Matched Filter

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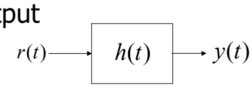
ECE4823

Maximum SNR Property

- Assume AWGN
- We want the filter that yields the highest signal-to-noise ratio (SNR) at its output
- Next, show that the optimum filter is the matched filter

Set Up

- Suppose $r(t) = s(t) + n(t)$, where $n(t)$ is WGN with spectral height $N_o/2$, and $s(t)$ is a signal with a finite duration T
- Let $r(t)$ be the input to a filter with impulse response $h(t)$
- Let $y(t)$ be the output



Signal and Noise Parts

- The signal part of the output is

$$y_s(t) = \int_0^t s(u)h(t-u)du$$

- The noise part of the output is

$$y_n(t) = \int_0^t n(u)h(t-u)du$$

Signal-to-noise Ratio

$$\text{SNR} = \frac{y_s^2(t)}{E\left[y_n^2(t)\right]}$$

$$= \frac{\left[\int_0^t s(u)h(t-u)du\right]^2}{E\left[\int_0^t n(u)h(t-u)du\right]^2}$$

Denominator

$$E\left[y_n^2(t)\right]$$

$$= E\left\{\left[\int_0^t n(u)h(t-u)du\right]\left[\int_0^t n(v)h(t-v)dv\right]\right\}$$

$$= \int_0^t \int_0^t E\{n(u)n(v)\}h(t-u)h(t-v)dudv$$

Invoke White Noise Model

$$E\left[y_n^2(t)\right]$$
$$= \int_0^t\int_0^t \frac{N_0}{2}\delta(u-v)h(t-u)h(t-v)\,du\,dv$$
$$= \frac{N_0}{2}\int_0^t h^2(t-u)\,du$$

SNR So Far

- To optimize the SNR, choose $h(u)$ to maximize the numerator

$$\mathrm{SNR} = \frac{\left[\int_0^t s(u)h(t-u)\,du\right]^2}{\frac{N_0}{2}\int_0^t h^2(t-u)\,du}$$

Cauchy-Schwarz Inequality

- Say S and Q are two points in a Hilbert space, then

$$\langle S,Q\rangle^2 \le |S|^2|Q|^2$$

with equality when Q=cS

Cauchy-Schwarz for Signals

- Let S and Q be points in the Hilbert space of square-integrable functions
- Then,

$$\left[\int_0^t s(u)q(u)\,du\right]^2 \le \int_0^t s^2(u)\,du\int_0^t q^2(u)\,du$$

- Equality is reached when $cs(u) = q(u)$

Apply Cauchy-Schwartz

- Recall numerator of SNR

$$\left[\int_0^t s(u)h(t-u)\,du\right]^2$$

- Pick $h(t-u)$ to be equal to $cs(u)$

Simplify Optimal SNR

- Substitute $h(t-u)=cs(u)$

$$\mathrm{SNR}^{\mathrm{opt}}(t) = \frac{\left[c\int_0^t s^2(u)\,du\right]^2}{\frac{N_0c^2}{2}\int_0^t s^2(u)\,du} = \frac{\int_0^t s^2(u)\,du}{\frac{N_0}{2}}$$

Optimize t

- If $s(t)$ has finite duration T , then SNR is maximized by setting $t=T$

$$\mathrm{SNR}^{\mathrm{opt}} = \frac{\int_0^T s^2(u)du}{\frac{N_0}{2}} = \frac{2\mathcal{E}_s}{N_0}$$

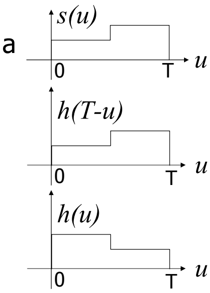
Max SNR Filter = Matched Filter

- Matched filter impulse response is a "flipped in place" version of signal

$$h(T-u) = cs(u)$$

or

$$h(u) = cs(T-u)$$

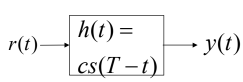


Inner Product Interpretation of Matched Filter

$$y(T) = \int_0^T h(u)r(T-u)du$$

$$= \int_0^T cs(T-u)r(T-u)du$$

$$= c\int_0^T s(u)r(u)du = c\langle \mathbf{s},\mathbf{r}\rangle$$



Matched Filter Frequency Response

- Take Fourier Transform of the Matched Filter impulse response

$$H(f) = \int_0^T h(u)e^{-j2\pi f u}du$$

$$= c\int_0^T s(T-u)e^{-j2\pi f u}du$$

Let $r=T-u$

$$H(f) = c\int_0^T s(T-u)e^{-j2\pi f u}du$$

$$= c\int_T^0 s(r)e^{-j2\pi f(T-r)}(-dr)$$

$$= ce^{-j2\pi fT}\int_0^T s(r)e^{j2\pi f r}dr = ce^{-j2\pi fT}\left[S(f)\right]^*$$

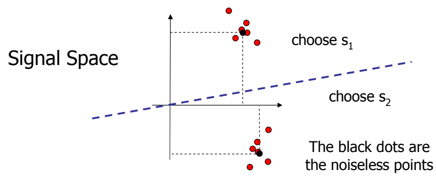
Matched in Frequency Domain

- The magnitude of the matched filter response is just a scaled version of the signal's F.T.

$$|H(f)| = c|S(f)|$$

Relation to Optimum Detection

- Recall optimum detector is the minimum distance detector



Alternative Form of Minimum Distance Detector

- Expand the signal space distance between the received vector $\mathbf{r}$ and the noiseless signal point $\mathbf{s}_m$

$$\|\mathbf{r}-\mathbf{s}_m\|^2 = \|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_m \rangle + \|\mathbf{s}_m\|^2$$

$$\hat{m}_{opt} = \arg \min_m \left[-2\langle \mathbf{r}, \mathbf{s}_m \rangle + \mathcal{E}_m \right]$$

If Signals are Equal Energy

- The minimum distance receiver can be implemented as a bank of matched filters

$$\begin{aligned} \hat{m}_{opt} &= \arg \min_m \left[-2\langle \mathbf{r}, \mathbf{s}_m \rangle + \mathcal{E}_m \right] \\ &= \arg \max_m \langle \mathbf{r}, \mathbf{s}_m \rangle \\ &= \arg \max_m \int_0^T r(t) s_m(t)\,dt \end{aligned}$$

Summary

- When the input is signal plus WGN, then the filter that maximizes the SNR is the matched filter
- The proof is an application of the Cauchy-Schwarz Inequality
- The filter "matches" (has the same shape as) the signal in magnitude in the frequency domain
- The minimum-distance receiver can be implemented as a bank of matched filters