## Geometry of Signals

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ECE4823

## Motivation

- Signals can be viewed as points in a space
- Notions of distance and angle have practical meaning
- Probability of bit error can be determined *graphically*

## Recall Euclidean Geometry

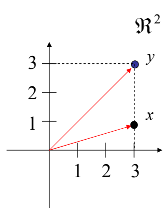
- The space: $\Re^2$
- The basis: $(1,0)$ and $(0,1)$
- Two points:

  $x = (3,1)$ and $y = (3,3)$

- Inner product:

  $\langle x, y \rangle = 3 \cdot 3 + 1 \cdot 3 = 12$

  $= |x||y|\cos\theta$

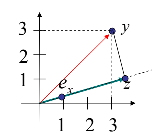



## Magnitude and Projection

- Magnitude: $|x| = \sqrt{\langle x, x \rangle} = \sqrt{3 \cdot 3 + 1 \cdot 1} = \sqrt{10}$
- Projection of $y$ onto $x$ :
  1. Normalize $x$ : $e_x = x/|x|$
  2. Magnitude of projection:

     $$\langle y, e_x \rangle = \frac{3 \cdot 3 + 3 \cdot 1}{\sqrt{10}}$$

  3. Finally, the projection: $z = \langle y, e_x \rangle e_x$




## Changing Coordinates

- New basis:

  $e_x = (3,2)/\sqrt{13}, \quad e_{\perp x} = (-2,3)/\sqrt{13}$

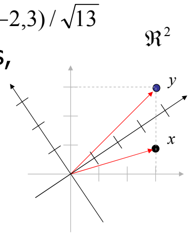
- In the new coordinates,

  $\tilde{x} = \langle x, e_x \rangle e_x + \langle x, e_{\perp x} \rangle e_{\perp x}$

  $= 3.05 e_x - 0.832 e_{\perp x}$

  $\tilde{y} = \langle y, e_x \rangle e_x + \langle y, e_{\perp x} \rangle e_{\perp x}$

  $= 4.16 e_x + 0.832 e_{\perp x}$




## Distance Between Points

- If  and  are points in the space, the distance between them is

$$|\tilde{x} - \tilde{y}| = \sqrt{\langle \tilde{x} - \tilde{y}, \tilde{x} - \tilde{y} \rangle}$$

$$\tilde{x} - \tilde{y} = (3.05 - 4.16)e_x + (-0.832 - 0.832)e_{\perp x}$$

$$= -1.11 e_x - 1.664 e_{\perp x}$$

$$|\tilde{x} - \tilde{y}| = \sqrt{(-1.11)^2 + (-1.664)^2} = 2.00$$

## The Space of Signals

- The space: All square-integrable (finite energy) functions, $L_2$
- Example basis functions:

$$e_1(t) = \sqrt{\frac{2}{T_S}}\cos(2\pi f_c t) \quad 0 \le t \le T_S$$

$$e_2(t) = \sqrt{\frac{2}{T_S}}\cos\left(2\pi f_c t + \frac{\pi}{2}\right) \quad 0 \le t \le T_S$$

## The Inner Product and Magnitude

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} x(t)y(t)dt$$

$$|x|^2 = \langle x, x \rangle = \int_{-\infty}^{+\infty} x^2(t)dt$$

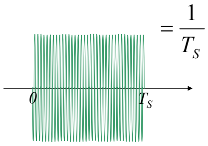
$$|x| = \sqrt{\int_{-\infty}^{+\infty} x^2(t)dt}$$

## The Basis Functions Should Have A Zero Inner Product

$$\langle e_1, e_2 \rangle = \frac{2}{T_S}\int_0^{T_S}\cos(2\pi f_c t)\cos\left(2\pi f_c t + \frac{\pi}{2}\right)dt$$

$$= \frac{2}{T_S}\int_0^{T_S}\frac{1}{2}\left[\cos\left(-\frac{\pi}{2}\right) + \cos\left(2\pi 2 f_c t + \frac{\pi}{2}\right)\right]dt$$

$$= \frac{1}{T_S}\int_0^{T_S}\cos\left(2\pi 2 f_c t + \frac{\pi}{2}\right)dt \approx 0$$




