EE 6082 HW 1

1. Identity element is unique.

Suppose not : $\Rightarrow \exists\, e' \neq e$ s.t. $\forall g \in G$, $e*g = e'*g = g$.

Let $e = g$, $\Rightarrow e = e'*e = e'$, a contradiction. Q.E.D.

2. Inverse a^{-1} of a is unique.

Suppose not : $\Rightarrow \exists\, a_1^{-1} \neq a_2^{-1}$ s.t. $\forall a \in G$

$a_1^{-1} * a = a_2^{-1} * a = e$.

By associativity, $(a_1^{-1} * a) * a_2^{-1} = a_1^{-1} * (a * a_2^{-1})$

$\Rightarrow e * a_2^{-1} = a_1^{-1} * e$, or $a_2^{-1} = a_1^{-1}$, a contradiction. Q.E.D.

(2) 6. The element 2 has no inverse, otherwise $2*2^{-1} = 1$ and

$$4 = 4*(2*2^{-1}) = (4*2)*2^{-1} = 0*2^{-1} = 0.$$

(3) 16. Since there are only 4 linearly independent vectors over GF(2), dimension = 4.

GF(2) is embedded in GF(4), therefore the dimension does not change, So dimension = 4.

Since there are 5 linearly independent vectors over GF(31), dimension = 5 in this case.

17. We need solutions $\underline{V}$ to $G\,\underline{V} = \underline{0}$, where

$$G = \begin{bmatrix} 1&1&1&0&0 \\ 0&1&1&1&0 \\ 0&0&1&1&1 \end{bmatrix}.$$ We use row reduction on G;

(5) $$G \rightarrow \begin{bmatrix} 1&0&0&1&0 \\ 0&1&0&0&1 \\ 0&0&1&1&1 \end{bmatrix} = \left[I_3 \,\vdots\, P \right].$$ We then perform

$$\left[-P^T \,\vdots\, I_2 \right] = \begin{bmatrix} 1&0&1&1&0 \\ 0&1&1&0&1 \end{bmatrix},$$ which gives a basis as

$\{(10110), (01101)\}$.

19. $(11010) = (10000) + (01000) + (00010)$

20. $(10111) = (10101) + (00010)$.

21. We essentially have the system

$$\left[\underline{V}_1^T \;\; \underline{V}_2^T \cdots \underline{V}_n^T \right] \left[\underline{a}^T \right] = \underline{V}^T \quad \text{where } \underline{a} = (a_1, a_2, \ldots, a_n)$$

& each $\underline{V}_i = (V_{i1}, V_{i2}, \ldots, V_{in})$. We can thus use Cramer's rule to solve for the $\{a_i\}$ by

$$a_i = \frac{\left| \underline{V}_1^T \ldots \overset{i \leftarrow \text{position } i}{\underline{V}^T} \ldots \underline{V}_n^T \right|}{\left| \underline{V}_1^T, \underline{V}_2^T \cdots \underline{V}_n^T \right|}.$$