Closed Form Throughput of a Slotted ALOHA Network Using LMMSE Receiver

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***Abstract*—This paper develops a framework for the analysis of a slotted ALOHA (S-ALOHA) network employing a two antenna LMMSE receiver. Throughput of such a network is analyzed under a Rayleigh channel condition. Two collision models are presented in this paper. One is signal to interference plus noise ratio (SINR) based and the other is average bit error rate (ABER) based. Coding is also considered when using ABER collision model. We compare the throughput of one and two antenna systems under Rayleigh fading channel and demonstrate the performance improvement. A closed form expression of the throughput of S-ALOHA network using a two antenna LMMSE receiver is derived.**

***Index Terms*—LMMSE, SINR, ABER, collision model, S-ALOHA, throughput.**

I. Introduction

THERE is great interest in defining a realistic physical (PHY) layer model, also called the collision model (CM), that enables a better analysis of wireless networks. One of the earliest and seminal work used a simple distance based CM [1]. However, the two most common CMs are based on: i) signal to interference ratio (SIR) [2–6] and ii) average bit error rate (ABER) [6–9]. A number of these models have been developed for Rayleigh fading channel. However, most of the models are for omni directional antennas. The authors in [3] presented a novel CM for multiple antenna linear minimum mean squared error (LMMSE) receivers.

Slotted ALOHA (S-ALOHA) networks have been very frequently used for developing a networking framework. SIR based CMs with Rayleigh fading with simple path loss model are the most commonly used in such framework [2-4]. Seminal work in [2] and [10] showed that independent fading channel of the interferers result in a higher throughput of S-ALOHA network than a line of sight channel (*only* path loss or distance based collision models). The work in [2] was extended in [4] to include the effect of number of interferers and the choice of modulation. Rician and Nakagami fading with additive white Gaussian noise (AWGN) channels were considered in [5] to compute the throughput. The authors in [6] use both the SIR and ABER based collision models under Nakagami fading. Effect of noise is considered in developing the ABER based CM. This paper also considers the effect of modulation and packet length on the throughput. More work on ABER based CMs can be found in [7–9]. The authors in [8–9] introduce the effect of coding along with modulation, packet length and signal to noise ratio (SNR).

In this paper, we present a framework for the analysis of slotted ALOHA network employing a two antenna receiver. The network model using a general CM is first established. Then SINR and ABER based CMs are developed using the authors earlier work on LMMSE receiver analysis [11]. These CMs are then used in the network model to obtain approximate closed form expressions for the throughput. The framework is then used to analyze the effects of coding, modulation, packet length, and SNR.

The rest of paper is organized as follows. Section II gives the network model that describes the S-ALOHA network and the assumptions therein. Bounds on throughput are also derived in this section. The CMs using the LMMSE receiver are presented in Section III. Simulation details and results are presented in section IV. Conclusions follow in Section V.

II. Network Model

A slotted ALOHA based traffic model is used to capture the effect of bursty packet transmissions in the network. Every packet in the network is of length τ and is transmitted only at the beginning of a slot period. An unsuccessful packet will be retransmitted after waiting a random number of slots. The channel is memoryless, i.e. a packet experiences collision uncorrelated with its previous attempts.

We assume that the total number of packets generated in the network (including retransmissions) be Poisson distributed. The validity of this assumption is important when one considers packet delays, since if one assumes a Poisson point process for packet originations, then the assumption that the combined process, of origination plus retransmissions, is poisson is valid only when one allows very large packet delays compared to the slot time. The mean generation rate from each node is λ packets per second. The mean offered channel traffic expressed in packets per time slot is $G = \lambda\tau$. The probability of an arbitrary test packet being overlapped by n other interferers is

$$R_n = \frac{G^n}{n!} e^{-G} \quad (1)$$

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Most of the studies of standard ALOHA networks assume that any collision results in a lost packet. It was shown in [2] that Rayleigh fading channel results in improvement in the network throughput. The advent of MIMO technology allows nodes to have multiple antennas. In this work we extend the work of [2] to multiple antenna receivers.

