A Quasi-Stationary Markov Chain Model of a Cooperative Multi-Hop Linear Network

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***Abstract*—We consider a quasi-stationary Markov chain as a model for a decode and forward wireless multi-hop cooperative transmission system that forms successive Opportunistic Large Arrays (OLAs). This paper treats a linear network topology, where the nodes form a one-dimensional horizontal grid with equal spacing. In this OLA approach, all nodes are intended to decode and relay. Therefore, the method has potential application as a high-reliability and low-latency approach for broadcasting in a line-shaped network, or unicasting along a pre-designated route. We derive the transition probability matrix of the Markov chain based on the hypoexponential distribution of the received power at a given time instant assuming that all the nodes have equal transmit power and the channel has Rayleigh fading and path loss with an arbitrary exponent. The state is represented as a ternary word, which indicates which nodes have decoded in the present hop, in a previous hop, or have not yet decoded. The Perron-Frobenius eigenvalue and the corresponding eigenvector of the sub-stochastic matrix indicates the signal-to-noise ratio (SNR) margin that enables a given hop distance.**

***Index Terms*—Stochastic modeling, quasi-stationary Markov chains, wireless network, cooperative transmission.**

I. Introduction

WIRELESS multi-hop communications, where radios forward the packets of other radios, has a wide variety of applications, not only in the cellular and sensor networking regimes, but in technologies like wireless computer networking and mobile computing. One promising, fast, and low-overhead wireless transmission technique is the Opportunistic Large Array (OLA) [1], in which all radios that decode a message relay the message together, very shortly after reception, without coordination with other relays. Slot and symbol time synchronization can be achieved based on a packet preamble that all cooperators receive [2], or from GPS or a network time synchronization protocol [3]. For OFDM, carrier and sample frequency synchronization can also be derived from a packet preamble that all cooperators receive [4]. Because only a minimal amount of inter-node coordination is needed, OLAs are particularly well suited for mobile networks, such as large groups of people with smart phones or swarms of robots. Especially when paired with a transmission threshold, OLA broadcasting is an energy-efficient candidate for large dense wireless sensor networks [5].

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In this paper, we model a special case of the decode and forward (DF) OLA network, where the nodes are uniformly spaced along a line. This topology can be considered a precursor to a strip shaped network or a uni-cast cooperative route for the finite density case. Typical examples include structural health monitoring and sensors employed in hallways of buildings in a linear fashion. The topology would also be consistent with a *plastic communication cable*, in which small wireless relays are embedded along a cable made of a non-conducting material [6]. Such "*plastic wires*" might find applications in areas of high electric fields. The wireless channel is modeled with path loss and flat Rayleigh fading. All the nodes that can decode the source packet correctly, and that have not relayed the packet before, relay the packet concurrently in orthogonal channels, thereby providing transmit diversity [2]. We do not address how the orthogonal channels are assigned, other than to suggest that in a line network, they can be assigned as a repeating sequence, e.g., $ABCDABCD...$, such that the sequence $ABCD$ is at least as long as the hop distance. Then, all the nodes that can decode that OLA transmission will relay in the next hop, and this process proceeds until it fails. Also, we assume that the distance between the source and the destination is long enough that the transmission reaches a kind of steady state. Specifically, we assume that the conditional probability that the kth node in a level decodes, given that the previous level had at least one node transmitting, is the same for each level. This allows us to apply the well-established theory of quasi-stationary discrete time Markov chains with an absorbing state [14]. The absorbing state is defined to be when all the nodes in one hop cannot decode the message, and the transmissions stop propagating. Once we have the quasi-stationary distribution, we can determine network performance, such as packet delivery ratio and latency over a given distance as a function of system parameters such as transmit power, inter-node distance and path loss exponent.

Many authors have studied multi-hop networks with cooperative transmission (CT). For the purpose of this paper, we will classify these previous works into non-OLA based ([7]–[9] and references therein) and OLA based. The OLA is distinguished from the other forms of CT mainly in the way the relays are selected. In an OLA, selection is achieved without any kind of cluster head, leader, or anchor node. Nodes autonomously decide to relay if they can decode the packet, and if they have not relayed that packet before. In other words, the radios that will participate in the OLA of the nth hop, for instance, cannot be determined in advance. Among the OLA based works, the authors in [1], [10]–[12], studied large dense networks, using the continuum assumption. Under this assumption, the number of nodes goes to infinity while the power per unit area is kept fixed. These papers derived conditions under which broadcasting over an infinite disk or strip is guaranteed. In contrast, in this paper, we obtain closed-form theoretical results without the continuum assumption, by deploying a simple one-dimensional network where the nodes are uniformly spaced on a grid. By applying the quasi-stationary Markov chain analysis, we show that there is no condition guaranteeing infinite propagation of OLAs. There is only a probability of successfully delivering a packet over a given distance. Although our analysis focuses on the delivery of only a single packet, in many applications, numerous packets, composing for example a video file, could be injected into such a cooperative route, one every few time slots, similarly to how they are injected in a non-cooperative route. In this paper, we do not address important network layer metrics, such as throughput. However, the interested reader may wish to see the first results of an OLA-based uni-cast protocol compared to ad hoc on-demand distance vector (AODV) routing on software defined radios in an indoor network [13].

The rest of the paper is organized as follows. In the next section, we define the network parameters. Section III proposes a model of the network via discrete time Markov chains (DTMC) and obtains a quasi-stationary distribution of this chain. In Section IV, we derive the transition probability matrix for the proposed model and we propose an iterative algorithm for optimizing the membership function in Section V. We will then validate the analytical results with those of numerical simulations in Section VI. The paper then concludes with certain recommendations of the future work in Section VII.

