz pencils have symplectic structures

The topology of symplectic manifolds, 2001

I 🥰 MATH

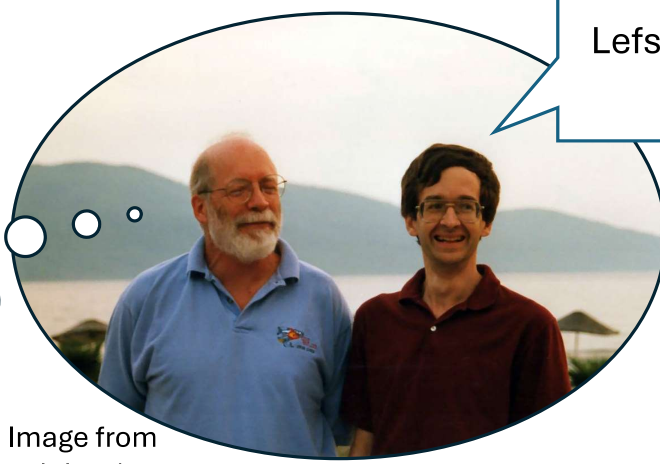


Image from
celebratio.org

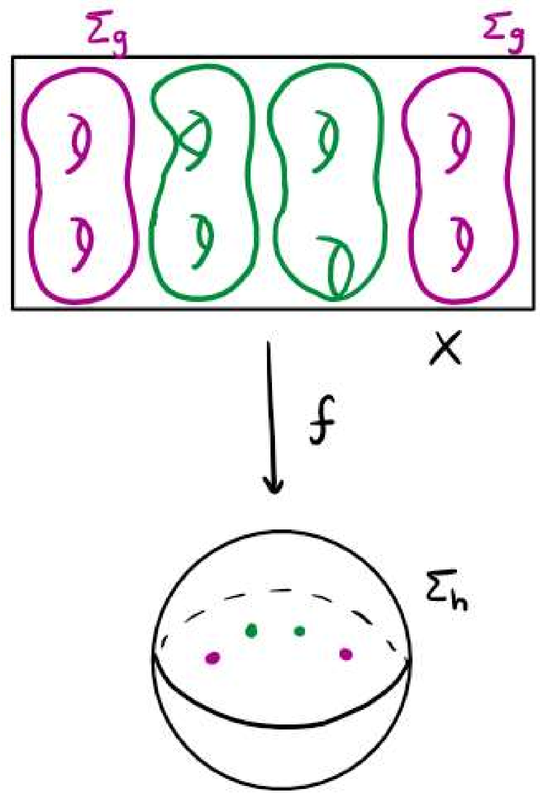
What are we talking about today?

- What is a Lefschetz fibration? 😛
- What's so special about genus-2 fibers? 🤯
- What do we know about them? 🤑
- What have I figured out? 😏
- What haven't I figured out? 😔

Part I

Definitions

Definition: Lefschetz fibration



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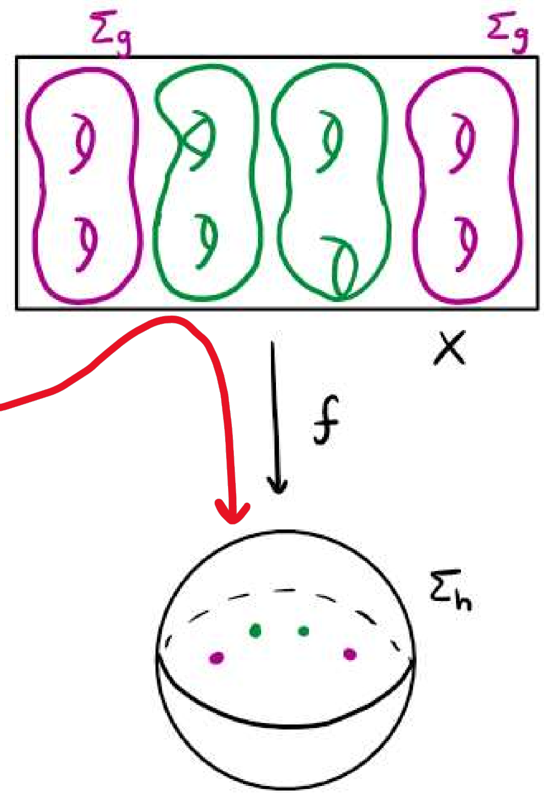

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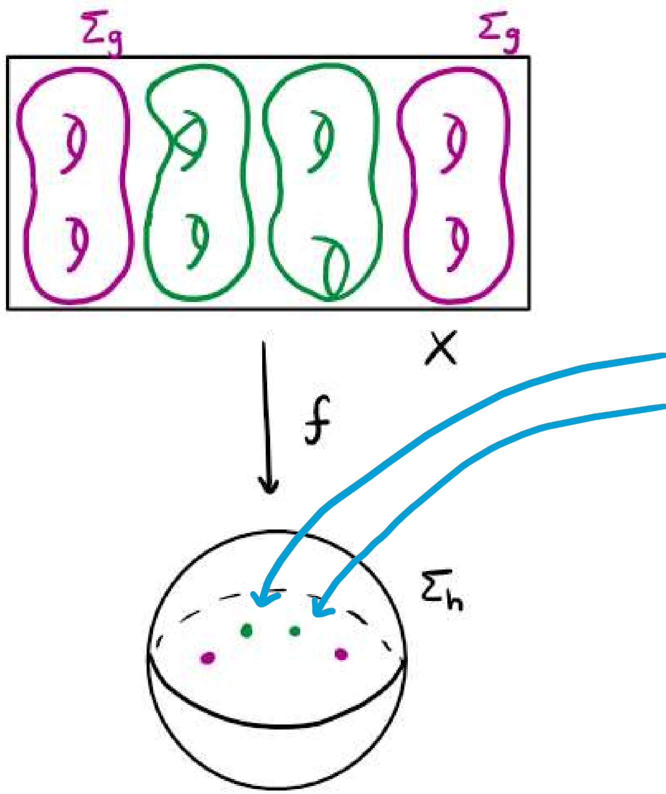
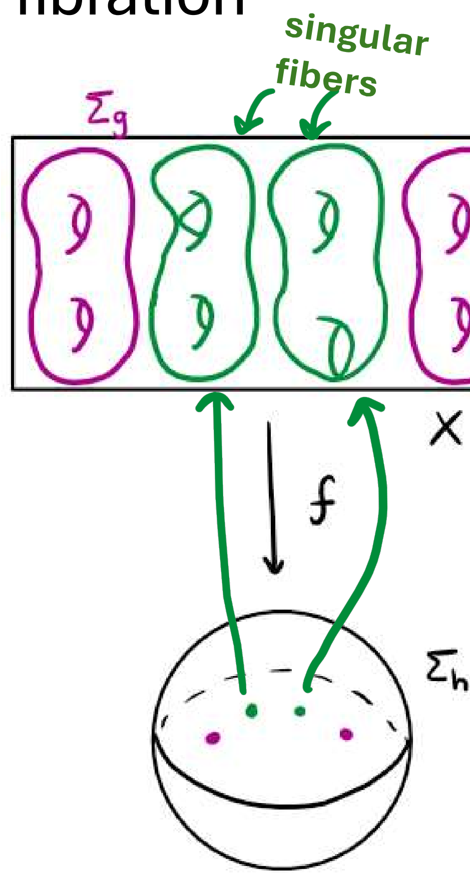


2 Finitely many critical values $q_1, \ldots, q_n$

Definition: Lefschetz fibration



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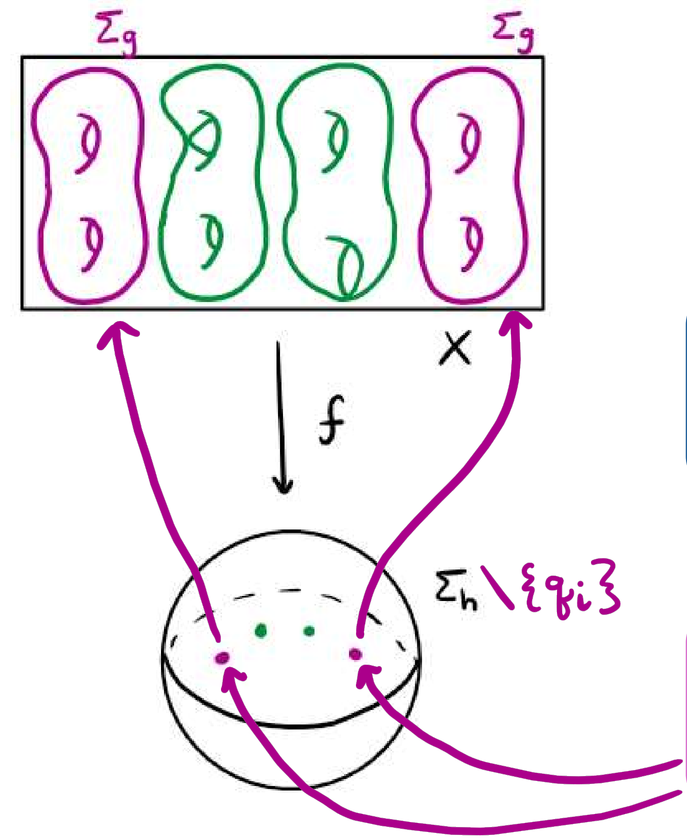
Definition: Lefschetz fibration



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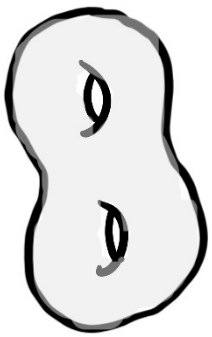
4 $f^{-1}(b)$ = regular fiber (genus-g surface)

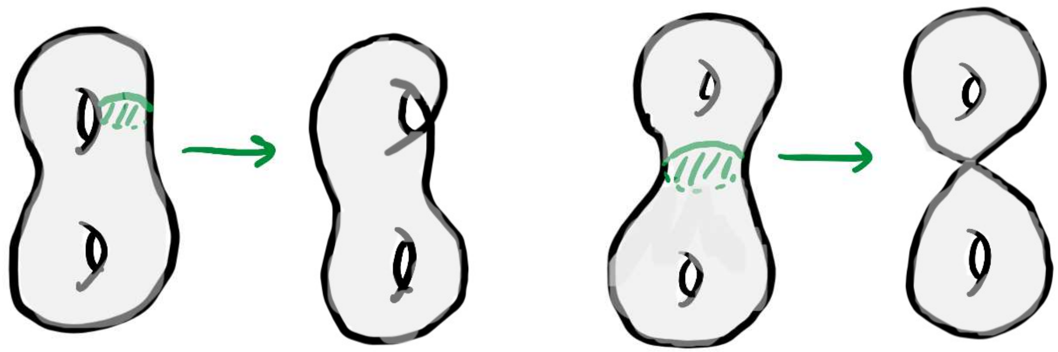
Definition: Lefschetz fibration

Regular fiber



genus 2

Singular fibers



non-separating   separating

Defn:  = vanishing cycle

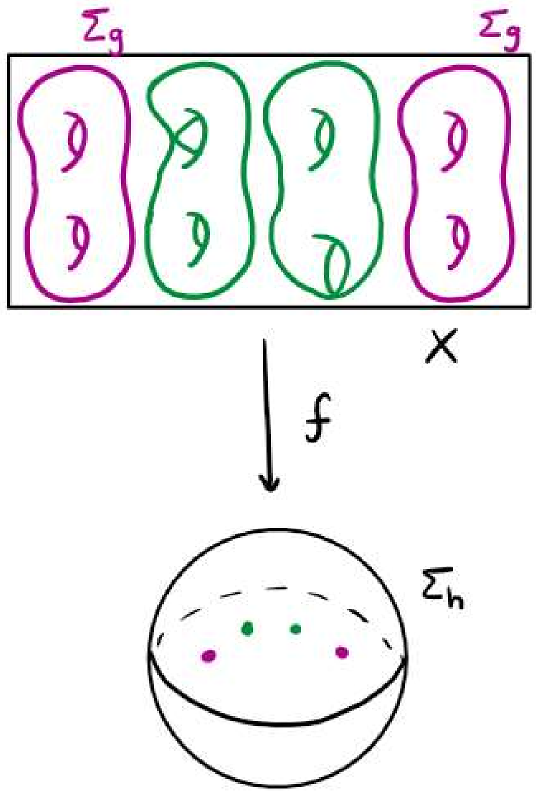
Definition: Lefschetz fibration

Remarks:

1. the genus of the Lefschetz fibration is the genus of a **regular fiber**

2. the **monodromy** determines the Lefschetz fibration

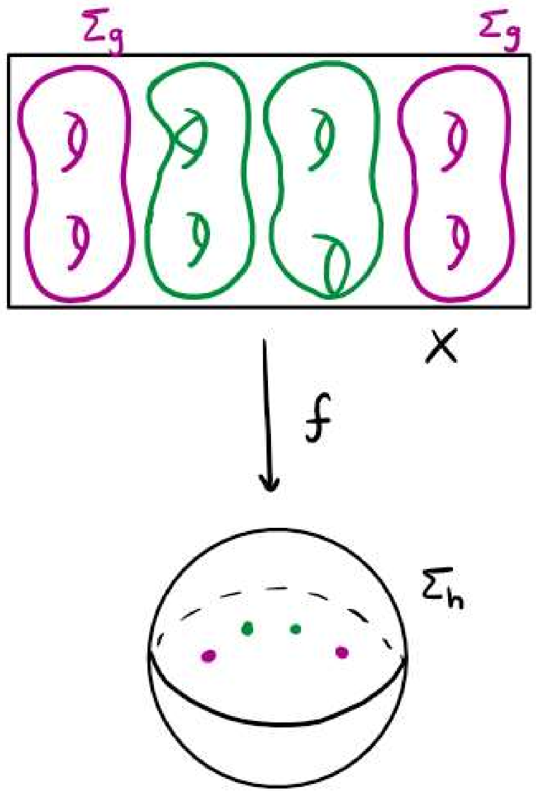
   - ...must be recording genus and vanishing cycles?