The probability of being able to capture the receiver in an arbitrary time slot is defined as

$$P_{cap} = 1 - \sum_{n=1}^{\infty} R_n P_{NC}^{CM}\left(n, \bar{\gamma}\right) \tag{2}$$

where $P_{NC}\left(n, \bar{\gamma}\right)$ is the probability of no-capture in presence of n interferers and average SINR $\bar{\gamma}$. This probability depends upon the type of CM represented in the superscript. Section III discusses these CMs in detail. Using the capture probability of Eq. 2 the channel throughput can be stated as

$$S = GP_{cap}\,. \tag{3}$$

The lower bound on the throughput will be obtained when $P_{NC}\left(n, \bar{\gamma}\right)$ is equal to one. In this case the probability Eq. 2 contains the infinite sum of the terms R_n. We then recover the classical expression for slotted ALOHA

$$S_{\min} = Ge^{-G}\,. \tag{4}$$

However, for any smaller value of $P_{NC}\left(n, \bar{\gamma}\right)$, the throughput will exceed Eq. 4, as determined by the relative strengths of the packet signals reaching the common receiver, the number of receiver antennas, modulation choice, etc. The upper bound on the throughput is obtained when P_{NC} is equal to zero. In this case the capture probability is one, hence

$$S_{\max} = G\,. \tag{5}$$

This implies that that all the received packets will be captured without any error.

III. Collision Model

Now that the traffic model has been identified we focus our attention on developing a good collision model to capture the PHY layer effect. The collision model used in this paper is same as the one in [3], however, that paper used a semi-analytical model. The authors derived a closed form collision model [11] for LMMSE receiver that will be used for developing the analytical framework for the network under consideration. The semi-analytical collision model developed in [3] used monte-carlo simulation for computing the required average eigenvalues of interference covariance matrix. The closed form model derived in [11] obtains an approximate expression for the average eigenvalues for two interferers and one desired user transmission for an arbitrary number of receive antennas, say M. The approximate average eigenvalues [11] are given in Eq. 1.

$$\begin{aligned}\bar{\alpha}_1 &= M\left(P_1 + P_2\right)\\ \bar{\alpha}_2 &= \left(M-1\right)\left(\tfrac{1}{P_1} + \tfrac{1}{P_2}\right)^{-1}\end{aligned} \tag{6}$$

If received power from both the users is same, $P_1 = P_2 =$ P, then the approximation becomes

$$\bar{\alpha}_1 = 2MP, \quad \alpha_2 = \left(M-1\right)\frac{P}{2} \tag{7}$$

To use the above result in S-ALOHA framework, which has more than two users in the network, we need to generalize it for an arbitrary number of users. We use the theorem that non-zero eigenvalues of a matrix and its conjugate transpose is the same. Hence, eigenvalues of interference covariance matrix for two antennas with M users is the same as those for M antennas and two users. This allows us to use the collision model developed in [11] for in the framework of S-ALOHA network. The average eigenvalues of interference-plus-noise covariance matrix becomes

$$\bar{\lambda}_1 = 2M + \sigma^2/P\,, \quad \bar{\lambda}_2 = \left(M-1\right)/2 + \sigma^2/P \tag{8}$$

The expression for the CDF of SINR can be obtained using [12, Eq. 25]

$$\begin{aligned}P(\gamma) &= \Pr\{SINR \le \gamma\}\\ &= \sum_{i=1}^{2} \frac{B_i}{\bar{\lambda}_i}\left[1 - \exp\left(-\bar{\lambda}_i \gamma\right)\right]\end{aligned} \tag{9}$$

where B_i is obtained using partial fraction expansion of [12, Eq. 20]. For all equal power interferers and desired user case, Eq. 9 can be re-written using the average eigenvalue approximations in Eq. 8 as

$$P\left(\gamma \mid M, \bar{\gamma}\right) = 1 - \frac{\bar{\lambda}_1 \bar{\lambda}_2}{\bar{\lambda}_1 - \bar{\lambda}_2}\left[\frac{e^{-\bar{\lambda}_2 \gamma}}{\bar{\lambda}_2} - \frac{e^{-\bar{\lambda}_1 \gamma}}{\bar{\lambda}_1}\right] \tag{10}$$