II. System Description

In this section, we describe our model for the signal-to-noise ratio (SNR) in each receiver, and state our other assumptions. Consider a line of nodes where adjacent nodes are a distance d apart from one another, as shown in Figure 1. We assume that the nodes transmit synchronously in OLAs or levels, and that a *hop* occurs when nodes in one level transmit a message and at least one node is able to decode the message for the first time. Correct decoding is assumed when a node's received SNR at the output of the diversity-combiner, from the previous level only, is greater than or equal to a modulation-dependent threshold, τ. Exactly one time slot later, all the nodes that just decoded the message relay the message. Thus, this type of cooperative transmission is similar to selection relaying in [7]. Once a node has relayed a message, it will not relay that message again. Let $p_n(m)$ be the *membership* probability that the mth node transmits in the nth level, given that at least one node transmitted in the $(n-1)$th level. Also let M be at least the width of the region of support of $p_n(m)$. In other words, there exists some M_0 such that $p_n(m) \geq 0$ for $M_0 \leq m \leq M_0 + M - 1$ and $p_n(m) = 0$ otherwise. As we will show later, the quasi-stationary property implies that there exists a hop distance, h_d, such that $p_{n-1}(m - h_d) = p_n(m)$. Hence h_d can be considered as a shift to the window of size M. A sample outcome of the transmissions is shown in Figure 1 where the window size, M, is 5 and hop distance or the shift in window, h_d, is 2. The nodes 1, 2, and 4 are able to decode

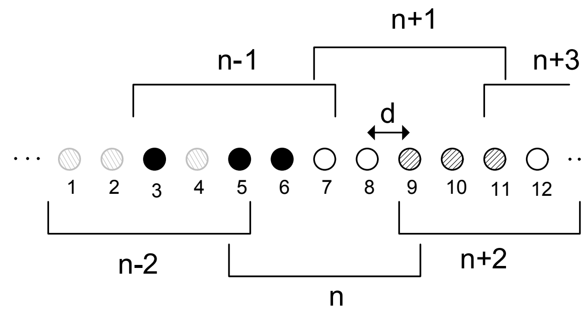

Fig. 1. A sample outcome of the transmission system with the overlapping windows; $M = 5$ and $h_d = 2$.

the message and become part of level $n-2$. These nodes will relay the message in the next time slot and only the nodes in level $n-1$ may decode that message. Since node 4 has already participated in level $n-2$, so it cannot be part of any other level including $n-1$. Thus the candidate nodes are 3, 5, 6, and 7, out of which nodes 3, 5, and 6 become DF nodes in level $n-1$ and this process continues.

We assume that all the nodes transmit with the same transmit power P_t. A node receives superimposed copies of the message signal from the nodes that decoded the message correctly in the previous level, over orthogonal fading channels using equal gain combining (EGC). Let us define $\mathbb{N}_n = \{1, 2, ..., k_n\}$, where k_n is the cardinality of the set $\mathbb{N}_n$ such that $\sup_n k_n = M$, to be the set of indices of those nodes that decoded the signal perfectly at the time instant (or hop) n. For example, from Figure 1, $\mathbb{N}_n = \{3, 4\}$ and $\mathbb{N}_{n+1} = \{3, 4, 5\}$. The received power at the jth node at the next time instant $n+1$ is given by

$$P_{r_j}(n+1) = \frac{P_t}{d^\beta} \sum_{m \in \mathbb{N}_n} \frac{\mu_{mj}}{|h_d - m + j|^\beta}, \quad (1)$$

where the summation is over the nodes that decoded correctly in the previous level. The flat fading Rayleigh channel gain from node m in the previous level to node j in the current level is denoted by $\mu_{mj} \in \mu$; the elements of μ are independently and identically distributed (i.i.d.) and are drawn from an exponential distribution with the parameter σ_μ^2=1; β is the path loss exponent with a usual range of 2-4. Consequently, the received SNR at the jth node is given as $\gamma_j = P_{r_j}/\sigma_j^2$, where σ^2 is the variance of the noise in the receiver. Throughout the paper, we will use the notation $P_{r_j}(n)$ as the power received at the jth node at the nth time instant. We assume perfect timing and frequency recovery at each receiver, and we also assume that there is sufficient transmit synchronization between the nodes of a level, such that all the nodes in a level transmit to the next level at the same time [2]. In other words, the transmissions only occur at discrete instants of time $n, n+1, ...$ such that the hop number and the time instants can be defined by just one index n. By the overlapping nature of the windows, we have the following proposition.

Proposition 1. *Given M and h_d, a node at a position x can become part of several levels n, such that $\forall x > M - h_d$*

$$\left\lceil \frac{x - M}{h_d} \right\rceil + 1 \leq n \leq \left\lfloor \frac{x-1}{h_d} \right\rfloor + 1. \quad (2)$$

Proof: Without the loss of generality, we can assume that the first node in the network is located at $x = 1$ and is a part of level $n = 1$. From the given geometry, the starting location of nth window is given by $(n-1)h_d + 1$, while the end location as $(n-1)h_d + M$. A node at any position x in this window, lies in between these locations, i.e.

$$(n-1)h_d + 1 \leq x \leq (n-1)h_d + M. \tag{3}$$

The above inequality can be broken into two, such that

$$(n-1)h_d \leq x - 1 \qquad \text{and} \qquad x - M \leq (n-1)h_d.$$

This implies, $x - M \leq (n-1)h_d \leq x - 1$. From the necessary condition derived in (3), we get (2). ■

Corollary 1. *$\forall x \leq M - h_d$, we have $n = 1, ..., \left\lceil \frac{x}{h_d} \right\rceil$.*

One goal of this study is to find the hop distance as a function of the values of system parameters such as relay transmit power and inter-node distance. However, because of the discrete nature of the hop distance, solving the problem is this manner is quite tedious. Hence in this paper, we follow the inverse approach, i.e., for a given hop distance, we will find the system parameters that generate this hop distance. We find the parameters that give the most compact OLAs.

III. Modeling by Markov Chain

At a certain time n, a node from the nth level will take part in the next transmission, if it has decoded the data perfectly at the current time, or it will not take part, if it did not decode correctly or it has already decoded the data in one of the previous levels. The decisions of all the nodes in the nth level can be represented as $X(n) = [\mathbb{I}_1(n), \mathbb{I}_2(n), ..., \mathbb{I}_M(n)]$, where $\mathbb{I}_j(n)$ is the ternary indicator random variable for the jth node at the nth time instant given as

$$\mathbb{I}_j(n) = \begin{cases} 0 & \text{node } j \text{ does not decode} \\ 1 & \text{node } j \text{ decodes} \\ 2 & \text{node } j \text{ has decoded at some earlier time} \end{cases} \tag{4}$$

Thus each node is represented by either 0, 1 or 2 depending upon the successful decoding of the received data. For example, from Figure 1, we have $\mathbb{I}_1(n) = \mathbb{I}_2(n) = 2$, $\mathbb{I}_3(n) = \mathbb{I}_4(n) = 1$ and $\mathbb{I}_5(n) = 0$. We observe that the outcomes of $X(n)$ are ternary M-tuples, each outcome constituting a state, and there are 3^M number of states, which are enumerated in decimal form $\{0, 1, ..., 3^M - 1\}$. Let i_n be the outcome at time n. For example, $i_n = [22110]$ in ternary, and $i_n = 228$ in decimal in Figure 1. Then we may write

$$\begin{aligned} \mathbb{P}\{X(n) = i_n | X(n-1) = i_{n-1}, ..., X(1) = i_1\} = \\ \mathbb{P}\{X(n) = i_n | X(n-1) = i_{n-1}\}, \end{aligned} \tag{5}$$

where $\mathbb{P}$ indicates the probability measure. Equation (5) implies that $X(n)$ is a discrete-time finite-state Markov Process. Assuming the statistics of the channel are same for all the hops in the network, the Markov chain can be regarded as a homogeneous one.