Definition: monodromy of a Lefschetz fibration

Definition:

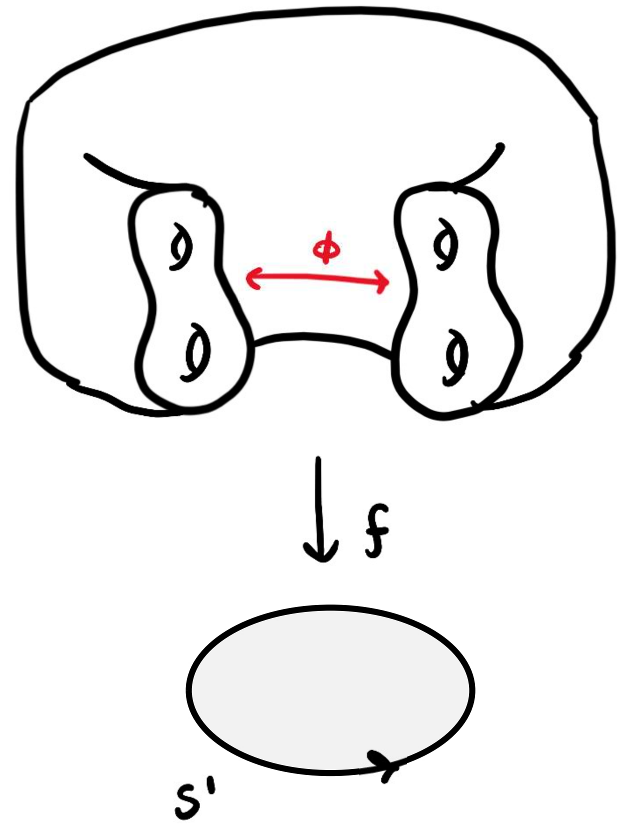
- embedded S^1 in base space
- pre-image is $S^1 \times \Sigma_g$
- the **monodromy** is the self-diffeo of a regular fiber Σ_g to itself



Definition: monodromy of a Lefschetz fibration

Remarks:

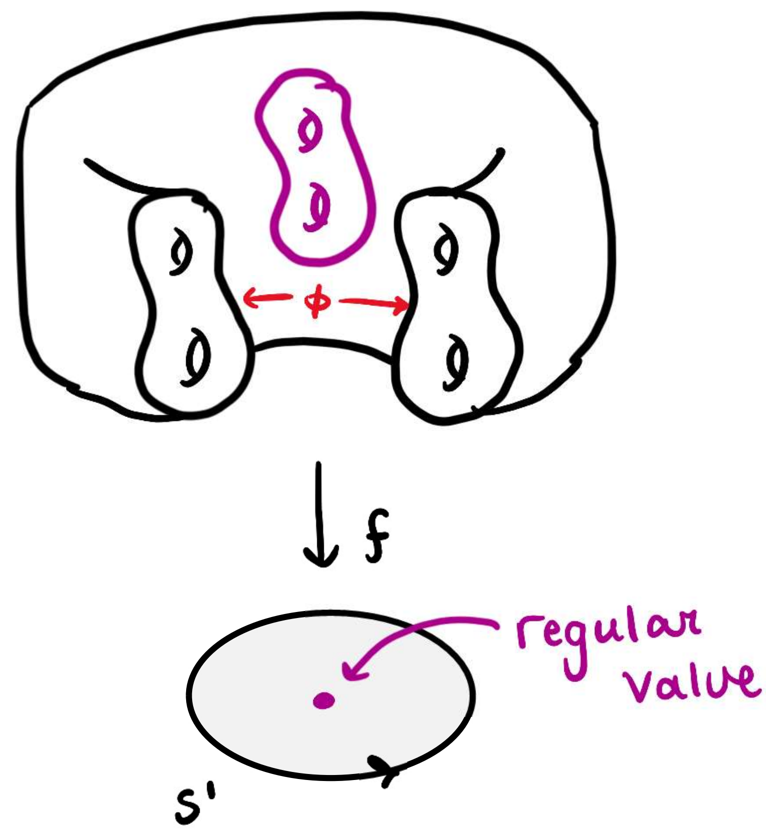
- the **monodromy** is the self-diffeo of a regular fiber Σ_g to itself
- Denoted ϕ
- $\phi \in Mod(\Sigma_g)$



Definition: monodromy of a Lefschetz fibration

No critical values in D^2

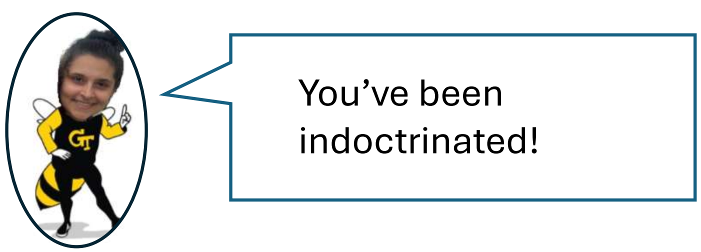
- $\phi =$ how to glue Σ_g to itself
- $\phi = Id$

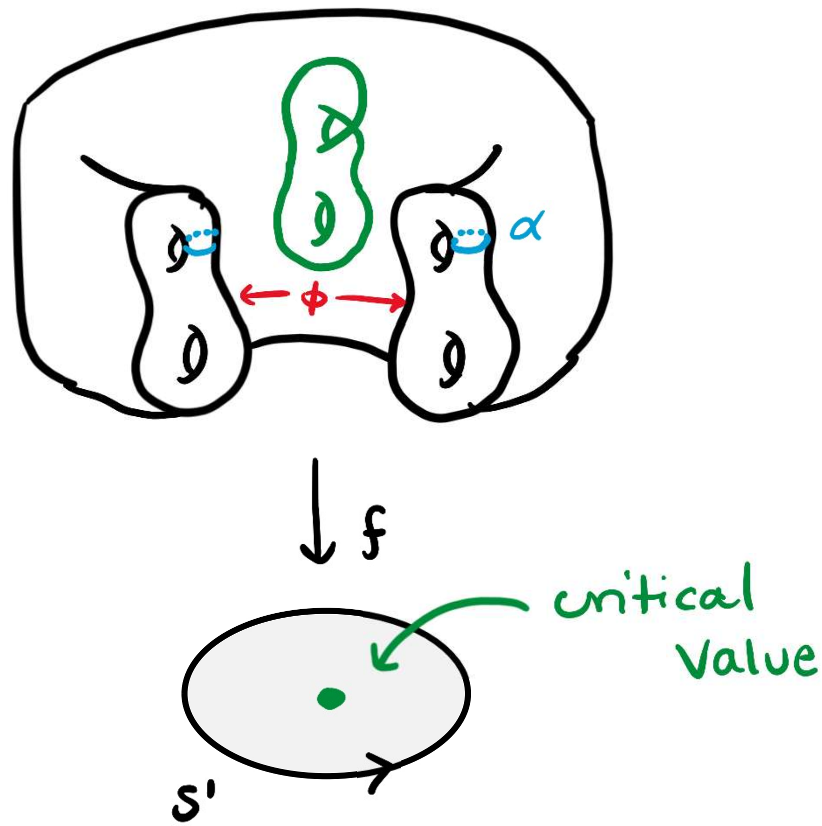


Definition: monodromy of a Lefschetz fibration

One critical value in D^2:

- $\phi = \tau_\alpha$
- Positive (left-handed) Dehn twist about vanishing cycle α



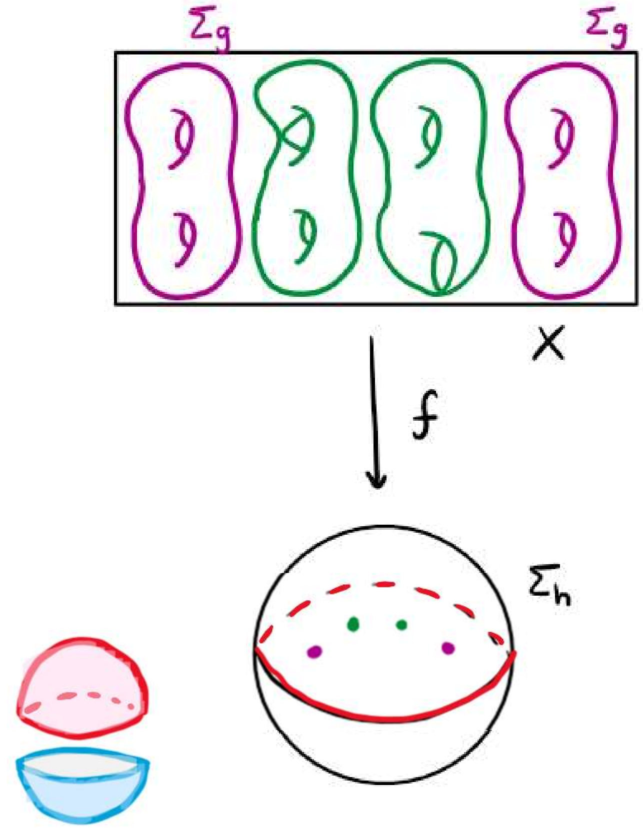


Monodromy is identity in $Mod(\Sigma_g)$ of a Lefschetz fibration

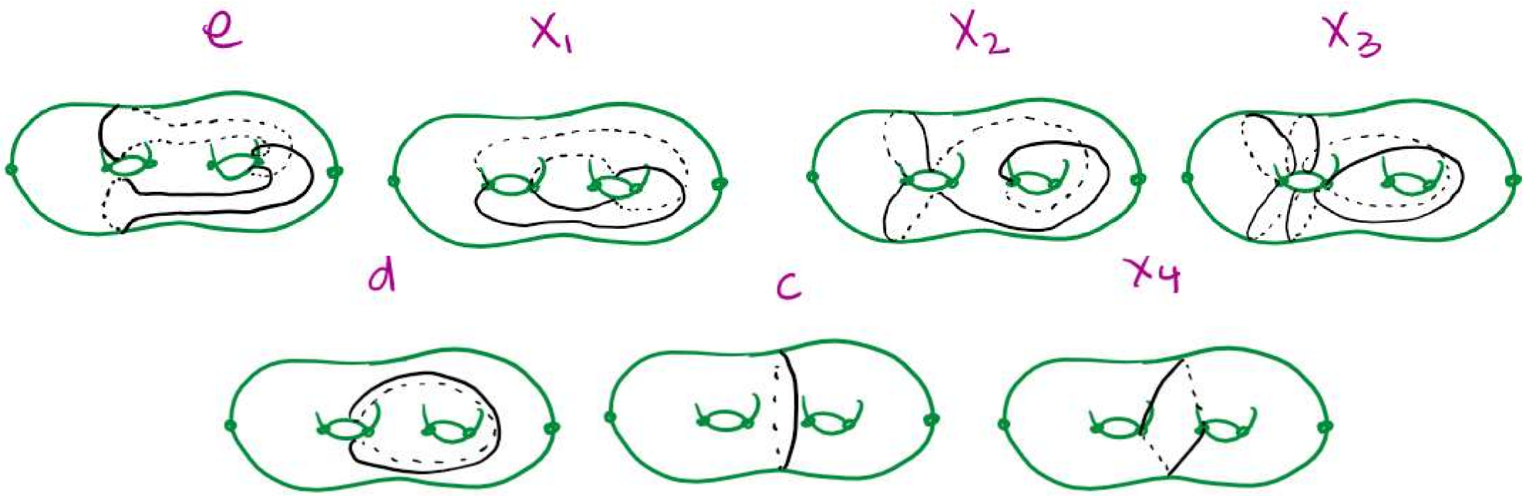
Why is $\phi \in Mod(\Sigma_g)$?