The probability of no-capture for SINR based CM can now be obtained by setting a SINR threshold, γ_{th}.

$$\begin{aligned}P_{NC}^{SINR}\left(M, \bar{\gamma}\right) &= \Pr\{SINR \le \gamma_{th}\}\\ &= 1 - \frac{\bar{\lambda}_1 \bar{\lambda}_2}{\bar{\lambda}_1 - \bar{\lambda}_2}\left[\frac{e^{-\bar{\lambda}_2 \gamma_{th}}}{\bar{\lambda}_2} - \frac{e^{-\bar{\lambda}_1 \gamma_{th}}}{\bar{\lambda}_1}\right]\end{aligned} \tag{11}$$

Next, we derive the ABER based collision model that is not only closer to providing a more realistic PHY layer abstraction but also allows a detailed analysis of the network performance because of different PHY parameters. In this work, we consider the following parameters: packet length, error control coding, and modulation.

Gaussian approximation on the interferers is used to simplify the BER collision model. Let $P_{eb}(\gamma)$ be the BER when the interferes are approximated to be Gaussian and equivalent SINR is γ. If the packet length is L then the probability that the packet is successfully received for the specified SINR is

$$P_s\left(L, \gamma\right) = \left(1 - P_{eb}\left(\gamma\right)\right)^L\,. \tag{12}$$

A flat fading channel is assumed such that all the bits in the packet undergo the same amount of fading. Hence, the probability of capture for a packet with an average SINR, $\bar{\gamma}$ is then

$$P_{NC}^{ABER}\left(M, \bar{\gamma}\right) = 1 - \int_0^{\infty} P_s\left(L, \gamma\right) p\left(\gamma \mid M, \bar{\gamma}\right) d\gamma\,. \tag{13}$$

where $p\left(\gamma \mid M, \bar{\gamma}\right)$ is the SINR probability density function for an LMMSE receiver that is obtained by differentiating Eq. 10 w.r.t γ. It can be observed from Eqs. 12 and 13 that the capture probability obtained using the BER collision model depends upon L and $P_{eb}(\gamma)$. The latter is a function of the type of modulation; hence, to model the effect of modulation we need to plug-in the appropriate expression of the BER.

A closed form solution of Eq. 13 is difficult to obtain and numerical computation results in inaccurate results at high SNR values because of very low $P_{eb}(\gamma)$ values involved. Hence, we separate the integration into two with different limits. The first integral, given in Eq. 14, will be computed numerically.

$$P_C^1(M,\bar{\gamma}) = \int_0^{\gamma_0} P_s(L,\gamma)\, p(\gamma \mid M,\bar{\gamma})\, d\gamma \,. \qquad (14)$$

The second uses two approximations: (i) $P_s(L,\gamma) \approx 1 - LP_{eb}(\gamma)$ at high SNRs and (ii) given in Eq. 15

$$Q(x) \approx \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right). \qquad (15)$$

The second approximation helps in obtaining a simple closed form expression for the second integral, given in Eq. 16, without much loss in accuracy. Using approximation (i) we get the second integral as

$$P_C^2 = \int_{\gamma_0}^{\infty} \left(1 - LP_{eb}(\gamma)\right) p(\gamma \mid M,\bar{\gamma})\, d\gamma \,. \qquad (16)$$