It can be further noticed that at any point in time, there is a probability that the Markov chain can go into an absorbing state, thus terminating the transmission. That can be a state when all the nodes at a particular hop cannot decode the message perfectly and thus Markov chain will be in the 0 state (decimal). It can be further noticed, that any possible combination of 0 and 2 will also make the state an absorbing state. Since we are enumerating the states using ternary words, the total number of states appears to be 3^M. But the following claim shows that the number of transient states in the Markov chain are less than 3^M.

Claim 1. *Given M and h_d, the possible number of states that can be reached during transitions is $\hat{N} = 3^{M-h_d} \times 2^{h_d}$, including 2^{M-h_d} number of absorbing states.*

Proof: Please see the Appendix A. ■

Hence we consider the Markov chain, X, on a state space $A \cup S$, where A is the set of absorbing states, and we have

$$\lim_{n \to \infty} \mathbb{P}\{X(n) \in A\} \nearrow 1 \quad \text{a.s.} \tag{6}$$

On the other hand, the states in S (where the cardinality of S is $|S| = \hat{N} - 2^{M-h_d}$) make an irreducible state space, i.e., there is always a non-zero probability to go from any transient state to another transient state. We will define two matrices to describe the Markov Chain. The first, $\tilde{\mathbf{P}}$, is the full transition probability matrix for all the states in the set $A \cup S$. Each row in $\tilde{\mathbf{P}}$ sums to one. The second matrix, $\mathbf{P}$, is the submatrix of $\tilde{\mathbf{P}}$ that is formed by striking each column and row that involves transitions to and from the absorbing states in A. Therefore, $\mathbf{P}$ is the matrix corresponding to the states in S. It can be noticed that the transition probability matrix $\mathbf{P}$ on the state space S is not right stochastic, i.e., the row entries of $\mathbf{P}$ do not sum to 1 because of the *killing probabilities* given as

$$\kappa_i = 1 - \sum_{j \in S} \mathbf{P}_{ij}, \qquad i \in S. \tag{7}$$

Since $\mathbf{P}$ is a square irreducible nonnegative matrix, then by the Perron-Frobenius theorem [17], there exists a unique maximum eigenvalue, ρ, such that the eigenvector associated with ρ is unique and has strictly positive entries. For the proof, please refer to [17] and [19]. Since $\mathbf{P}$ is not right stochastic, $\rho < 1$. Also since all states in S are transient and not strictly self-communicating, $\rho > 0$ [15]. Overall our assumptions imply that

$$0 < \rho < 1. \tag{8}$$

From the theory of Markov chains [19], we know that a distribution $\mathbf{u} = (u_i, i \in S)$ is called ρ-invariant distribution if $\mathbf{u}$ is the left eigenvector of the transition matrix $\mathbf{P}$ corresponding to the eigenvalue ρ, i.e.

$$\mathbf{uP} = \rho\mathbf{u}. \tag{9}$$

We are now interested in the limiting behavior of this Markov chain as time proceeds. Since $\forall n$, $\mathbb{P}\{X(n) \in A\} > 0$, eventual killing is certain. But we are interested in finding the distribution of the transient states, before the killing occurs. The so-called limiting distribution is called the quasi-stationary distribution of the Markov chain, which is independent of the initial conditions of the process. From [14] and [15], this unique distribution is given by the ρ-invariant distribution for the one step transition probability matrix of the Markov chain on S. We can find the quasi-stationary distribution by getting the *maximum* eigenvector, $\hat{\mathbf{u}}$ of $\mathbf{P}$, then defining $\mathbf{u} = \hat{\mathbf{u}}/\sum_{i=1}^{\hat{N}} \hat{\mathbf{u}}_i$ as a normalized version of $\hat{\mathbf{u}}$ that sums to one.

Thus we can define the unconditional probability of being in state j at time n as

$$\mathbb{P}\{X(n) = j\} = \rho^n u_j, \quad j \in S, \quad n \geq 0. \tag{10}$$

We also let $T = \inf\{n \geq 0 : X(n) \in A\}$ denote the end of the survival time, i.e., the time at which killing occurs. It follows then,

$$\mathbb{P}\{T > n + m | T > n\} = \rho^m, \tag{11}$$

while the quasi-stationary distribution of the Markov chain is given as

$$\lim_{n\to\infty} \mathbb{P}\{X(n) = j | T > n\} = u_j, \quad j \in S. \tag{12}$$

We also note that the membership probability can be expressed as

$$p_n(m) = \sum_{j\in\theta} u_j, \tag{13}$$

where $\theta = \{X(n) \in S : \mathbb{I}_m(n) = 1\}$.

IV. Formulation of the Transition Probability Matrix

In this section, we will find the state transition matrix **P** for our model, the eigenvector of which will give us the quasi-stationary distribution. Let i and j denote a pair of states of the system such that $i, j \in S$, where each i and j are the decimal equivalents of the ternary words formed by the set of indicator random variables. Now for each node m, the probability of being able to decode at time n given that it failed to decode in the previous level is given as

$$\mathbb{P}\{\mathbb{I}_m(n) = 1 | \mathbb{I}_{h_d+m}(n-1) = 0\} = \mathbb{P}\{\gamma_m(n) > \tau\}. \tag{14}$$

Similarly, the probability of outage or the probability of $\mathbb{I}_m(n) = 0$ is given as $1 - \mathbb{P}\{\gamma_m(n) > \tau\}$ where

$$\mathbb{P}\{\gamma_m(n) > \tau\} = \int_\tau^\infty p_{\gamma_m}(y)dy. \tag{15}$$

$p_{\gamma_m}(y)$ is the probability density function (PDF) of the received SNR at the mth node. From (4), we note that a node can have three possible states, where the initial state of a node is always 0. A node can make the transitions shown in Figure 2. Hence each individual node is a state machine, and $\mathbb{I}_m(n)$ is a non-homogeneous Markov chain itself; the probabilities of transition for a single node are non-zero only at certain times. P_{01} from Figure 2, i.e., the conditional probability of success of the mth node in the nth level, is given as

$$P_{01} = \mathbb{P}\{\gamma_m(n) > \tau | \mathbb{I}_{h_d+m}(n-1) = 0; X(n-1) \in S\}. \tag{16}$$

Hence the probability of perfect decoding is based on the PDF of the received power which can be obtained as follows.