- We are only considering Lefschetz fibrations $f: X \to S^2$
- Let $S^1 \subset S^2$ enclose all critical values $q_1, q_2, \dots, q_N$
- Then,

$\phi = \tau_{q_1} \circ \tau_{q_2} \circ \cdots \circ \tau_{q_N} = id \in Mod(\Sigma_g)$



Example of a genus-2 Lefschetz fibration



$$\phi = \tau_e \tau_{x_1} \tau_{x_2} \tau_{x_3} \tau_d \tau_c \tau_{x_4}$$

Remarks:

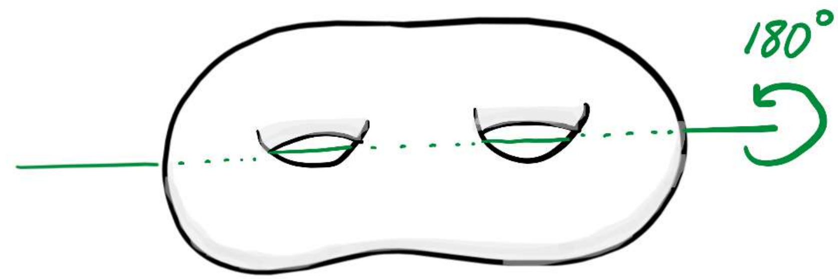
- Vanishing cycles e, d, and c are separating, whereas x_1, x_2, x_3, and x_4 are non-separating
- This Lefschetz fibration is said to be **length 7** and of type $(4{,}3) = (n, s)$

Part II

Motivation

What's special about genus 2?

- All vanishing cycles of a genus-2 Lefschetz fibration are loops on Σ_2



- All embedded loops on Σ_2 are hyperelliptic

- If $f: X \to S^2$ is of type (n, s), then

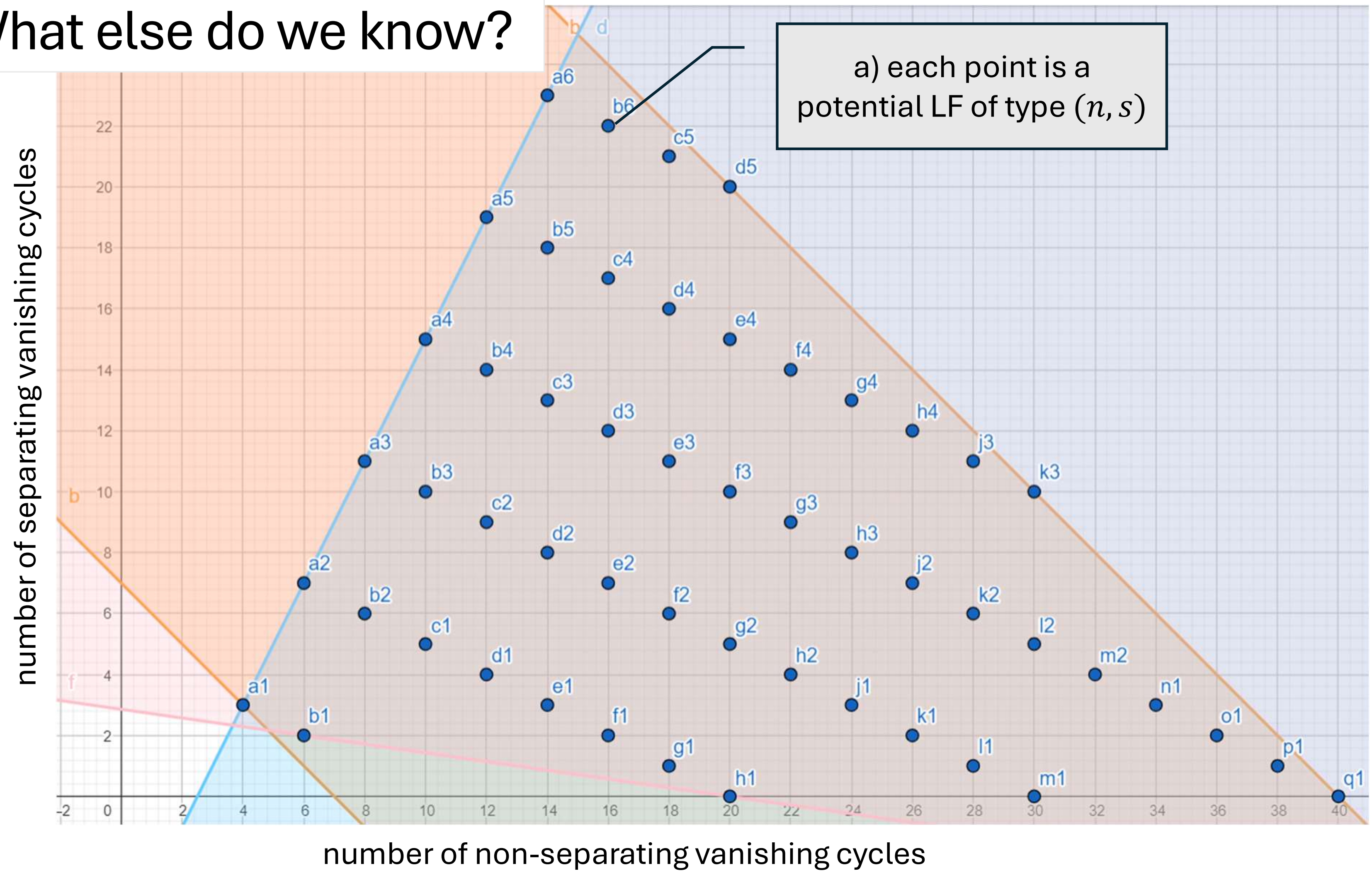
  a) $n + 12s \equiv 0 \ (mod\ 10)$
  b) $2n - s \geq 3$
  c) $n + 7s \geq 20$

  } Baykur-Korkmaz, 2016

  d) a) + b) $\Rightarrow 2n - s \geq 5$

  } Knavel

What else do we know?



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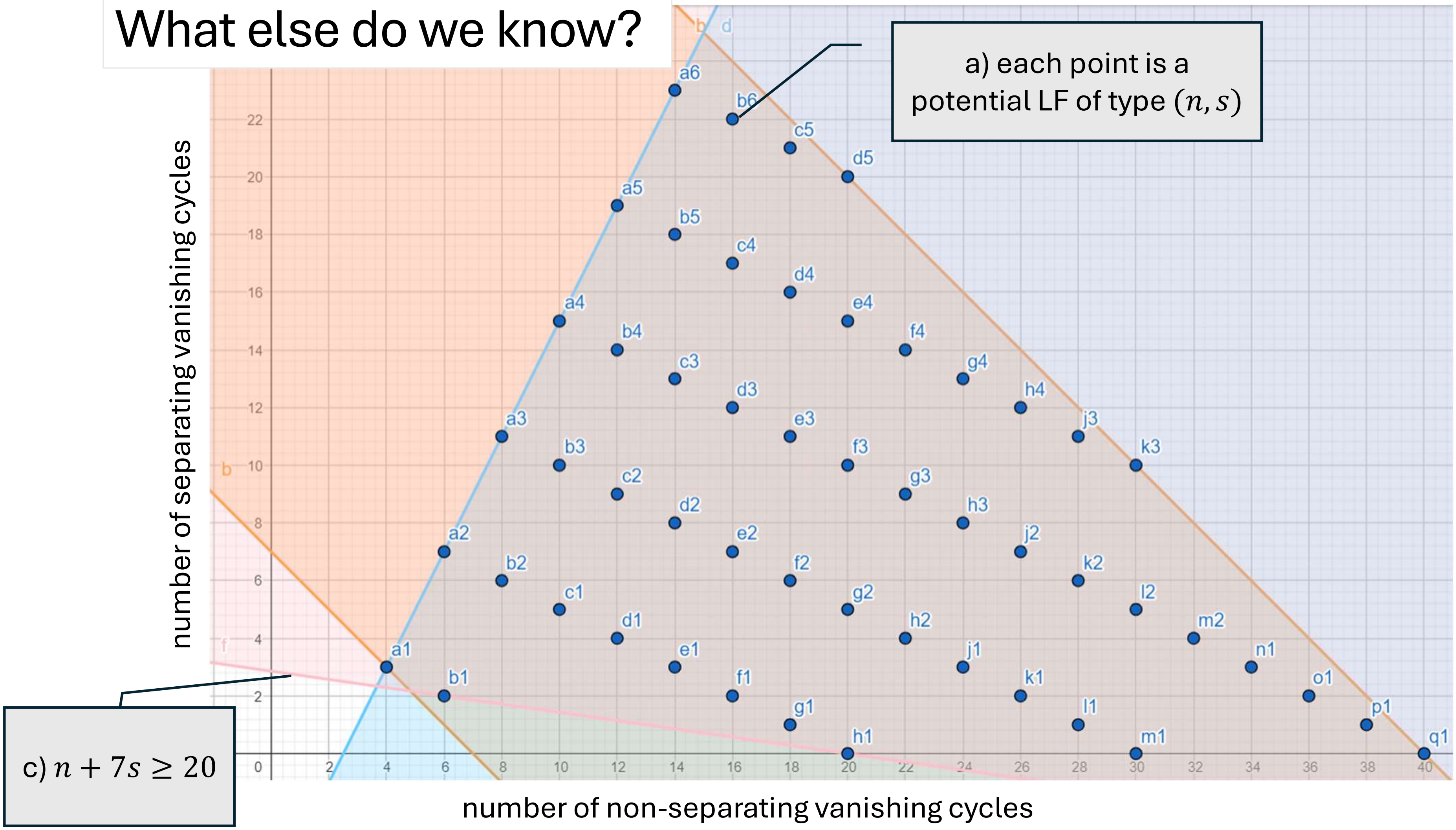

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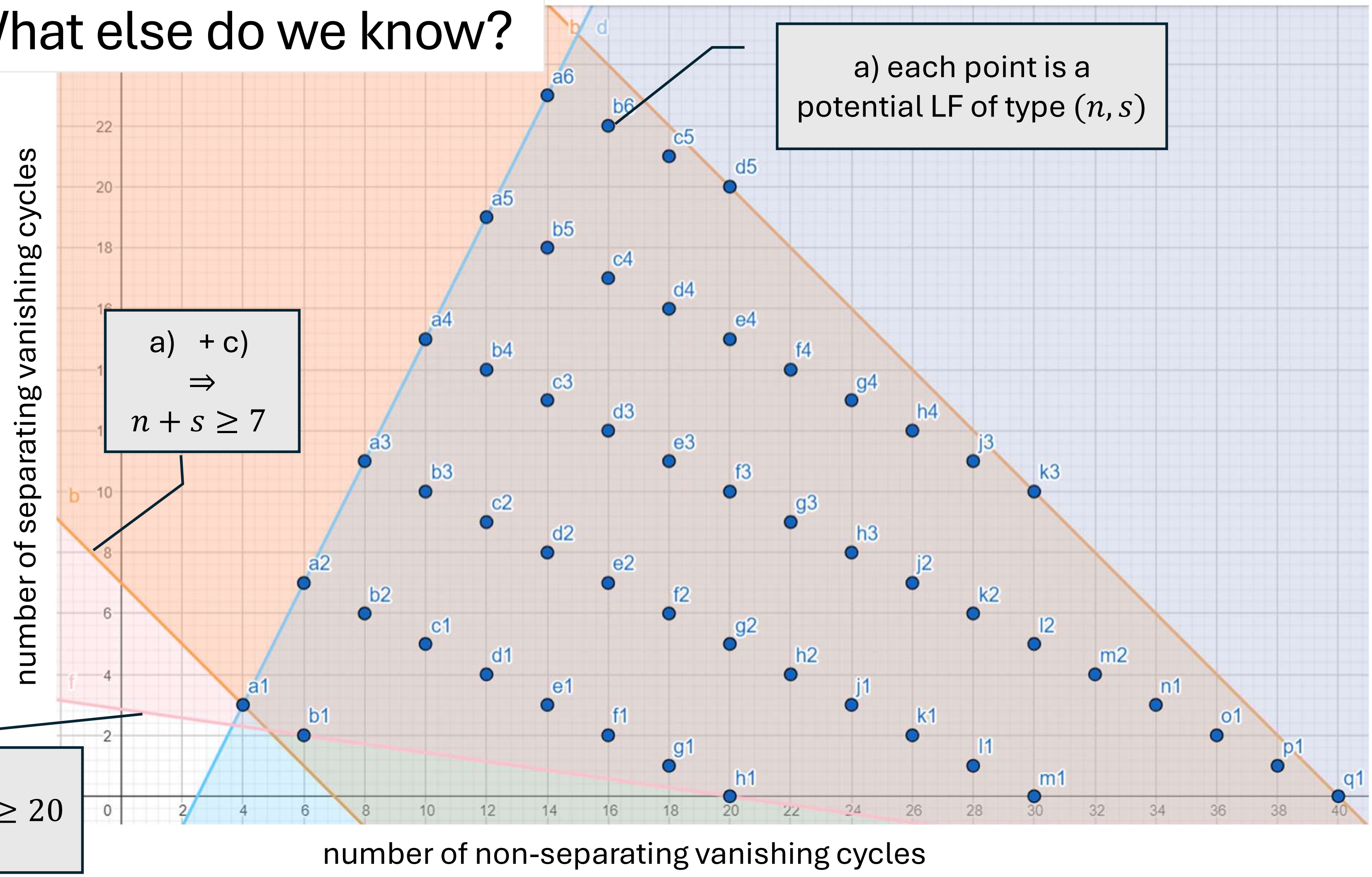