The closed form solution of Eq. !6 is derived in the Appendix. To model different modulations $P_{eb}(\gamma)$ should be selected accordingly [13, pp-217]. Using Eqs. 14 and 16 the ABER based CM can now be written as

$$P_{NC}^{ABER}(M,\bar{\gamma}) = 1 - P_C^1 - P_C^2 \,. \qquad (17)$$

Coding can be used to improve the probability of capture for a packet. The BCH codes are examined to determine the effects of coding in presence of interfering packets. For a given probability of bit error, the probability of survival of a packet of length L is given by

$$P_{ep}(n \mid \gamma) = \sum_{i=0}^{t} \binom{L}{i} \left(1 - P_{be}(n \mid \gamma)\right)^{L-i} \left(P_{be}(n \mid \gamma)\right)^{i} \,, \qquad (18)$$

where t is the number of bits the error correction code can correct. For an un-coded system $t = 0$ and the equation reduces to Eq. 12.

IV. Simulation and Results

In this section we use the two CMs developed in Section III in the S-ALOHA network model presented in Section II. This framework allows us to study the network throughput as a function of different physical layer parameters. The parameters for SCM are: SNR per user and SINR threshold. The parameters for ACM are: modulation, packet length, SNR per user, and coding. In all the results presented in this section the network employs a two antenna LMMSE receiver.

Fig. 1 shows the throughput results using SCM in the S-ALOHA network. Throughput is computed for different received SNRs, $\bar{\gamma}$ (5 and 30 in dB), and SINR thresholds, γ_{th} (2, 3, 5, and 10 in dB). We observe that the throughput is higher for higher SNR. However, this increase is not monotonic. Increasing the SNR beyond 30 dB does not result in further increase in throughput. This is because the interferers become dominant beyond this level and hence the effect of noise is reduced. Also, we note that the gain in throughput because of SNR is reduced when we choose a lower γ_{th}. This gain finally reduces to zero when γ_{th} is around 10 dB. The throughput curve in this case corresponds to the standard S-ALOHA throughput, i.e. a transmission is successful only when there is no collision. This behavior is analytically explained by Eq. 4. We also observe that reducing γ_{th} results in significant throughput improvement. Hence, we conclude that a PHY layer design that can suppress the interferers even with a lower post LMMSE processing SINR will improve network throughput. A good example for such a design is CDMA. Let us assume an IEEE 802.11b signal that is spread using a 11-chip Barker sequence. Because of the good autocorrelation property of the sequence, any asynchrony between the desired user and an interferer results in around 10 dB gain in SINR.

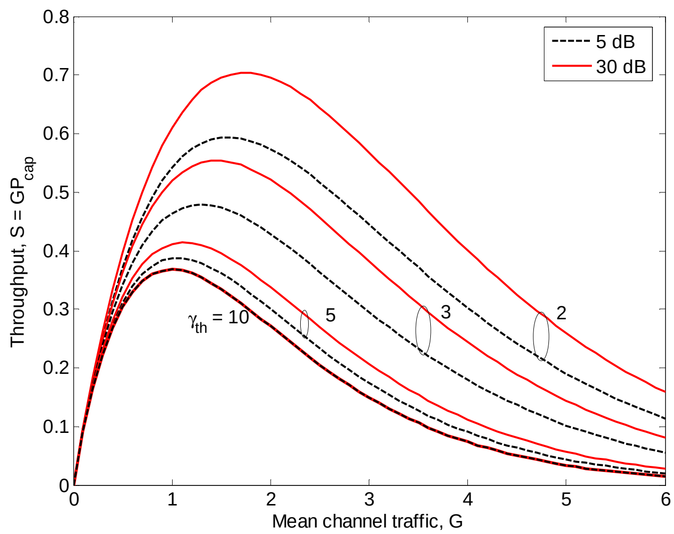

Figure 1. Throughput of S-ALOHA network employing two antenna receiver with LMMSE receiver processing, SINR collision model