Lemma 1. *If $h_d = M$, the conditional PDF of the received power, conditioned on which nodes transmit, is hypoexponential.*

Proof: It can be seen that the power at a certain node is the sum of the finite powers from the previous level nodes, each of which is exponentially distributed. Thus for K independently distributed exponential random variables with respective parameters λ_k, where $k = 1, 2, ..., K$, the resulting distribution of the sum of these random variables is known as hypoexponential distribution [16] which is given as

$$p_Y(y) = \sum_{k=1}^{K} C_k \lambda_k \exp(-\lambda_k y), \tag{17}$$

where

$$C_k = \prod_{\zeta\neq k} \frac{\lambda_\zeta}{\lambda_\zeta - \lambda_k}. \tag{18}$$

Although $\int_0^\infty p_Y(y)dy = 1$, it should not be thought that C_k are probabilities, because some of them will be negative. ■

For $1 \leq h_d < M$, we consider the following lemma.

Lemma 2. *For two independent exponential random variables with parameters λ and $\lambda + \epsilon$, the complementary CDF (tail probability) of their sum approaches that of a Gamma distribution, $\Gamma(2, \lambda)$, as $\epsilon \to 0$.*

Proof: The CCDF of sums of two independent exponential random variables is given as

$$F_x(x) = \sum_{k=1}^{2} C_k \exp(-\lambda_k x); \tag{19}$$

where $C_1 = \frac{-\lambda}{\epsilon}$ and $C_2 = \frac{\lambda}{\epsilon} + 1$. Thus the CCDF is given as

$$F_x(x) = \exp(-\lambda x)\lambda \left[-\frac{\exp(-\epsilon x)}{\epsilon} + \frac{1}{\epsilon}\right] + \exp(-\lambda x). \tag{20}$$

Taking $\lim_{\epsilon\to 0}$ and using L'Hospital's rule, we get

$$F_x(x) = \exp((-\lambda x))(1 + \lambda x) \tag{21}$$

which is the CCDF of $\Gamma(2, \lambda)$. ■

With the help of these lemmas, let's consider the following theorem.

Theorem 1. *The received power at any node in the network, conditioned on a certain pattern of nodes transmitting in the previous level, is always hypoexponentially distributed.*

Proof: If $h_d = M$, the resulting distribution is hypoexponential from Lemma 1. For $h_d < M - 1$, a node will receive powers from adjacent nodes that are either hypoexponentially distributed (if their respective parameters are different) or they are received as pairs of Gamma distributed variables. Thus the power received will be sum of exponential random variables such that there will be (groups of) two variables having same parameters and rest having distinct parameters. But using Lemma 2, the power received at any node is hypoexponential. ■

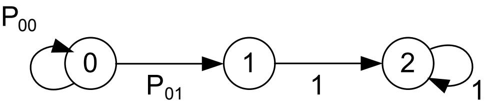

Fig. 2. State transition diagram of a node.

Let us define a set which consists of all those nodes that decoded the data perfectly in the previous hop as $\mathbb{N}_{n-1} =$

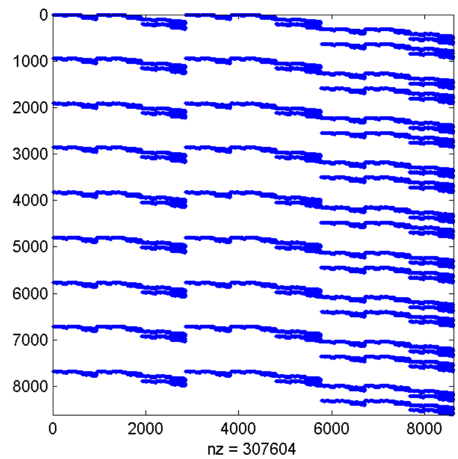

Fig. 3. Sparse structure of the transition probability matrix with $M = 9$ and $h_d = 2$.

$\{m_i : \mathbb{I}_{m_i}(n-1) = 1\}$ $\forall i = 1, 2, ...M$, then from Theorem 1, P_{01} from (16) is given as

$$P_{01} = \sum_{k \in \mathbb{N}_{n-1}} C_k \exp\left(-\lambda_k^{(m)} \tau\right), \tag{22}$$

where $\lambda_k^{(m)}$ is given as

$$\lambda_k^{(m)} = \frac{d^\beta \left|h_d - k + m\right|^\beta \sigma^2}{P_t}. \tag{23}$$

To determine the possible destination states in a transition from level $n-1$ to level n, it is helpful to distinguish between two mutually exclusive sets of nodes in the nth level: 1) the nodes that were also in the M-node window of the $(n-1)$th level, i.e., nodes that are in the h_d overlap region of the two consecutive windows, and 2) the remaining $M - hd$ nodes that are not in the overlap region. We denote these two sets of nodes as $\mathbb{N}_{OL}^{(n)}$ and $\overline{\mathbb{N}}_{OL}^{(n)}$, respectively, where *OL* stands for *overlap*.

Suppose node k in $\mathbb{N}_{OL}^{(n)}$ decoded in the previous $(n-1)$th level; this would be indicated by $\mathbb{I}_{h_d+k}(n-1) = 1$. This node will not decode again, and therefore $\mathbb{I}_k(n) = 2$. Similarly, if that node decoded prior to the $(n-1)$th level, then $\mathbb{I}_{h_d+k}(n-1) = 2$. In this case also, we must have $\mathbb{I}_k(n) = 2$. Alternatively, if the node has not previously decoded, then $\mathbb{I}_{h_d+k}(n-1) = 0$, and $\mathbb{I}_k(n)$ can equal 0 or 1, depending on the previous state and the channel outcomes; $\mathbb{I}_k(n) = 2$ is not possible. If the node k is in the $\overline{\mathbb{N}}_{OL}^{(n)}$, then there is no previous level index for this node, and, again we can have $\mathbb{I}_k(n) \in \{0, 1\}$ depending on the previous state and channel outcomes, but we may not have $\mathbb{I}_k(n) = 2$.