## The Basis Functions Should Have Unit Norm

$$\langle e_1, e_1 \rangle = \frac{2}{T_S}\int_0^{T_S}\cos^2(2\pi f_c t)dt$$

$$= \frac{2}{T_S}\int_0^{T_S}\frac{1}{2}[1 + \cos(2\pi 2 f_c t)]dt$$

$$\approx 1$$

## Two Points in the Space

$$x(t) = \sqrt{\frac{2\mathcal{E}_b}{T_S}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right) \quad 0 \le t \le T_S$$

$$y(t) = \sqrt{\frac{2\mathcal{E}_b}{T_S}}\cos\left(2\pi f_c t - \frac{\pi}{6}\right) \quad 0 \le t \le T_S$$

## Signal Energy

- The energy of $x(t)$ is

$$\int_0^{T_S} x^2(t)dt = |x|^2 = \frac{2\mathcal{E}_b}{T_S}\int_0^{T_S}\cos^2\left(2\pi f_c t + \frac{\pi}{4}\right)dt = \mathcal{E}_b$$

- The energy of $y(t)$ is the same
- The energy of a signal is its norm squared

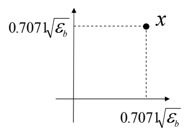
## Find Coordinates of $x$

$$\langle x, e_1 \rangle = \frac{2\sqrt{\mathcal{E}_b}}{T_S}\int_0^{T_S}\cos\left(2\pi f_c t + \frac{\pi}{4}\right)\cos(2\pi f_c t)dt$$

$$= \frac{2\sqrt{\mathcal{E}_b}}{T_S}\int_0^{T_S}\frac{1}{2}\left[\cos\left(\frac{\pi}{4}\right) + \cos\left(2\pi 2 f_c t + \frac{\pi}{4}\right)\right]dt$$

$$= \sqrt{\frac{\mathcal{E}_b}{2}} = 0.7071\sqrt{\mathcal{E}_b}$$

$$\langle x, e_2 \rangle = \sqrt{\frac{\mathcal{E}_b}{2}} = 0.7071\sqrt{\mathcal{E}_b}$$



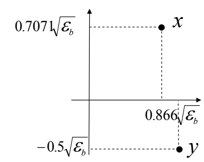

## Find Coordinates of $y$

$$\langle y, e_1 \rangle = \frac{2\sqrt{\mathcal{E}_b}}{T_S}\int_0^{T_S}\cos\left(2\pi f_c t - \frac{\pi}{6}\right)\cos(2\pi f_c t)dt$$

$$= \frac{2\sqrt{\mathcal{E}_b}}{T_S}\int_0^{T_S}\frac{1}{2}\left[\cos\left(-\frac{\pi}{6}\right) + \cos\left(2\pi 2 f_c t - \frac{\pi}{6}\right)\right]dt$$

$$= \sqrt{\mathcal{E}_b}\cos\left(-\frac{\pi}{6}\right) = 0.866\sqrt{\mathcal{E}_b}$$

$$\langle y, e_2 \rangle = \sqrt{\mathcal{E}_b}\cos\left(\frac{2\pi}{3}\right) = -0.5\sqrt{\mathcal{E}_b}$$




## Distance Between the Points

- Can find using the integral:

$$|x - y| = \sqrt{\langle x - y, x - y \rangle} = \sqrt{\int_{-\infty}^{+\infty}[x(t) - y(t)]^2}$$