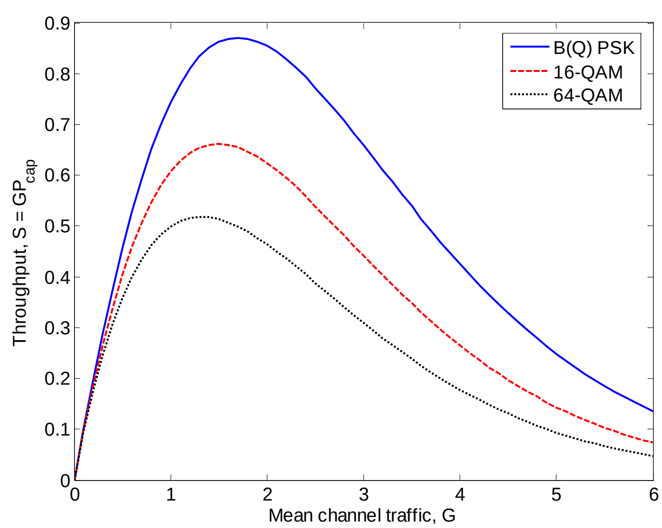

Figure 2. Throughput of S-ALOHA network employing two antenna receiver with LMMSE receiver processing, ABER collision model without coding at 20 dB SNR, L = 128

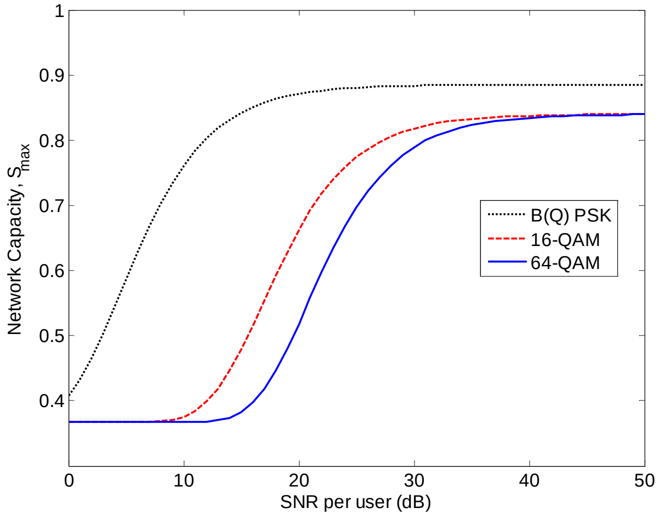

Figure 3. Maximum throughput of S-ALOHA network Vs. per user SNR for different modulation schemes, L = 128

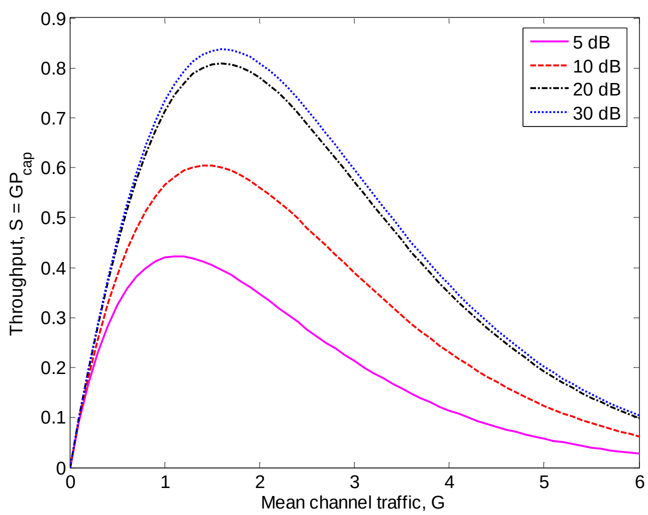

Figure 4. ABER collision model without coding parameterized on SNR, BPSK modulation, L = 8192

Fig. 2 shows the effect of modulation on throughput using ACM. All the packets have same number of bits (L = 128) irrespective of the modulation. Each symbol has same time period. Hence, the slot length reduces when using higher order modulation. The throughput is expressed in terms of packets per slot in Fig. 2. We compare throughput for BPSK, QPSK, 16-QAM and 64-QAM modulations. Since, BPSK and QPSK have the same BER for a given SNR per bit they have the same throughput in packets per slot. We observe that the throughput of 16-QAM and 64-QAM is lower than BPSK in packets per slot because of higher BER of these modulations. However, when considering the same length of time, systems using BPSK, QPSK, 16-QAM, and 64-QAM will have 1, 2, 4, and 8 times their throughput in packets/slot, respectively.