Let a superscript on the indicator functions show the value of the indicator given the ith state. For example, if $i = \{22110\}$, then $\mathbb{I}_5^{(i)}(n) = 0$.Therefore, considering the above discussion, one-step transition probability going from the state i in level $n-1$ to state j in level n is always 0 when either of the following conditions is true:

Condition I: $\mathbb{I}_k^{(j)}(n) \in \{0, 1\}$ and $\mathbb{I}_{h_d+k}^{(i)}(n-1) \in \{1, 2\}$,

Condition II: $\mathbb{I}_k^{(j)}(n) = 2$ and $\mathbb{I}_{h_d+k}^{(i)}(n-1) = 0$.

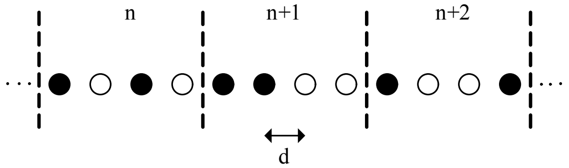

Fig. 4. Arrangement of nodes on a grid with non-overlapping windows; $M = 4$ and $h_d = 4$.

Thus the one step transition probability for going from state i to state j is 0 if condition I or II holds; otherwise it is given as

$$P_{ij} = \prod_{k \in \mathbb{N}_n^{(j)}} \left(\sum_{m \in \mathbb{N}_{n-1}^{(i)}} C_m \exp\left(-\lambda_m^{(k)} \tau\right) \right) \bullet \prod_{k \in \overline{\mathbb{N}}_n^{(j)}} \left(1 - \sum_{m \in \mathbb{N}_{n-1}^{(i)}} C_m \exp\left(-\lambda_m^{(k)} \tau\right) \right) \tag{24}$$

where $\mathbb{N}_n^{(j)}$ and $\overline{\mathbb{N}}_n^{(j)}$ are the indices of those nodes which are 1 and 0, respectively, in state j at level n. Thus it can be seen that the transition probability matrix will contain a large number of zeros. The smaller the hop distance, the larger are the number of zeros in the matrix. Thus the resulting matrix is highly sparse which helps in evaluating the Perron-Frobenius eigenvalue quickly. A sample sparse structure of this matrix that results from $M = 9$ and $h_d = 2$ is shown in Figure 3. It can be seen that there are more than 95% of zeros in the matrix. Another interesting observation is that the matrix entries start to repeat after 2/3 of the matrix. This is because there is no difference in calculating transmissions if the first node in the window is 0 or 2. Thus the calculations are further reduced by a factor of 1/3.

A. A Special Case: Non-Overlapping Windows, $M = h_d$

A special case of the transmission system is that when the hop distance becomes equal to the window size. Thus in this process, we constrain the clusters to be contained in a pre-specified non-overlapping sets of nodes. Each cluster or OLA is still opportunistic in the sense that only the nodes in the set that can decode will be part of the OLA. An example of the cluster to cluster transmission is given in Figure 4, where the correctly decoding nodes are shown as filled black circles. Since no overlap is involved, at a certain time n, each node from the nth level will take part in the next transmission, if it has decoded the data perfectly, or it will not take part, if it did not decode correctly. The decisions of all the nodes in a level can be represented as *binary* indicator random variables, $\mathbb{I}_j(n)$, taking value 1 for successful decoding and 0 for a failure decoding. Hence the considered Markov chain, X, is defined on a state space $0 \cup S$, where S is a finite transient irreducible state space, $S = \{1, 2, ..., 2^M - 1\}$, and 0 being the absorbing state. The resulting sub-stochastic transition probability matrix $\mathbf{P}$ is a $(2^M - 1) \times (2^M - 1)$ corresponding to the states in S. For M nodes in a level, let us define the index sets corresponding to the ith state as

$\mathbb{N}_n^{(i)} = \{1, 2, ..., k_n\}$ and $\overline{\mathbb{N}}_n^{(i)} = \{1, 2, ..., M\} \setminus \mathbb{N}_n^{(i)}$, to be the sets of those nodes which are 1 and 0, respectively, in state i. Then the one step transition probability for going from state i to j is the same as given in (24), where the distribution of received power at a single node is hypoexponential from Lemma 1 and $\lambda_k^{(m)}$ is given as

$$\lambda_k^{(m)} = \frac{d^\beta (M-k+m)^\beta \sigma^2}{P_t}. \quad (25)$$

It should be noticed that in this case, there are no conditions that would lead to zero probability of transition from state i to state j and hence the matrix is not sparse.

V. Iterative Approach

In Section IV, we showed how to compute the quasi-stationary distribution and the membership probabilities for a given specification of system parameters, such as transmit power, path loss exponent, inter-node distance, hop distance, and for the one artificial constraint, the window width. Therefore, an infinite variety of possible solutions exist, depending on the choices of these parameters. In this section, we eliminate the artificial constraint and show how the design space dimension can be further reduced through parameter normalization and by optimizing the shape of the membership probability function.

M is an *artificial* constraint because there is no real physical need for it, however, it strongly impacts the size of the state space and therefore the computational complexity of finding the quasi-stationary distribution. Therefore, we would like for M to be as small as possible without significantly impacting the system performance results. The transmissions from nodes at the trailing edge of a large window will have only a small contribution to the formation of the next OLA, because of disparate path loss (especially in a line-shaped network), and therefore, their contribution can be neglected. This suggests that an energy efficient solution will be a unimodal membership probability function with a narrow region of support, and therefore a small M can support it. We note that the number of nodes that relay in each hop determines the diversity order in this finite density scenario, so the most narrow membership function (a Kronecker delta) is not desirable. A final consideration is that for the broadcast application, ideally, we want every node to decode the message, and so, under our assumption that every node that decodes for the first time also relays, we have that for a hop distance of h_d, we want at least h_d nodes to relay in each hop.

Based on all of these considerations, we decided to choose the solution that yields a membership probability function that most closely resembles a square pulse of unit height that is h_d nodes wide, and takes the value of zero everywhere else on a window that is M nodes wide. This can be interpreted as corresponding to the most compact (i.e., shortest length) OLA. We find M by increasing it until the one-hop success probability (i.e., the Perron-Frobenius eigenvalue) ceases to change significantly.

To further decrease the design space dimension, we observe that the transition matrix in (24) depends on the product $\lambda_m^{(k)}\tau$, from which we can extract the normalized parameter

$$\Upsilon = \frac{\gamma_0}{\tau} = \frac{P_t}{d^\beta \sigma^2}\frac{1}{\tau}, \quad (26)$$

which can be interpreted as the SNR margin from a single transmitting node a distance d away. However, Υ is not the only independent parameter, because β and h_d also separately impact the value of $\lambda_m^{(k)}\tau$, in (23) through the factor $|h_d - k + m|^\beta$.