a) each point is a potential LF of type (n, s)

a) + c) $\Rightarrow$ $n + s \geq 7$

c) $n + 7s \geq 20$

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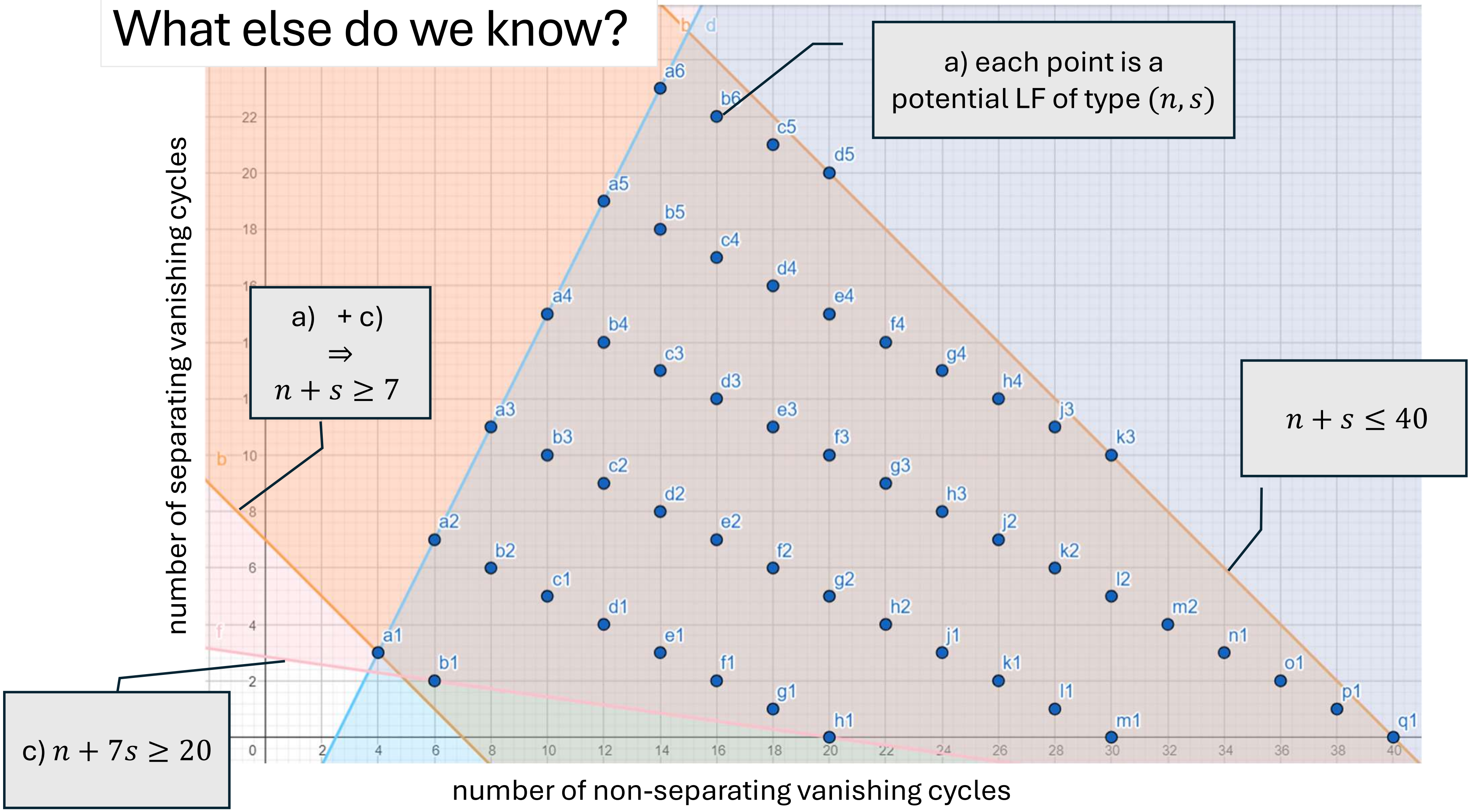

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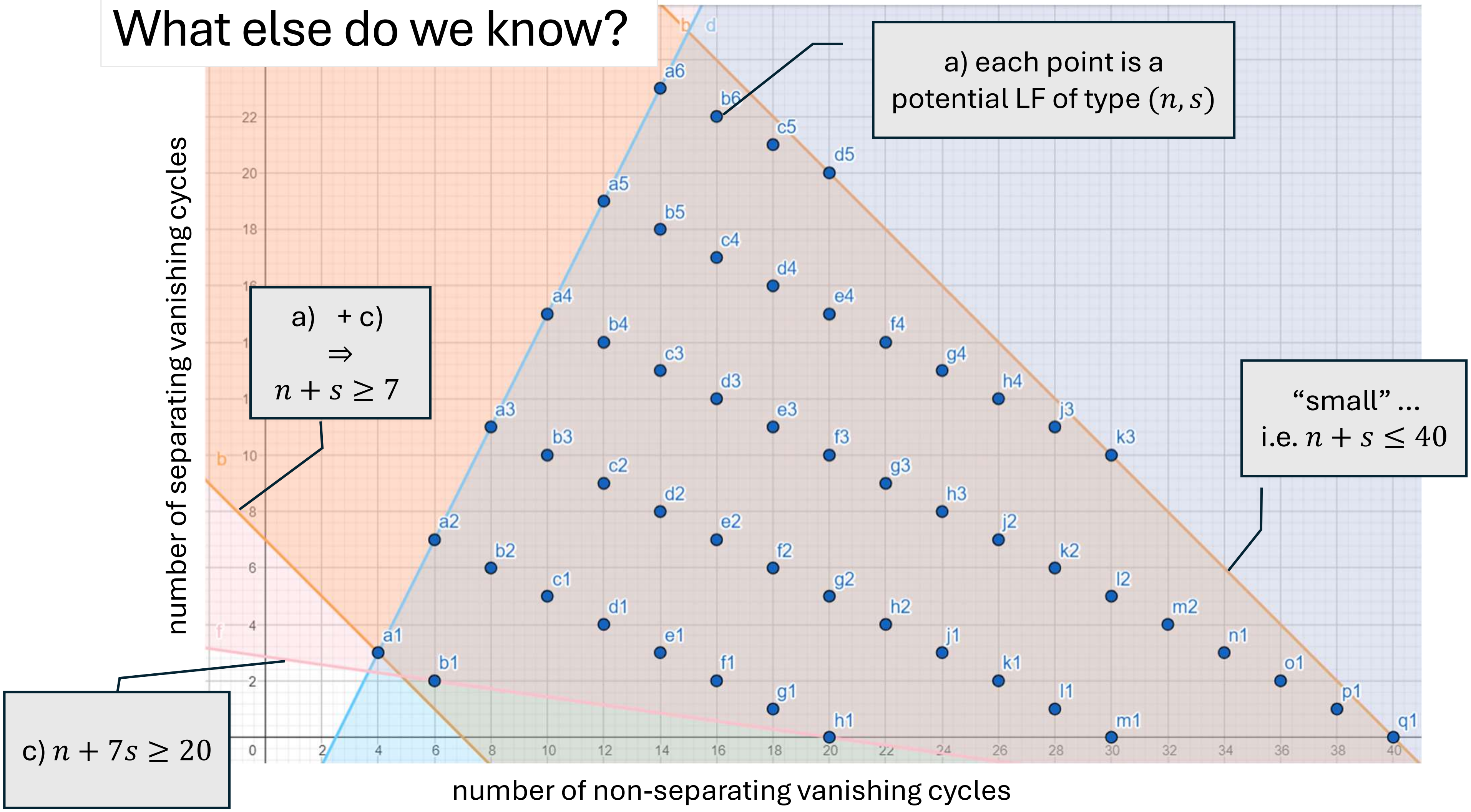

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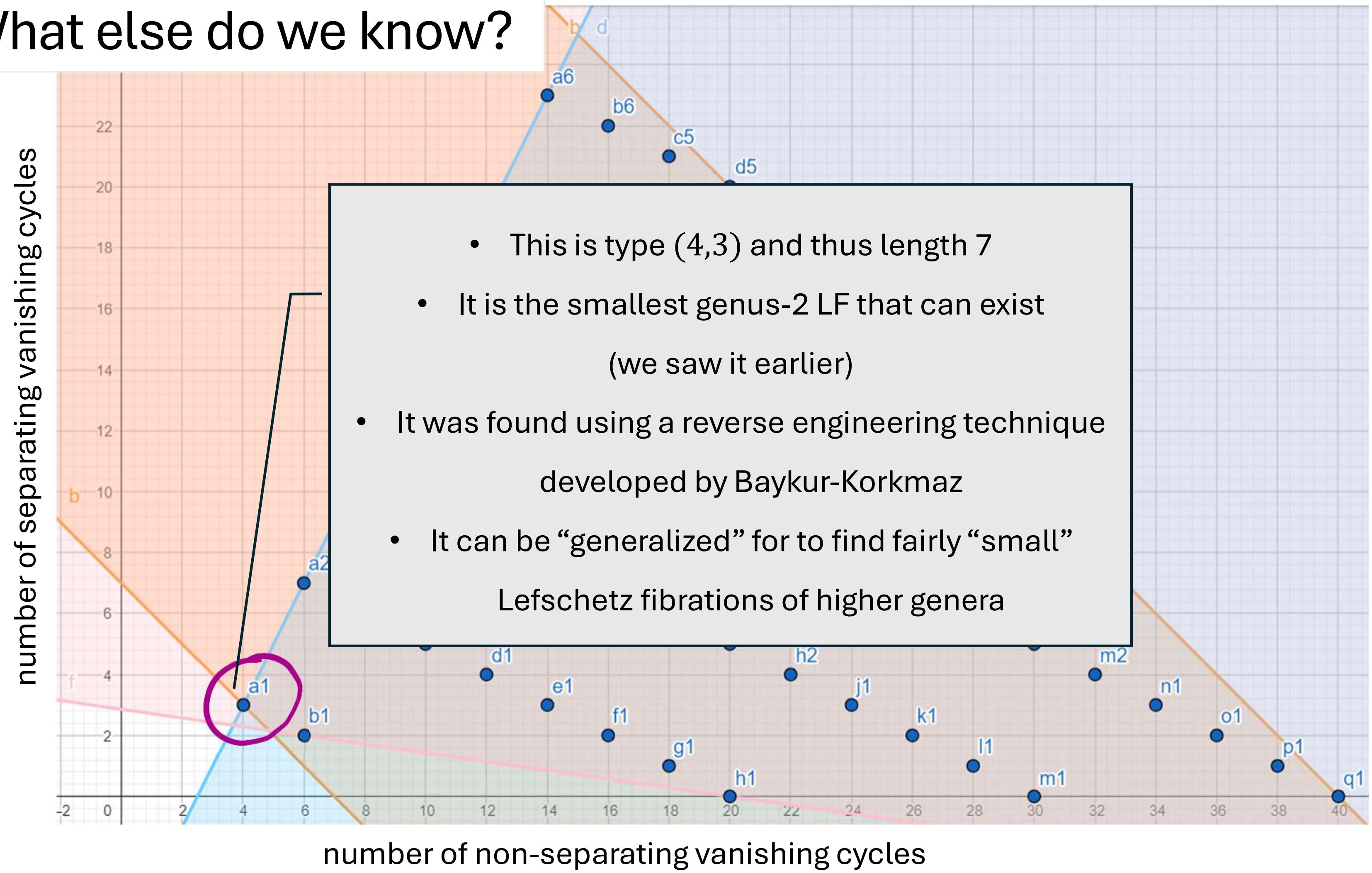
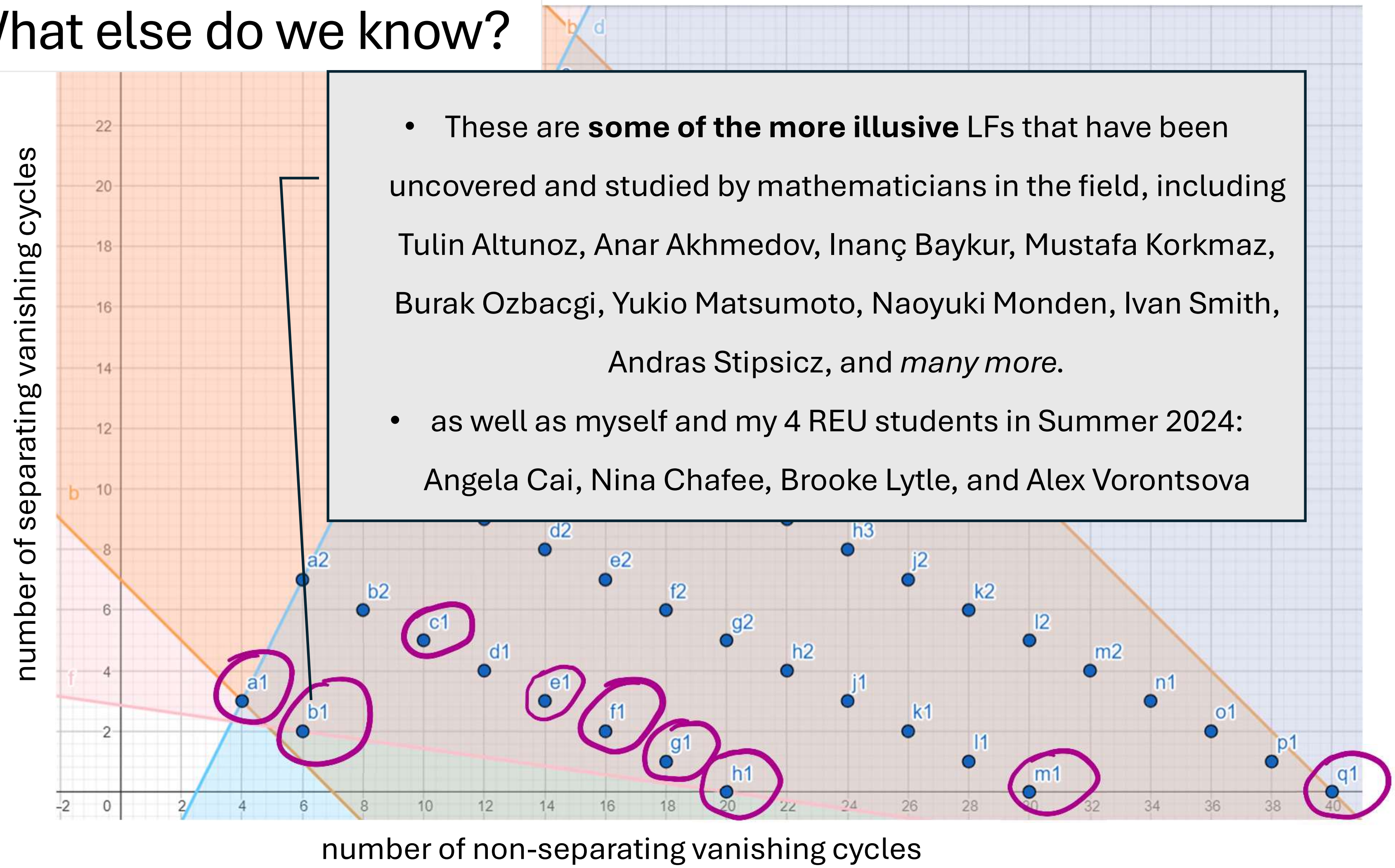