We observe from Fig. 1 that only a limited range of SNR affects throughput. Below this range noise dominates the desired signal and above that it is interference dominated. Fig. 3 shows these results more explicitly using ACM for different modulation schemes. In this figure we plot the network capacity as a function of SNR and parameterized on modulation. Network capacity is defined as the maximum throughput of the network for all possible mean channel traffic. We observe the throughput limits for different modulations at low and high SNRs. In the linear dependence range BPSK and QPSK have around 12 dB gain over 16-QAM and around 17 dB gain over 64-QAM. The throughput is in terms of packets per slot. It should be noted that the lower limit of throughput at low SNRs assumes that the desired user can be decoded at an arbitrarily low SNR if there are no interferers present. The CMs should be redefined to better capture the effect of noise in absence of interferers.

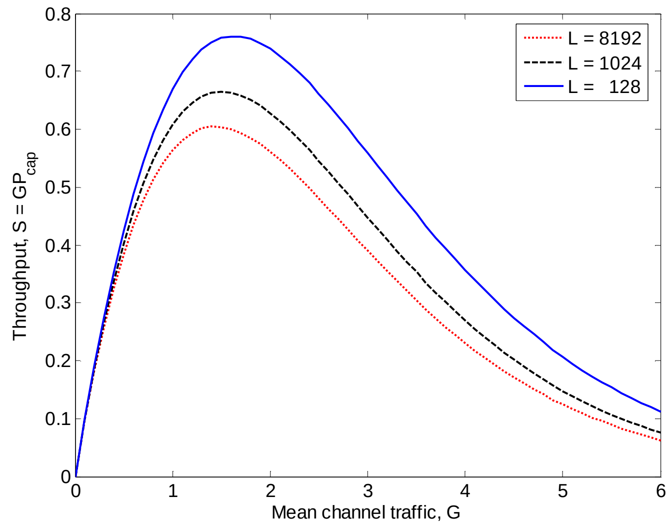

Figure 5. ABER collision model without coding, paratemerized on packet length for BPSK modulation

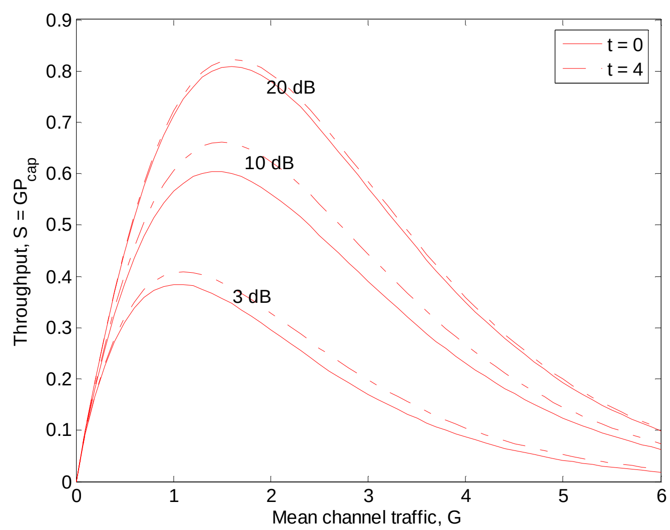

Figure 6. ABER collision model with coding parameterized on correction code and SNR, BPSK modulation

Fig. 4 shows the throughput parameterized on SNR (5, 10, 20, and 30 dB). Packets are of length 8192 bits and use BPSK modulation.

Fig. 5 shows the throughput parameterized on packet length (8192, 1024, and 128 bits). The SNR per user is 10 dB and again BPSK modulation is used. We can clearly observe that longer packets result in reduction in throughput because of higher packet error rate.