We now formally describe our optimization procedure. We define our ideal membership probability function as

$$\hat{q}(k) = u(k-a) - u(k-(a+h_d-1)) \quad k \geq 1, \quad (27)$$

where u is the unit step function and $a = \left\lfloor \frac{M-h_d}{2} \right\rfloor + 1$. We can express the membership probabilities for a given level in vector form as $\mathbf{q} = \{p_{m_1}, p_{m_2}, ..., p_{m_M}\}$, where the values of $p_{m_k}(n)$ can be found using either (13) or as

$$\begin{aligned} p_{m_k}(n) &= \mathbb{P}\{\mathbb{I}_{m_k}(n) = 1\} \\ &= \sum_{j=1}^{\hat{N}} \mathbb{P}\{\mathbb{I}_{m_k} = 1 | X(n) = j\} \mathbb{P}\{X(n) = j\} \quad (28) \\ &\forall k = \{1, 2, ..., M\} \text{ and } j \in S. \end{aligned}$$

Then the problem of finding the best Υ can be formulated as

$$\min_{\Upsilon > 0} \ \Xi = \frac{1}{M} \|\mathbf{q} - \hat{\mathbf{q}}\|^2. \quad (29)$$

The iterative algorithm is this case is given as follows.

Algorithm 1 Iterative Method

1) Given h_d, initialize the algorithm with a window size of $M = 2h_d$.
2) Compute the Perron-Frobenius eigenvalue, $\rho(M)$, over a range of SNR margins.
3) Increment the window size by one, and compute $\rho(M+1)$ using Step 2.
4) If $|\rho(M+1) - \rho(M)| < \epsilon$, for some small $\epsilon > 0$, M is the desired window size and the convergence is achieved. Otherwise go to Step 3.

By using the iterative technique, we are able to find the optimal window size M over a range of SNR margins. To choose the SNR margin that gives a close approximation to (27), minimize (29) over the SNR margin range to get the best value of SNR margin where we achieve the minimization. This value of Υ is the one that yields a given h_d with maximum probability.

VI. Results and System Performance

In this section, we compare the analytical results with those of numerical simulations for different sets of parameters and we investigate system performance as a function of certain parameters. For the purpose of the simulations, we calculate the received power at each node based on the previous state (assuming an initial distribution of nodes at the first hop), which is used to set the indicator functions as either 0,1 or 2 depending upon the threshold criterion. These indicator functions will form the current state and the process continues.

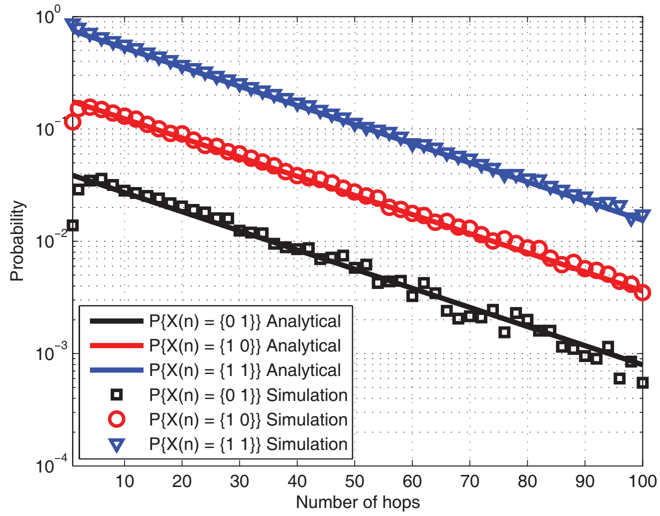

Fig. 5. Distribution of the states for $M = 2$ and $h_d = 2$ for non-overlapping windows.

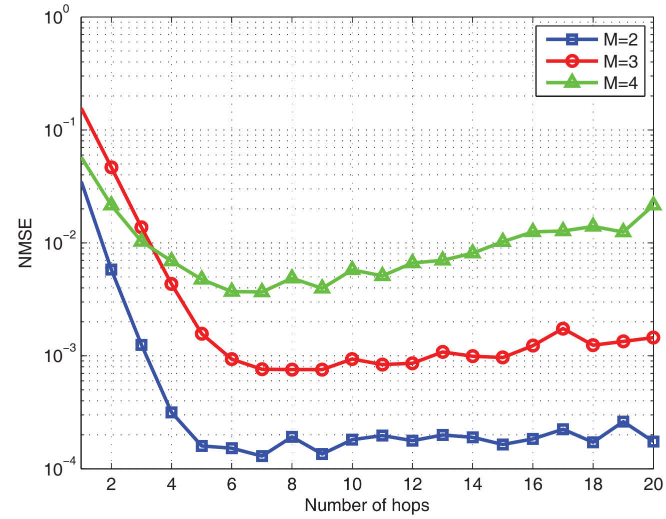

Fig. 6. NMSE between the quasi-stationary distributions from analysis and simulations.

We finally obtain the distribution of the chain by simulating over 20,000 trials. The Perron-Frobenius eigenvalue of $\mathbf{P}$ has been found using [18].

Figure 5 shows the state probabilities of the Markov chain as a function of hop n, when both the window size and the hop distance are assumed to be two, i.e., $M = h_d = 2$. The SNR margin is 12dB with a path loss exponent of 2. Thus, it can be seen that the analytical results are quite close to that of the simulations. It can be further noticed that as we increase the hop number, the probability of being in a transient state decreases, which asserts the relationship as described in (6). Figure 6 shows the normalized mean squared error (NMSE) between the quasi-stationary distribution assuming different values of non overlapping window, M, where the NMSE is defined as

$$NMSE = \frac{1}{2^M - 1} \frac{\|\mathbf{u} - \hat{\mathbf{u}}\|_2^2}{< \mathbf{u} >^2}, \qquad (30)$$

where $\hat{\mathbf{u}}$ is the quasi-stationary distribution obtained from simulation, $||.||_2^2$ is the squared Euclidean norm and $< . >$ is the mean value of the vector. The figure shows that as we increase the hop number, we approach the quasi-stationary distribution quite fast. As we increase M, the NMSE starts to increase and these deviations in the numerical and analytical results can be attributed as the precision errors while calculating the eigenvalues of larger matrices.