- This is type $(4,3)$ and thus length 7
- It is the smallest genus-2 LF that can exist (we saw it earlier)
- It was found using a reverse engineering technique developed by Baykur-Korkmaz
- It can be "generalized" for to find fairly "small" Lefschetz fibrations of higher genera

What else do we know?

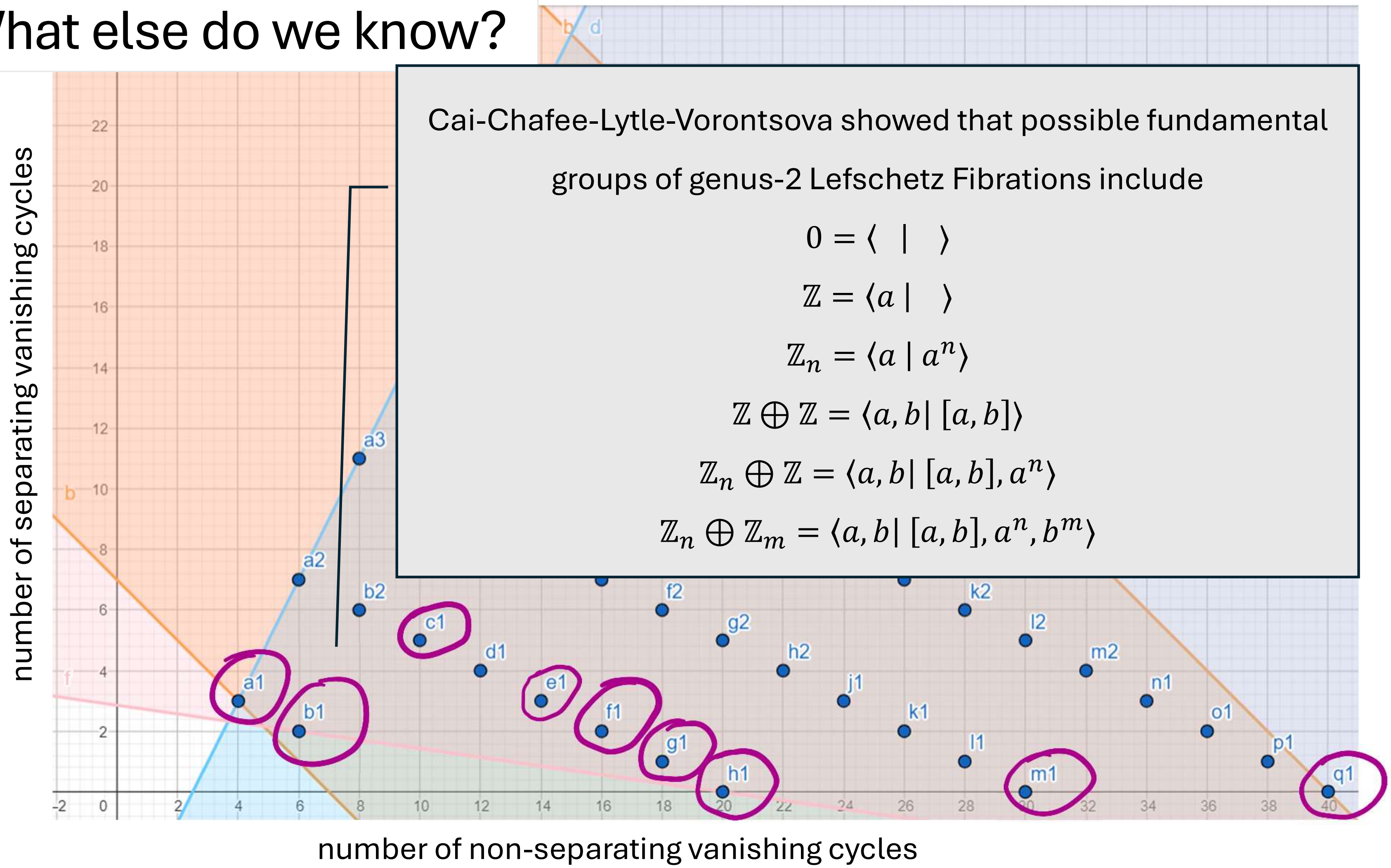
- These are **some of the more illusive** LFs that have been uncovered and studied by mathematicians in the field, including Tulin Altunoz, Anar Akhmedov, Inanç Baykur, Mustafa Korkmaz, Burak Ozbacgi, Yukio Matsumoto, Naoyuki Monden, Ivan Smith, Andras Stipsicz, and *many more*.
- as well as myself and my 4 REU students in Summer 2024: Angela Cai, Nina Chafee, Brooke Lytle, and Alex Vorontsova



Part III

Some results

What else do we know?



Cai-Chafee-Lytle-Vorontsova showed that possible fundamental groups of genus-2 Lefschetz Fibrations include

$$0 = \langle \ \mid \ \rangle$$

$$\mathbb{Z} = \langle a \mid \ \rangle$$

$$\mathbb{Z}_n = \langle a \mid a^n \rangle$$

$$\mathbb{Z} \oplus \mathbb{Z} = \langle a, b \mid [a, b] \rangle$$

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coo

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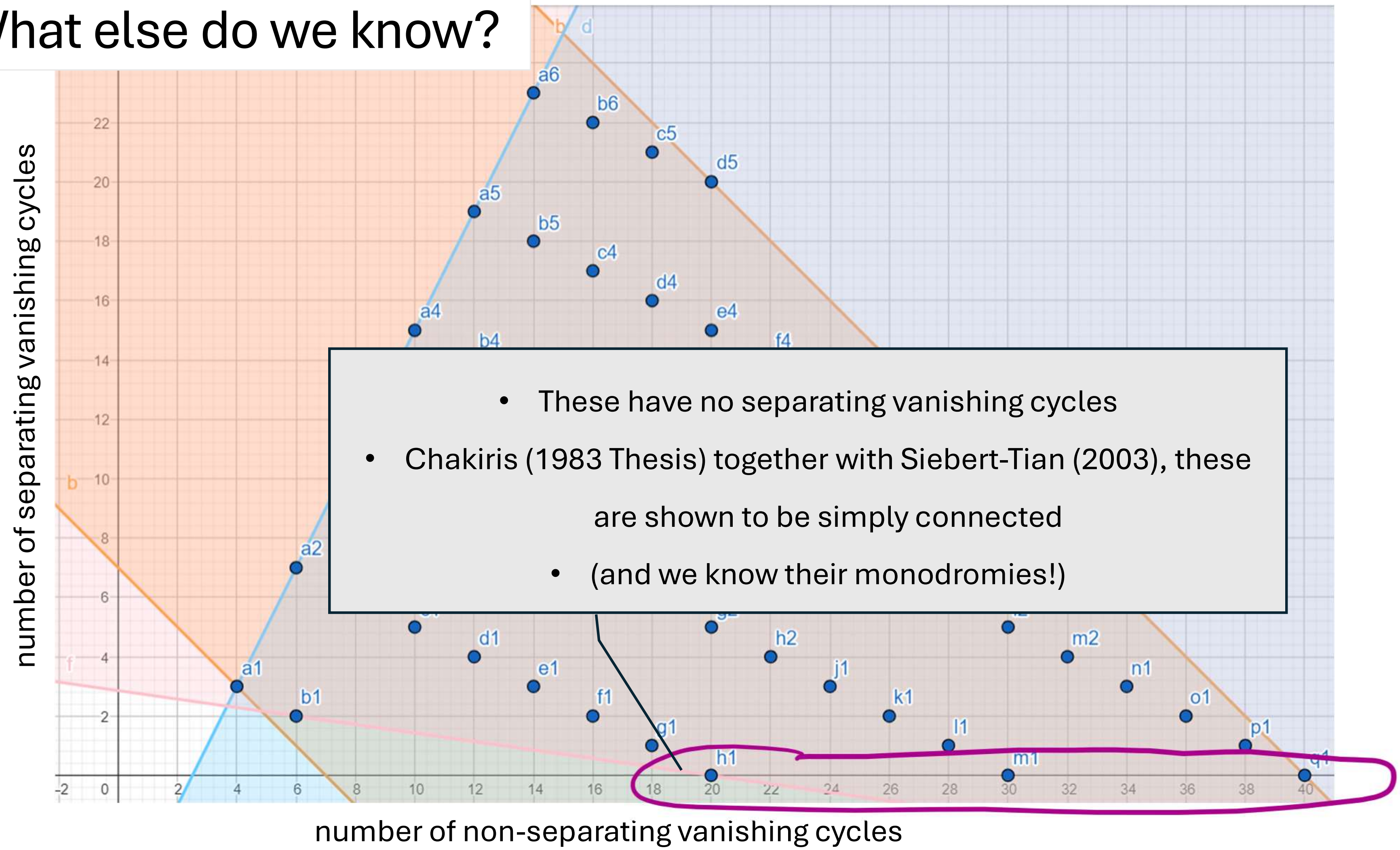
Hear ye, hear ye!
By proclamation of this Holy Hypothesis:

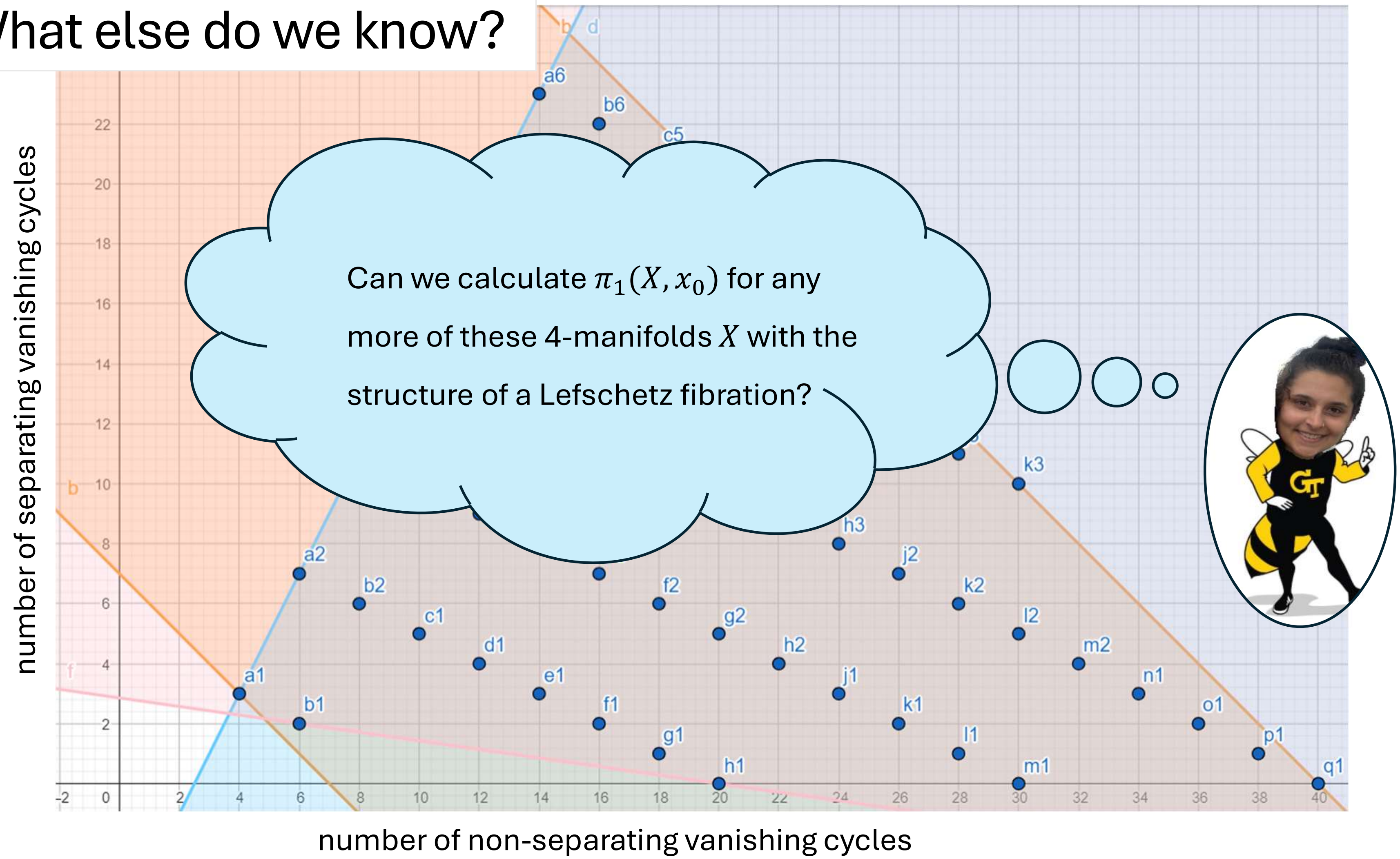
Lo, it is thusly declared that no genus-2 Lefschetz fibration doth exist wherein three or more mighty generators do cometh forth to bear the burden of its fundamental group. Verily, it is also perchance possible that, in all such cases, the fundamental group be evermore Abelian in nature!

What else do we know?



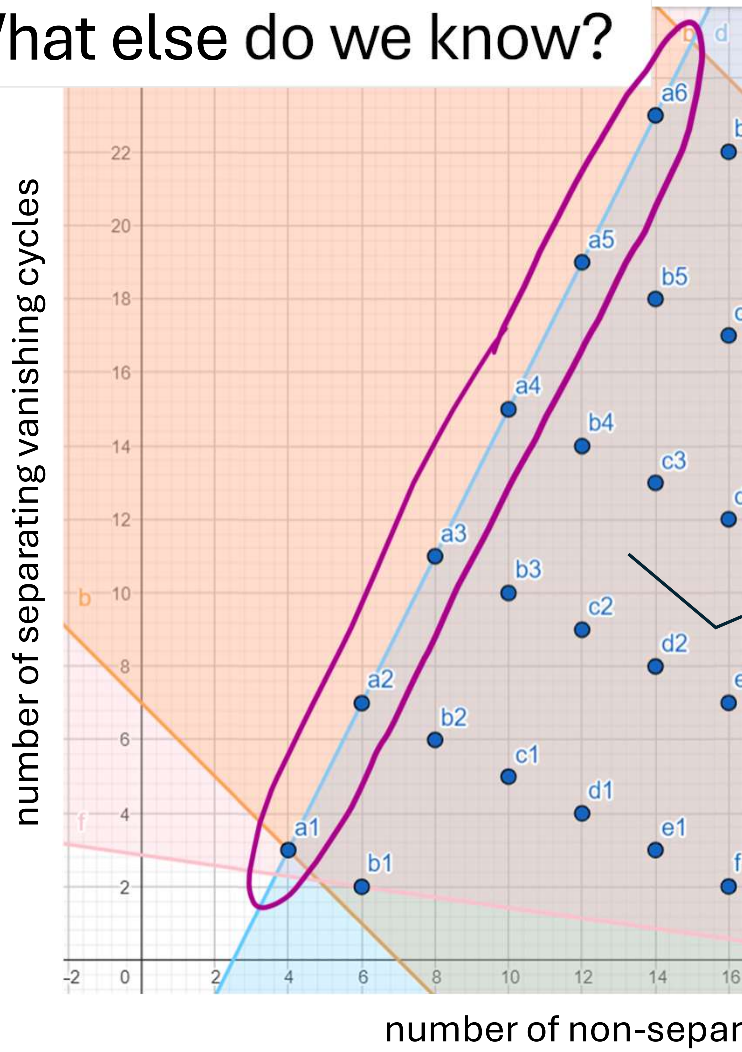
- These have no separating vanishing cycles
- Chakiris (1983 Thesis) together with Siebert-Tian (2003), these are shown to be simply connected
- (and we know their monodromies!)

What else do we know?



Can we calculate $\pi_1(X, x_0)$ for any more of these 4-manifolds X with the structure of a Lefschetz fibration?

What else do we know?



<u>Claim</u>: These are guaranteed not simply connected

Proof:

Suppose $b_1(X) = 0$. Then,

$$b_2^+(X) = 2\chi_h(X) + 2b_1(X) - 1 = \frac{1}{2}\big(e(X) + \sigma(X)\big) + b_1 - 1$$
$$= \frac{1}{2}\left(\frac{2}{5}n + \frac{4}{5}s\right) - 3 = \frac{1}{5}n + \frac{2}{5}s - 3$$

Recall: $\sigma(X) = b_2^+ - b_2^-$, so $b_2^- = b_2^+ - \sigma(X)$

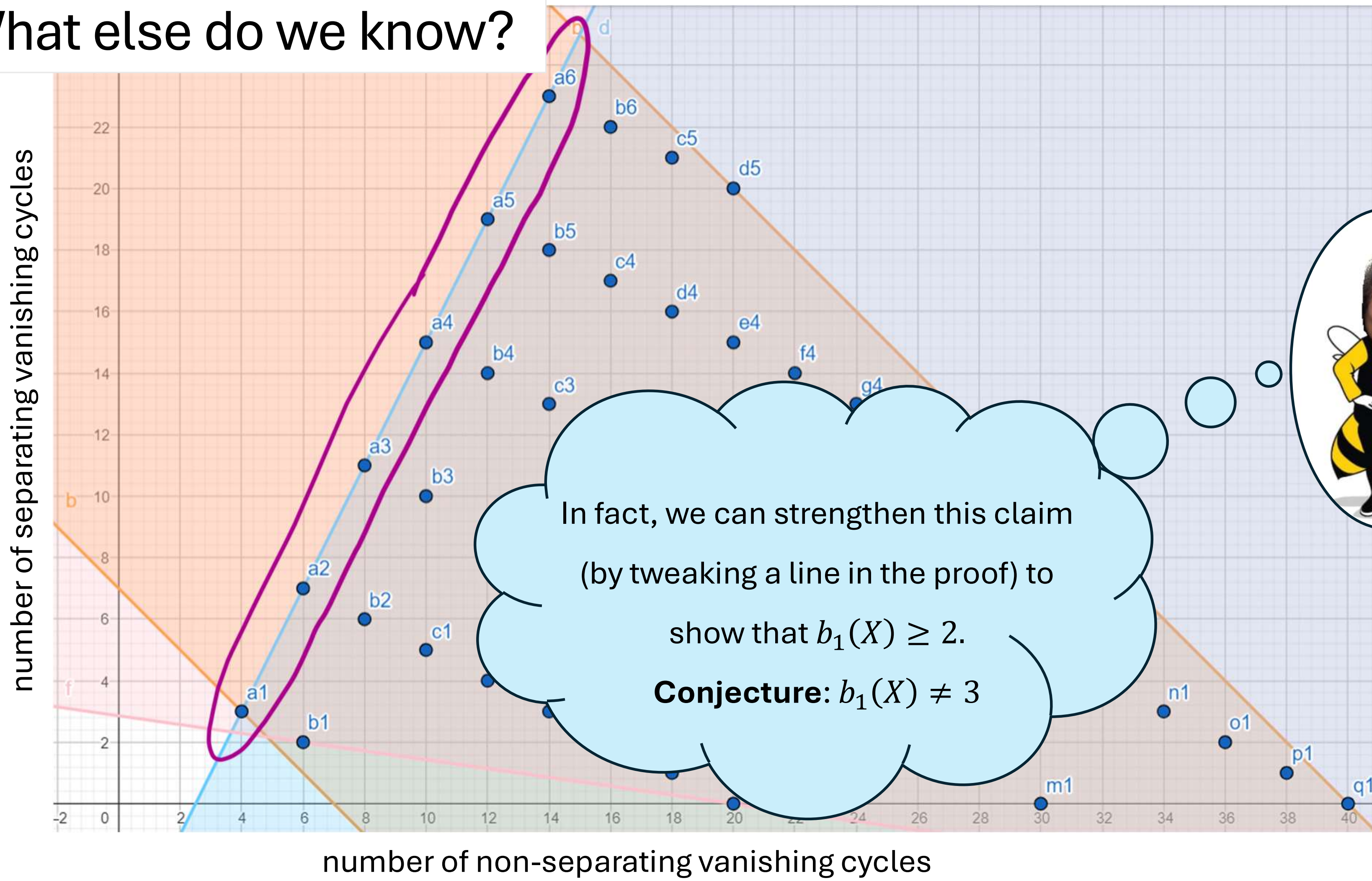
Therefore $b_2^- = \frac{1}{5}n + \frac{2}{5}s - 3 + \frac{3}{5}n + \frac{1}{5}s = \frac{2}{5}(2n) + \frac{3}{5}s - 3$

Remark: these sit on the line $2n - s = 5$, so

$$b_2^- = \frac{2}{5}(s+5) + \frac{3}{5}s - 3$$
$$= s - 1$$

But, it is known that each separating vanishing cycle contributes to $H_2(X)$. In fact, $b_2^- \geq s + 1$. So contradiction.

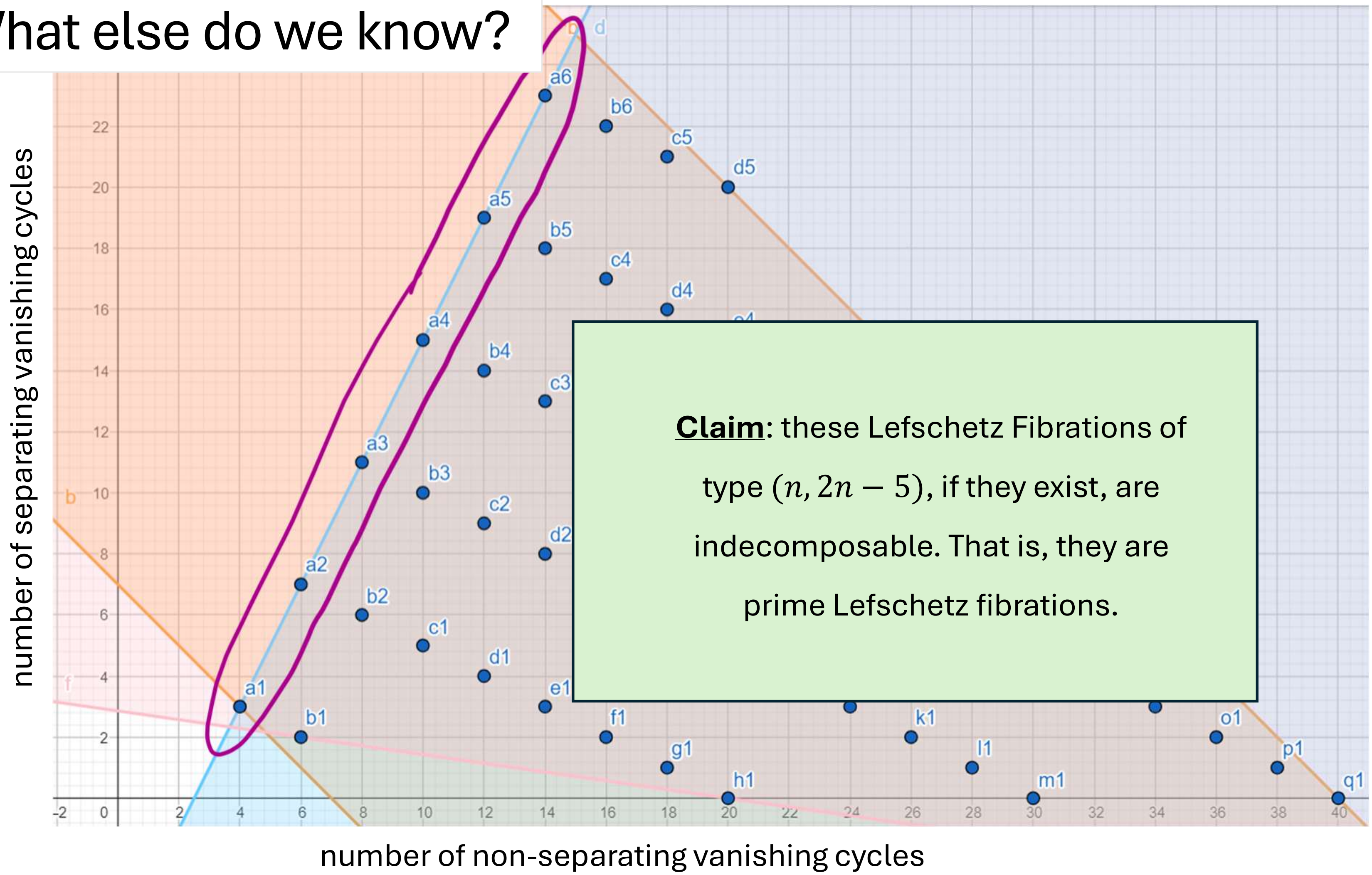
What else do we know?