V. Conclusion

We have developed a framework for the analysis of S-ALOHA network employing a two antenna LMMSE receiver. The main contribution of this paper is derivation of SINR and ABER based CMs. Using SCM in the framework brings out the interesting fact that a PHY processing that requires lesser SINR threshold for successful reception of a packet, e.g. CDMA, results in a higher network throughput. Using ACM we observe that higher order modulation reduces the throughput in terms of packets per slot, however, when considering same time duration higher order modulation results in greater throughput. The current framework assumes that the desired user can be decoded at an arbitrarily low SNR if there is no interference.

Appendix I

The intermediate steps of deriving Eq. 17 from Eq. 16 are given in this section. Rewriting Eq. 16 as

$$\begin{aligned} P_C^2 &= \int_{\gamma_0}^{\infty} p(\gamma \mid M, \bar{\gamma}) d\gamma - \int_{\gamma_0}^{\infty} L P_{eb}(\gamma) p(\gamma \mid M, \bar{\gamma}) d\gamma \\ &= P_C^{21} - P_C^{22} \end{aligned} \qquad \text{(A1)}$$

where P_C^{21} and P_C^{22} are first and second integrals, respectively. P_C^{21} is the complimentary CDF of the SINR evaluated at γ_0, which can be obtained using Eq. 10 as

$$P_C^{21} = 1 - P(\gamma_0 \mid M, \bar{\gamma}) \qquad \text{(A2)}$$

To simplify P_C^{22} we use the high SNR approximation of the Q-function as given by Eq. 15. $P_{eb}(\gamma)$ can be written in general form as

$$P_{eb}(\gamma) = \alpha Q\left(\sqrt{\beta\gamma}\right) \qquad \text{(A3)}$$

with $\alpha = \beta = 1$ for BPSK/QPSK and $\alpha = 2\left(1 - 1/\sqrt{M}\right)$, $\beta = 3/(M-1)$ for M-QAM. After substituting Eq. A3 and PDF of SINR, i.e. derivative of Eq. 10, we obtain

$$P_C^{22} = K \int_{\gamma_0}^{\infty} \frac{1}{\sqrt{\gamma}} e^{-\beta\gamma} \left(e^{-\bar{\lambda}_2 \gamma} - e^{-\bar{\lambda}_1 \gamma} \right) d\gamma \qquad \text{(A4)}$$

where $K = L\alpha / \left(2\sqrt{\beta\pi}\right) \times \bar{\lambda}_1 \bar{\lambda}_2 / \left(\bar{\lambda}_1 - \bar{\lambda}_2\right)$. Rewriting A4 as follows

$$P_C^{22} = K \left[\int_{\gamma_0}^{\infty} \gamma^{\frac{1}{2}-1} e^{-\left(\beta + \bar{\lambda}_2\right)\gamma} d\gamma - \int_{\gamma_0}^{\infty} \gamma^{\frac{1}{2}-1} e^{-\left(\beta + \bar{\lambda}_1\right)\gamma} d\gamma \right] \qquad \text{(A5)}$$

We can clearly see that the integrals in Eq. A5 are a form of incomplete Gamma function. Hence, after substituting

$$\Gamma(m, \gamma_0) = \int_{\gamma_0}^{\infty} \gamma^{m-1} e^{-\gamma} d\gamma \qquad \text{(A6)}$$

into Eq. A5, we get

$$\begin{aligned} P_C^{22} = {} & \frac{K}{\sqrt{\beta + \bar{\lambda}_2}} \Gamma\left(\frac{1}{2}, \left(\beta + \bar{\lambda}_2\right)\gamma_0\right) - \\ & \frac{K}{\sqrt{\beta + \bar{\lambda}_1}} \Gamma\left(\frac{1}{2}, \left(\beta + \bar{\lambda}_1\right)\gamma_0\right) \end{aligned} \qquad \text{(A7)}$$

Hence, we obtain the closed form solution of Eq. 16 using Eqs. A1, A3, and A7.

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