Figure 7 depicts the trend of eigenvalues as we increase the SNR margin for different window sizes and a hop distance of 2. The behavior is quite obvious that increasing SNR margin increases the probability of survival of the transmissions. It can be further noticed that for a given value of SNR margin, the curves start to converge as we increase the window size, thereby indicating that after a specific window size, even if we increase M, there is no change in the transmissions outcome which agrees with the iterative algorithm that we discussed in Section V.

Figure 8 shows the error surfaces for the overlapping window case, generated by (29) for a hop distance of 2 and different window sizes. It can be seen that the error surface is convex that contains a minimum for a particular value of SNR margin, Υ. It can be further noticed, that as we increase the window size the difference between the errors becomes smaller in the same vicinity of Υ. Thus, for a window size of 10 and a hop distance of 2, we can select the SNR margin of around 6dB to give us desired membership probability function. Figure 9 shows the numerical simulation result for conditional membership probabilities of the nodes to different levels, where the values Υ and M are taken from the iterative algorithm. It can be seen that the distance between the peaks of any two membership functions is always 2. Thus a window size of 10 seems reasonable to get a hop distance of 2 with an SNR margin of approximately 6dB. The sub-figure in the right top corner shows the analytical membership function obtained from (28) by using the quasi-stationary distribution.

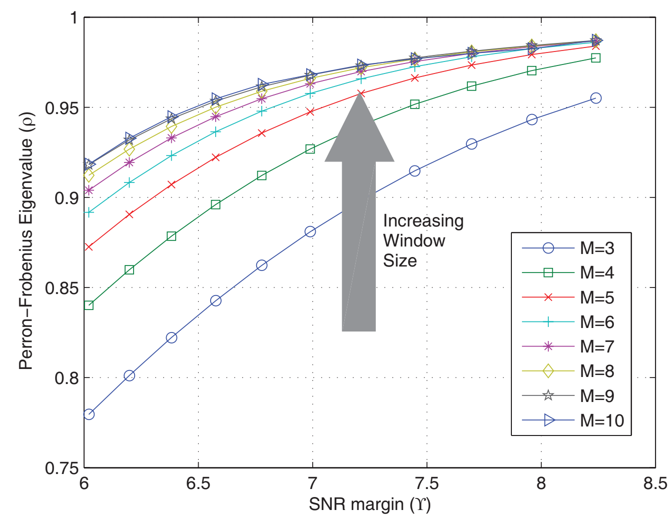

Fig. 7. Behavior of Perron-Frobenius Eigenvalues as M increase for a hop distance of 2 and $\beta = 2$.

Figure 10 shows the effect of increasing the path loss exponent on the Perron eigenvalue for a hop distance of 3. It can be noticed that for the same value of success probability, we require more SNR margin. The convergence of the iterative algorithm can also be seen in this figure. Also it can be noticed

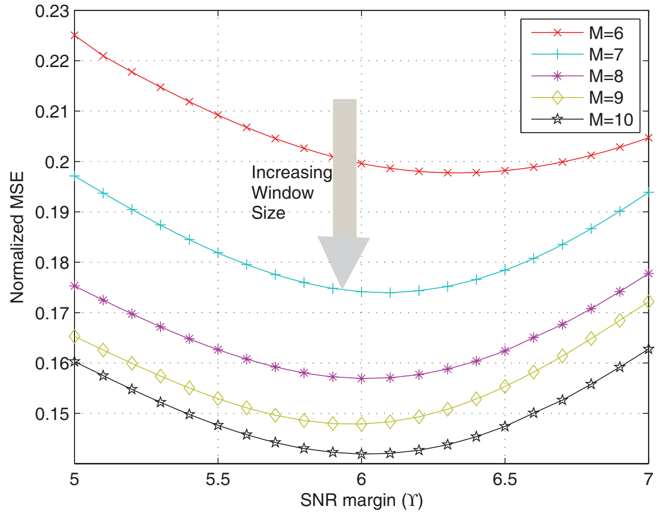

Fig. 8. Error curves for different window sizes for a hop distance of 2 and $\beta = 2$.

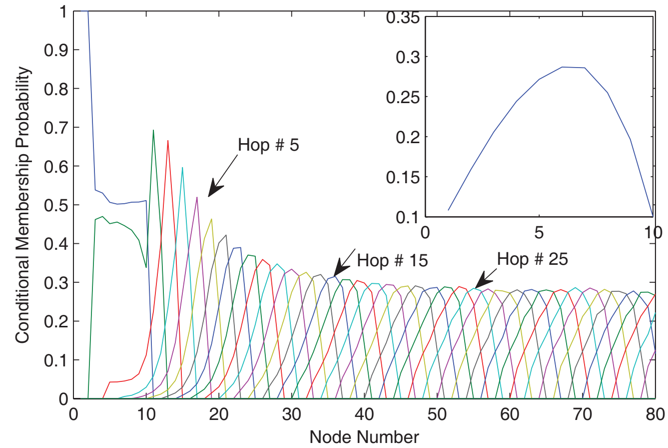

Fig. 9. Conditional membership probabilities of the nodes for $h_d = 2$ for a window size of 10 and $\Upsilon = 6dB$. The sub-figure shows the analytical membership function.

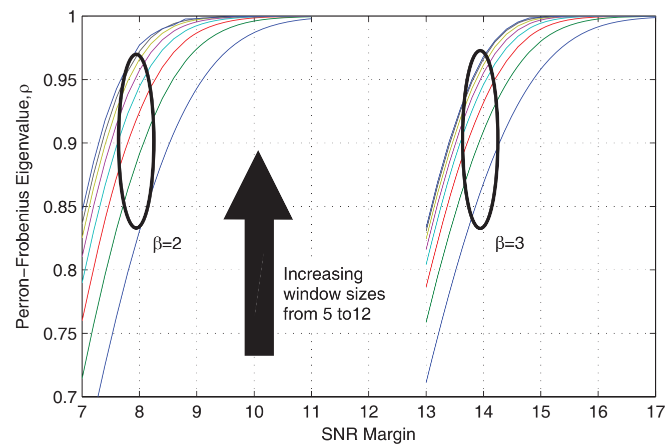

Fig. 10. Effects of path loss exponent on the convergence of eigenvalues for a hop distance of 3.