In fact, we can strengthen this claim (by tweaking a line in the proof) to show that $b_1(X) \geq 2$.

Conjecture: $b_1(X) \neq 3$

What else do we know?



<u>Claim</u>: these Lefschetz Fibrations of type $(n, 2n-5)$, if they exist, are indecomposable. That is, they are prime Lefschetz fibrations.

What else do we know?

Claim: these Lefschetz Fibrations of type $(n, 2n - 5)$, if they exist, are indecomposable. That is, they are prime Lefschetz fibrations.

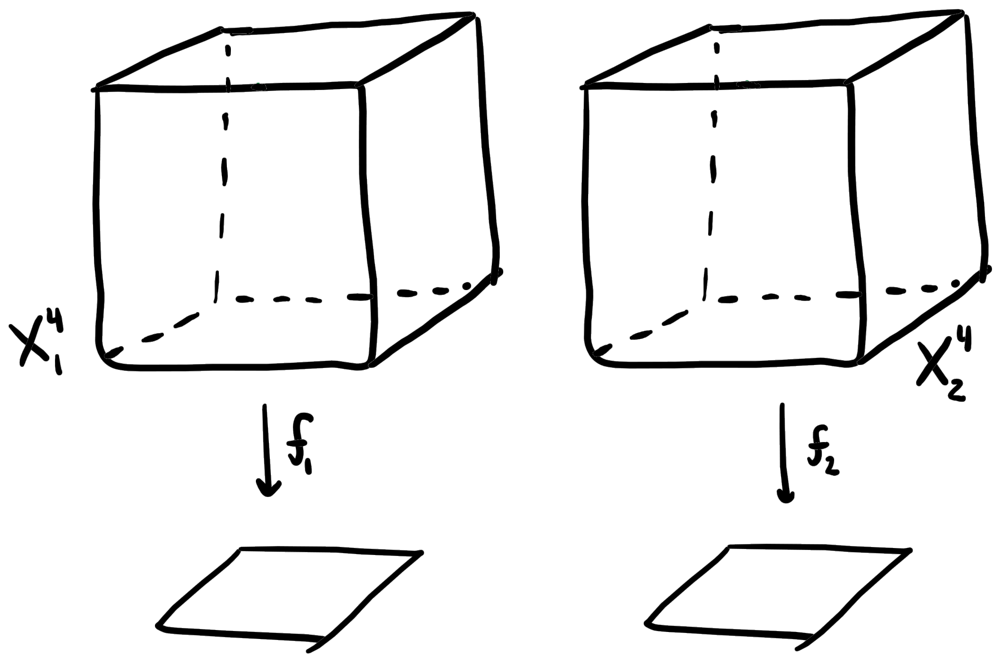
A Lefschetz fibration is indecomposable if it cannot be realized as the fiber sum of two nontrivial Lefschetz fibrations

The proof of this claim requires understanding the **fiber sum.**

Fiber Summing two Lefschetz fibrations outputs a new Lefschetz fibration.

The “addition” respects the fiber direction and thus is only defined when the genera of the fibrations agree.

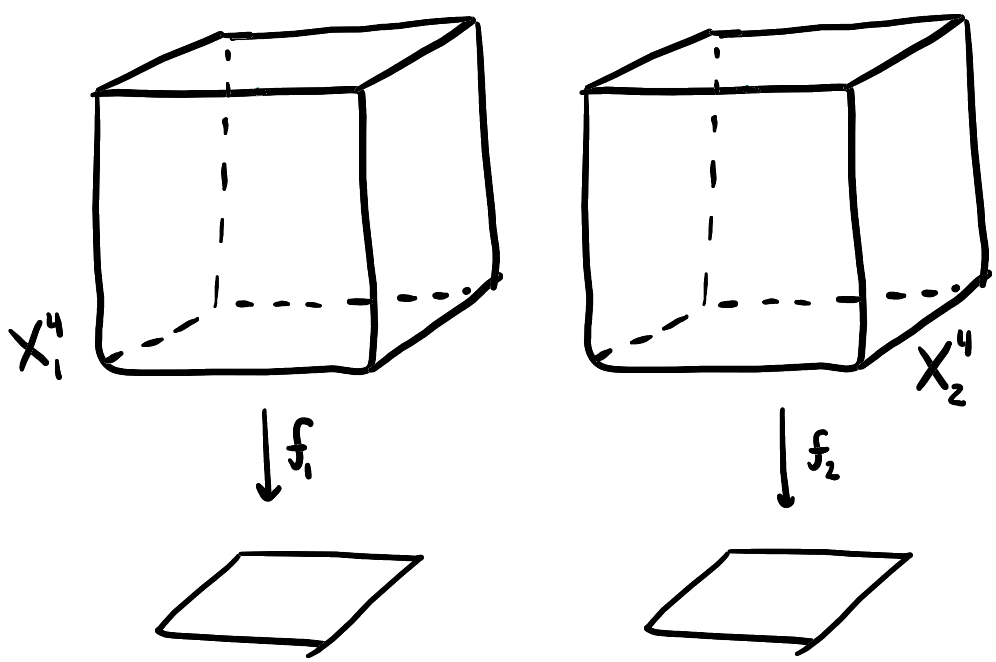
Fiber Sum



Fiber Sum $X_1 \#_f X_2$

1. Let $C_1 = F \times D^2$ in X_1

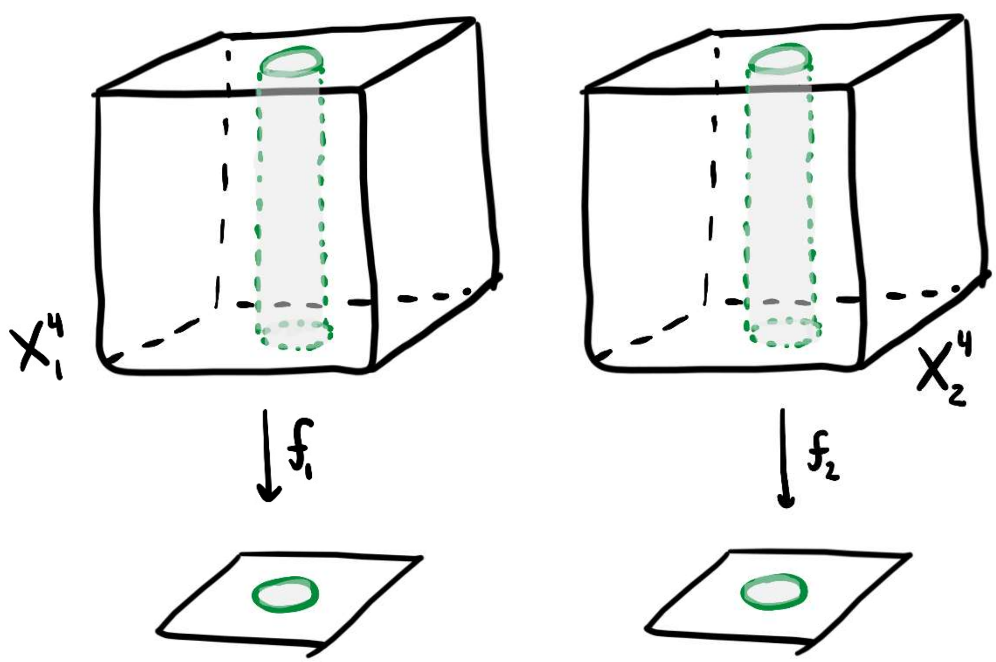
Fiber Sum



Fiber Sum $X_1 \#_f X_2$

1. Let $C_1 = F \times D^2$ in X_1
2. And $C_2 = F \times D^2$ in X_2

Fiber Sum



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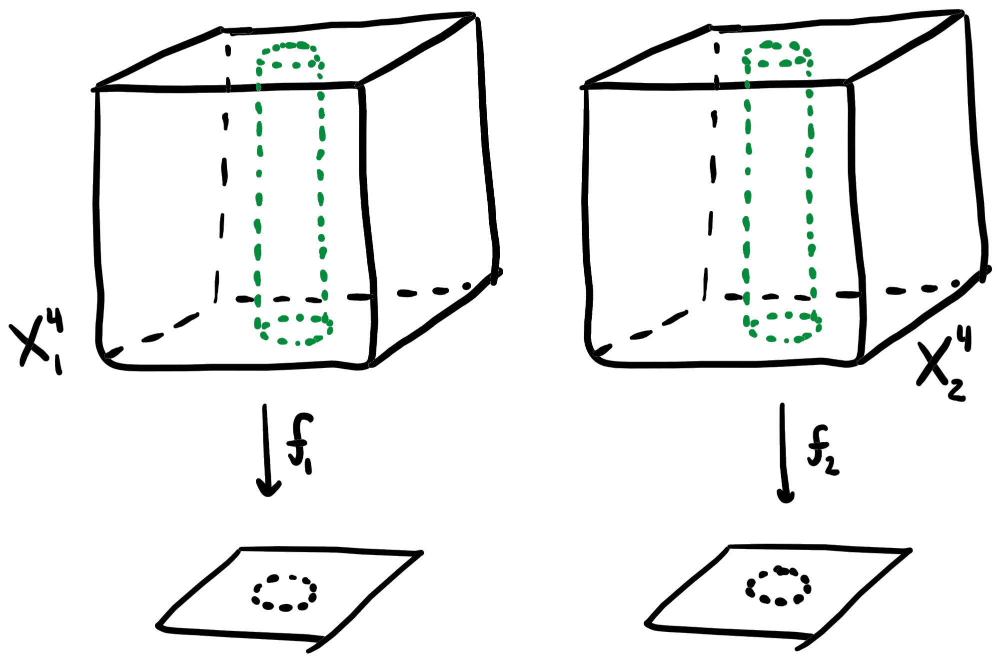

Fiber Sum $X_1 \#_f X_2$

1. Let $C_1 = F \times D^2$ in X_1
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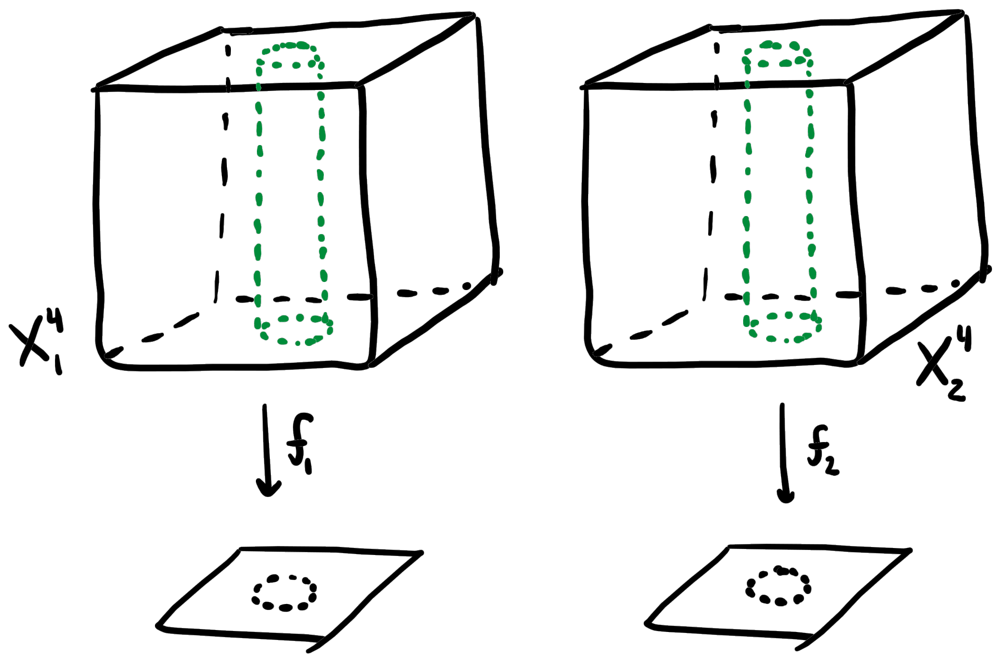
3. Remove C_i from X_i