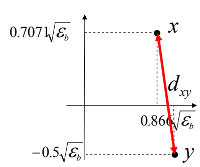
- or using the signal space diagram:

$$x - y = (0.7071 - 0.866)\sqrt{\mathcal{E}_b}e_1 + (0.7071 - [-0.5])\sqrt{\mathcal{E}_b}e_2$$

$$= -1.589\sqrt{\mathcal{E}_b}e_1 + 1.2071\sqrt{\mathcal{E}_b}e_2$$

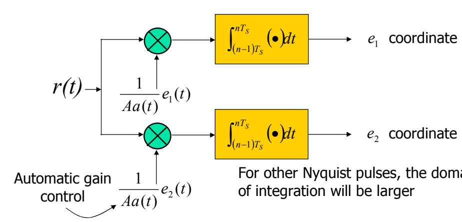
$$|x - y| = \sqrt{(-1.589)^2\mathcal{E}_b + (1.2071)^2\mathcal{E}_b}$$

$$= 1.751\sqrt{\mathcal{E}_b} = d_{xy}$$




## Demodulation

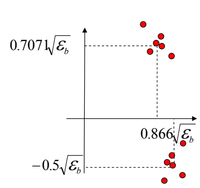
- The demodulator computes the coordinates of each received symbol




For other Nyquist pulses, the domain of integration will be larger

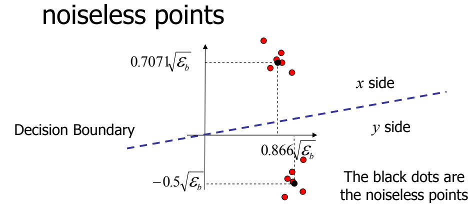
## Receiver Signal Space

- Because of noise, the received points are scattered




## Optimal Detection

- The optimal binary receiver effectively draws a threshold line between the two noiseless points



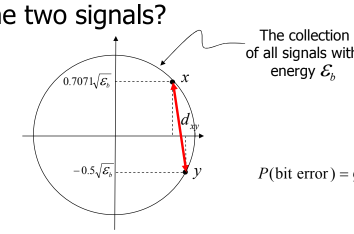
## Binary Probability of Bit Error

- The probability of the received signal point being on the wrong side of the line
- It depends on the distance between the two noiseless points and the noise spectral height

$$P(\text{bit error}) = Q\left(\frac{d_{xy}}{\sqrt{2N_0}}\right)$$

## Energy-Efficient Signals

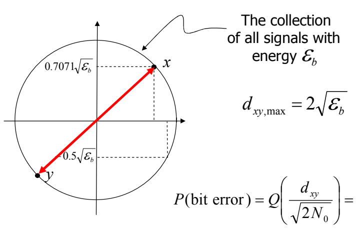
- If we constrain each signal to have energy $\mathcal{E}_b$ , what is the best choice for the two signals?




$$P(\text{bit error}) = Q\left(\frac{d_{xy}}{\sqrt{2N_0}}\right)$$

## Maximize the Distance

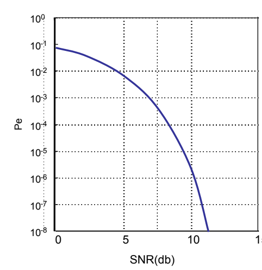
- $y(t) = -x(t)$ (BPSK)




$$d_{xy,\max} = 2\sqrt{\mathcal{E}_b}$$

$$P(\text{bit error}) = Q\left(\frac{d_{xy}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

## Approximate Graph of BER for BPSK




$$SNR = \frac{\mathcal{E}_b}{N_0}$$

## M-ary Modulation

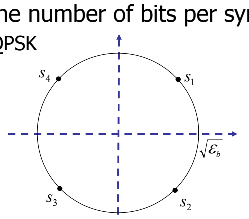
- Can have $M$ signals or symbols in the set

$$M = 2^k$$

where k is the number of bits per symbol

- Example: QPSK

$k=2, M=4$




## Union Bound

- When M>2, a union bound may be used for the probability of symbol error

$$P(\text{symbol error}) \le \frac{1}{M}\sum_{i=1}^{M} P(\text{symbol error} \mid s_i \text{ sent})$$

$$= \frac{1}{M}\sum_{i=1}^{M}\sum_{\substack{j=1 \\ j\neq i}}^{M} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

## Summary

- Received signals can be plotted as points in a signal space
- Noise makes the points clustered around the “noiseless” points
- Decision boundaries bisect the lines between adjacent points
- Probability of detection error depends on distance from boundary and the average noise power (spread of cluster)

## References


- [Rapp, ’02] T.S. Rappaport, *Wireless Communications*, Prentice Hall, 2002