that for higher path loss exponent, the curves converge fast as compared to smaller path loss exponent. This effect can be attributed to the fact that if path loss exponent is higher, adding a new node to window will not increase the success probability as the transmissions are weaker to reach there. The converse holds true for a small path loss exponent. From the deployment perspective of the network, it is sometimes desirable to determine the values of certain parameters like transmit power of relays or distance between them to obtain a certain quality of service (QoS), η. In other words, we are interested in finding the probability of delivering the message at a certain distance without having entered the absorbing state, and we desire this probability to be at least η where $\eta \sim 1$ ideally. Thus (11) gives us a nice upper bound on the value of m (the number of hops) one can go with a given η, i.e. $\rho^m \geq \eta$, which gives

$$m \leq \frac{\ln \eta}{\ln \rho}. \tag{31}$$

Thus if the destination is far off, we require more hops, which will require a larger value of ρ. Now ρ is a nonlinear function of the SNR margin, Υ, where a large SNR margin corresponds to a large node degree, whereas an SNR margin of 1 implies a node degree of exactly two in this line-network. Figure 11 shows the relationship between required SNR margin to reach the destination node at a particular normalized distance for different values of hop distance. The normalized distance, which is the true distance divided by d, is defined as the product of h_d and the number of hops (made to reach the destination).We have taken three values of the quality of service, η to show our result. We observe that the performance of all the cooperative cases exceeds that of non-cooperative case for a particular value of SNR margin, in terms of the normalized distance. It can be further noticed that the transmissions with cooperative case can reach a particular point in two ways, i.e., keeping both the hop distance and SNR margin small or having a higher hop distance with a higher SNR margin, where the latter has lower latency, i.e., fewer hops, and higher QoS, η. The results are also plotted for a higher path loss exponent, i.e., $\beta = 3$. However, from Figure 10 we know that a high SNR margin is required to get the same value of success probability. Thus we observe that if we increase the path loss exponent and also the SNR margin, we get results that are close to the case of small path loss exponent with small SNR margin. The non-cooperative results show that we can reach a small distance with a considerably small success probability when we use the same SNR margin for the high path loss exponent.

Figure 11 also supports our expectation that fixing the transmit power, while lowering the data rate, will increase the range that can be obtained for a given packet delivery ratio (PDR). Lowering the data rate implies lowering the decoding threshold, which implies from (26) a higher SNR margin. Figure 11 shows that for $\beta = 2$, lowering the decoding threshold by 3.4dB (i.e., increasing Υ from 6 to 9.4) increases the distances by nearly a factor of 7 for a PDR of 90% ($\eta = 0.9$).

From the broadcast perspective, another important parameter is to find the fraction of nodes that have decoded in the network. If we assume that the Markov chain is in the quasi-stationary state, and has not entered the absorbing state over a linear network of interest, then the fraction of decoded nodes

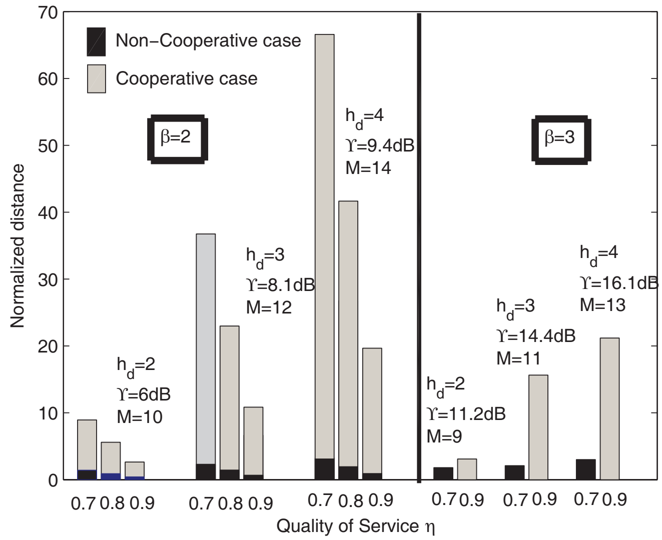

Fig. 11. Normalized distance for various cooperative vs. non-cooperative cases.

in the network is the same as the fraction of the nodes in any one hop. From Figure 9, we can see that we do not exactly get a rectangular membership function, which implies that not all the nodes in the network may have decoded the data. Let N_d be a random variable that denotes the number of forwarding nodes such that n_{d_j} are the realizations of this variable where $j = 1, 2, ...|S|$. Hence the average number of the nodes that have decoded the data is given as

$$\mathbb{E}(N_d) = \sum_{j=1}^{|S|} n_{d_j} u_j \tag{32}$$

where n_{d_j} is the number of DF nodes in the jth state and u_j is the quasi-stationary probability of that state. Hence for the cases that are described in Figure 11, the results are summarized in Table I. It can be seen that as we increase the hop distance (and the SNR margin consequently), we get more nodes that are able to decode in a given hop.

VII. Conclusion and Recommendations

In this paper, we have shown that a one-dimensional multi-hop network that does opportunistic large array transmission can be modeled as a Markov chain in discrete time and we derived the sub-stochastic matrix of this chain. The Perron-Frobenius eigenvalue and the corresponding eigenvector of this matrix helps in determining different parameters for achieving better performance in delivering the message to a destination. As an extension to this work, it is recommended to obtain a framework where the nodes are aligned on a two dimensional grid, which will mimic a strip shaped OLA network. The objective in this case can be to find the best path in the network as done in [20] and [21]. The node locations in this study are kept fixed, it is again an interesting situation to study the effects of path loss if the node locations are random on a one-dimensional or two-dimensional network.

TABLE I
Fraction of DF nodes for various hop distances

Hop distance, h_d	2	3	4
% of nodes decoded, $\beta = 2$	92.30	94.67	97.02
% of nodes decoded, $\beta = 3$	93.54	95.98	98.21

Appendix A
Proof of Claim 1

Let $N = 3^M$, where N is the possible number of states for a window size of M using ternary M-tuples. By construction, the window overlap size is $M - h_d$, thus we split a window such that

$$M = \underbrace{M - h_d}_{overlap} + \underbrace{h_d}_{shift}. \tag{33}$$

The *shift* part is always receiving transmissions from the previous window, thus the nodes contained in *shift* can either go to 0 or 1 but never 2. Thus the possible combinations in *shift* are those of 0 and 1 which make a total of 2^{h_d}. The *overlap* part can contain any combination and thus is 3^{M-h_d}. Using the multiplication rule of independent events, the effective states are given as

$$\hat{N} = 3^{M-h_d} \times 2^{h_d}. \tag{34}$$

If all the elements of the window form any combination of 0 and 2, the system will be in absorbing state. The effect of 2 in the *shift* has already been taken into account from above equation. Thus we want the additional combination of 0 and 2 to be excluded from the *overlap* part which makes a total of 2^{M-h_d}. ■

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