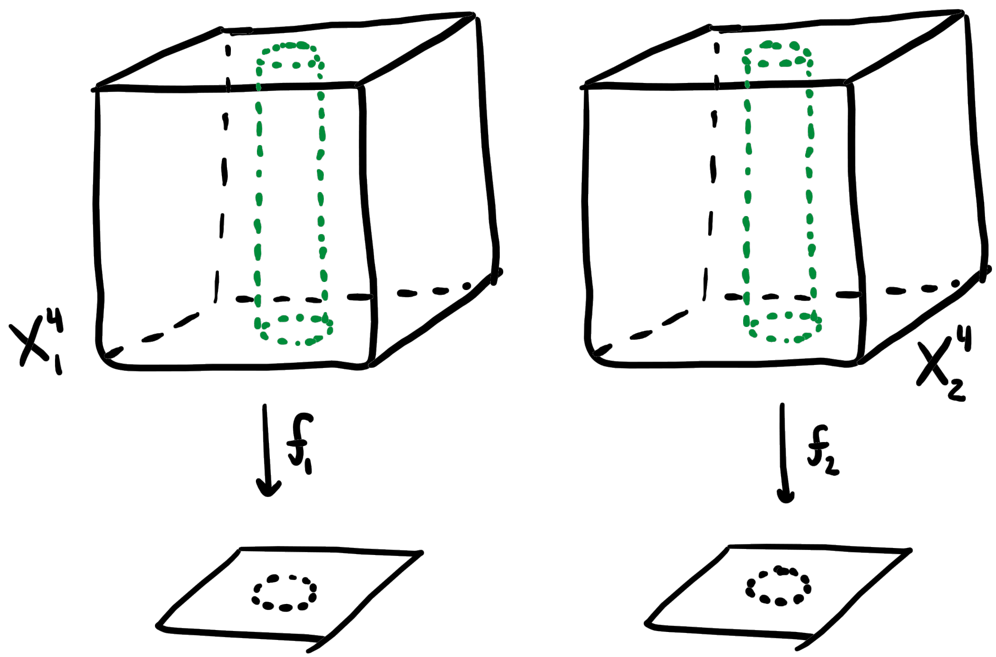
Fiber Sum



Fiber Sum $X_1 \#_f X_2$

1. Let $C_1 = F \times D^2$ in X_1
2. And $C_2 = F \times D^2$ in X_2
3. Remove C_i from X_i
4. Glue $\partial(X_1 - C_1)$ to $\partial(X_2 - C_2)$ by a fiber-preserving, orientation reversing diffeo

Fiber Sum



Fiber Sum $X_1 \#_f X_2$

Remark:

New monodromy is

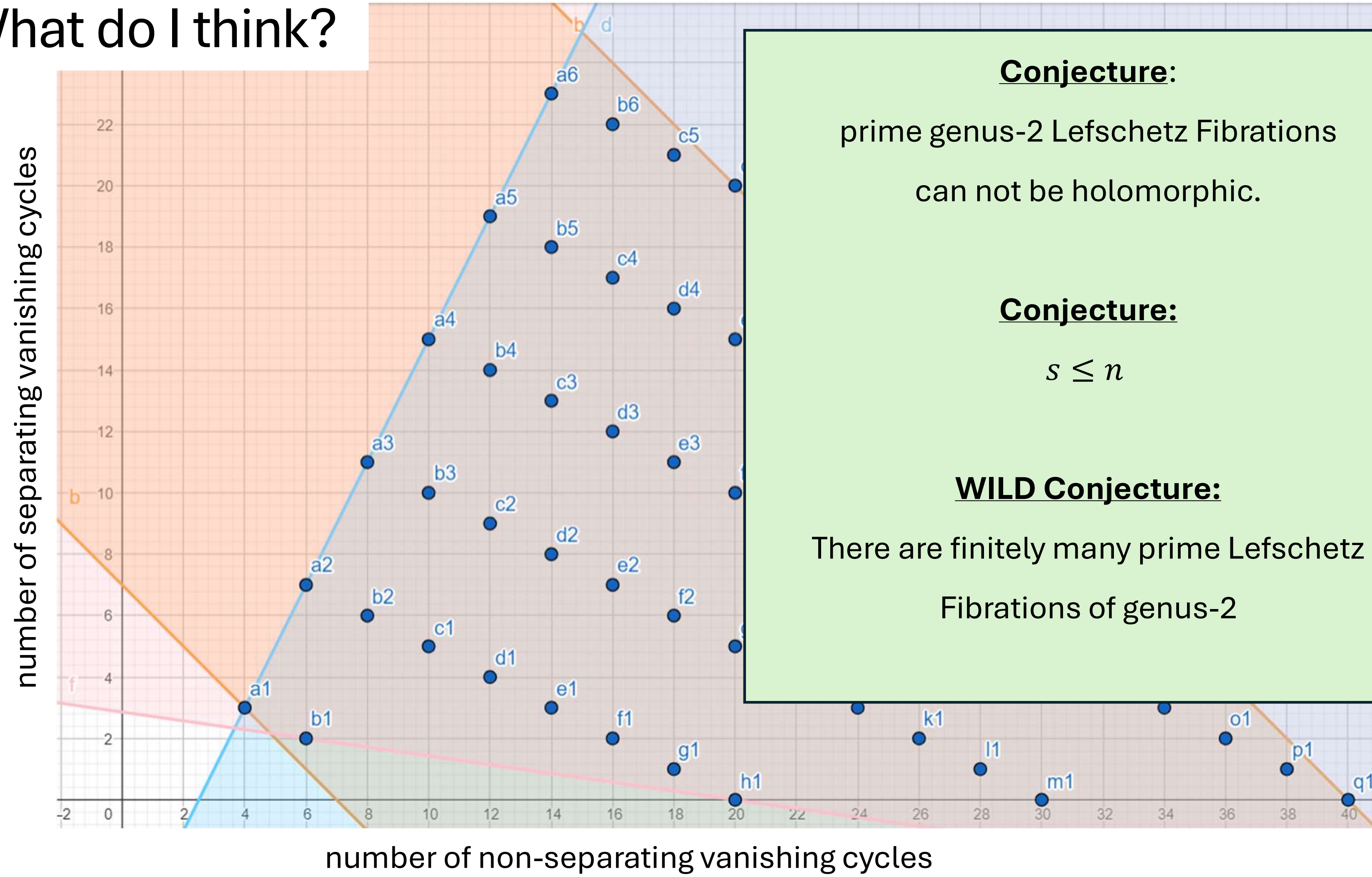
$$\phi_1\phi_2 = Id$$

Part III

Some final thoughts

(yes, it’s almost over)

What do I think?



Conjecture:

prime genus-2 Lefschetz Fibrations

can not be holomorphic.

Conjecture:

$s \leq n$

WILD Conjecture:

There are finitely many prime Lefschetz

Fibrations of genus-2

Thanks for listening!

n	s	true if s >= n	n+s	2n-s-3>=0	n+7s-20>=0	n+7s<30?	n+7s-40>=0 (minimal)	euler e(X) = n+s-4	signature =3n/5-1s/5	new inequality from signatu 2n-s>=5	b1 lower bound 4/5n+2/5s	c1^2 <=6e(X)-3 (ozbagci)	c1^2 = 3signature+2e(X) (ozbagci)	Chi_h = .25(e+sig)	b+ = .5(e + sig) + b1 - 1	b- = b+ - sig	is b- >= s+1?	b+	b-	is b- >= s+1?	b+	b-	is b- >= s+1?	b+	b-	
4	3	FALSE	7	2	5	-5	-15	3	-3	0	2	15	-3	0	2	5	TRUE	1	4	TRUE	0	3	FALSE	-1	2	FALSE
6	2	FALSE	8	7	0	-10	-20	4	-4	5	0	21	-4	0	2	6	TRUE	1	5	TRUE	0	4	TRUE	-1	3	TRUE
6	7	TRUE	13	2	35	25	15	9	-5	0	2	51	3	1	4	9	TRUE	3	8	TRUE	2	7	FALSE	1	6	FALSE
8	6	FALSE	14	7	30	20	10	10	-6	5	0	57	2	1	4	10	TRUE	3	9	TRUE	2	8	TRUE	1	7	TRUE
8	11	TRUE	[illegible]	2	65	55	45	15	-7	0	2	87	9	2	6	13	TRUE	5	12	TRUE	4	11	FALSE	3	10	FALSE
10	5	FALSE	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11	-7	10	0	63	1	1	4	11	TRUE	3	10	TRUE	2	9	TRUE	1	8	TRUE
10	10	TRUE	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	16	-8	5	0	93	8	2	6	14	TRUE	5	13	TRUE	4	12	TRUE	3	11	TRUE
10	15	TRUE	[illegible]	[illegible]	[illegible]	85	75	21	-9	0	2	123	15	3	8	17	TRUE	7	16	TRUE	6	15	FALSE	5	14	FALSE
12	4	FALSE	16	17	20	10	0	12	-8	15	0	69	0	1	4	12	TRUE	3	11	TRUE	2	10	TRUE	1	9	TRUE
12	9	FALSE	21	12	55	45	35	17	-9	10	0	99	7	2	6	15	TRUE	5	14	TRUE	4	13	TRUE	3	12	TRUE
12	14	TRUE	26	[illegible]	90	80	70	22	-10	5	0	129	14	3	8	18	TRUE	7	17	TRUE	6	16	TRUE	5	15	TRUE
12	19	TRUE	31	[illegible]	[illegible]	115	105	27	-11	0	2	159	21	4	10	21	TRUE	9	20	TRUE	8	19	FALSE	7	18	FALSE
14	3	FALSE	17	[illegible]	[illegible]	5	-5	13	-9	20	0	75	-1	1	4	13	TRUE	3	12	TRUE	2	11	TRUE	1	10	TRUE
14	8	FALSE	22	[illegible]	50	40	30	18	-10	15	0	105	6	2	6	16	TRUE	5	15	TRUE	4	14	TRUE	3	13	TRUE
14	13	FALSE	27	12	85	75	65	23	-11	10	0	135	13	3	8	19	TRUE	7	18	TRUE	6	17	TRUE	5	16	TRUE
14	18	TRUE	32	7	120	110	100	28	-12	5	0	165	20	4	10	22	TRUE	9	21	TRUE	8	20	TRUE	7	19	TRUE
14	23	[illegible]	37	2	155	145	135	33	-13	0	2	195	27	5	12	25	TRUE	11	24	TRUE	10	23	FALSE	9	22	FALSE
16	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14	-10	25	0	81	-2	1	4	14	TRUE	3	13	TRUE	2	12	TRUE	1	11	TRUE
16	7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	19	-11	20	0	111	5	2	6	17	TRUE	5	16	TRUE	4	15	TRUE	3	14	TRUE
16	12	[illegible]	28	[illegible]	[illegible]	[illegible]	[illegible]	24	-12	15	0	141	12	3	8	20	TRUE	7	19	TRUE	6	18	TRUE	5	17	TRUE
16	17	TRUE	33	12	115	105	[illegible]	29	-13	10	0	171	19	4	10	23	TRUE	9	22	TRUE	8	21	TRUE	7	20	TRUE
16	22	TRUE	38	7	150	140	130	34	-14	5	0	201	26	5	12	26	TRUE	11	25	TRUE	10	24	TRUE	9	23	TRUE
16	27	TRUE	43	2	185	175	165	39	-15	0	2	231	33	6	14	29	TRUE	13	28	TRUE	12	27	FALSE	11	26	FALSE
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18	6	FALSE	24	27	40	30	20	20	-12	25	0	117	4	2	6	18	TRUE	5	17	TRUE	4	16	TRUE	3	15	TRUE
18	11	FALSE	29	22	75	65	55	25	-13	20	0	147	11	3	8	21	TRUE	7	20	TRUE	6	19	TRUE	5	18	TRUE
18	16	FALSE	34	17	110	100	90	30	-14	15	0	177	18	4	10	24	TRUE	9	23	TRUE	8	22	TRUE	7	21	TRUE
18	21	TRUE	39	12	145	135	125	35	-15	10	0	207	25	5	12	27	TRUE	11	26	TRUE	10	25	TRUE	9	24	TRUE
18	26	TRUE	44	7	180	170	160	40	-16	5	0	237	32	6	14	30	TRUE	13	29	TRUE	12	28	TRUE	11	27	TRUE
18	31	TRUE	49	2	215	205	195	45	-17	0	2	267	39	7	16	33	TRUE	15	32	TRUE	14	31	FALSE	13	30	FALSE
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22	4	FALSE	26	37	30	20	10	22	-14	35	0	129	2	2	6	20	TRUE	5	19	TRUE	4	18	TRUE	3	17	TRUE
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22	34	TRUE	56	7	240	230	220	52	-20	5	0	309	44	8	18	38	TRUE	17	37	TRUE	16	36	TRUE	15	35	TRUE
22	39	TRUE	61	2	275	265	255	57	-21	0	2	339	51	9	20	41	TRUE	19	40	TRUE	18	39	FALSE	17	38	FALSE