ECE 6605

HW#3, Assigned Sept 15, Due Sept 24.

1. (10 points) Consider a stationary 3-state Markov chain $\{X_i\}$ where X_i is an element of $\mathcal{X} = \{1, 2, 3\}$ with probability transition matrix

$$P = \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ p & 1-p & 0 \end{bmatrix}$$

a) Draw the state transition diagram and find the stationary distribution □.
b) Find the entropy rate H(X) as a function of p and evaluate for p=1/2 .

2. For a stationary random process $X_1, X_2, ... X_n$ and all n, show that:

a) $\dfrac{H(X_1, X_2, ... X_n)}{n} \leq \dfrac{H(X_1, X_2, ... X_{n-1})}{n-1}$

b) $\dfrac{H(X_1, X_2, ... X_n)}{n} \geq H(X_n \mid X_{n-1}, X_1)$

3) Let $X_{-n} ..., X_{-1}, X_0, X_1, ... X_n$ be a stationary (not necessarily Markov) stochastic process. Which of the following statements are true? Prove or provide a counterexample.

(a) $H(X_n \mid X_0) = H(X_{-n} \mid X_0)$
(b) $H(X_n \mid X_0) \geq H(X_{n-1} \mid X_0)$
(c) $H(X_n \mid X_1, ..., X_{n-1}, X_{n+1})$ is nonincreasing in n.
(d) $H(X_{n+1} \mid X_1, ..., X_n, X_{n+2}, ... X_{2n+1})$ is nonincreasing in n.

4) This problem deals with mutual information and testing independence. Let $\underline{Z}_i = (X_i, Y_i)$ be a an IID sequence (the vectors $\underline{Z}_i$'s are IID but X_i and Y_i are not necessarily independent) with density $p(x, y)$. If we form the (log likelihood ratio) function

$$\frac{1}{n}\log\frac{p(X_1,X_2,...,X_n)p(Y_1,Y_2,...,Y_n)}{p(X_1,X_2,...,X_n,Y_1,Y_2,...,Y_n)}$$

What is its limit (that is, evaluate as n tends to   )? Provide an interpretation.

5) a) In class we saw that conditioning reduces entropy, i.e. H(X) ≥ H(X|Y). Use this property to show that (in general) conditioning <u>does not</u> reduce mutual information. For example, try to prove I(X;Y|Z) ≤ I(X;Y) then indicate why this cannot be true in general.

b) Let X| | Y | | Z be a Markov chain in that order. Prove that conditioning does reduce the mutual information, namely, prove I(X;Y|Z) ≤ I(X;Y).

6) *Entropy rate of dog looking for a bone.* A dog walks from one integer to the next, possible reversing his direction at each step with probability p=0.1 . Let X_i denote his position at time i. Let X_0 = 0 and his first step is equally likely to be positive or negative. A typical walk might be:

$$(X_0,X_1,X_2,X_3,...) = (0,1,2,1,0,□1,□2,□3,□2,...)$$

Find $H(X_1,...,X_n)$ and the entropy rate $\mathrm{H}(\mathcal{X})$