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25 (a) $p(y) = \int_{-\infty}^{\infty} p_\theta(y)\, w(\theta)\, d\theta = \int_0^1 \frac{6(y^2+\theta y)}{2+3\theta}\, d\theta, \qquad 0 \le y \le 1$

$p(y) = 0, \qquad y < 0 \text{ or } y > 1$

$$\begin{array}{l|l} 6y\theta + 6y^2 & 3\theta + 2 \\ \hline 6y\theta + 4y & 2y \\ \hline 6y^2 - 4y & \end{array} \quad \Rightarrow \quad \frac{6(y^2+\theta y)}{2+3\theta} = 2y + \frac{6y^2-4y}{2+3\theta}$$

$$\therefore\ 0 \le y \le 1 \Rightarrow p(y) = 2y + (6y^2-4y)\int_0^1 \frac{d\theta}{2+3\theta}$$

$$= 2y + \frac{1}{3}(6y^2-4y)\left[\ln(2+3\theta)\right]_0^1$$

$$= 2y + \frac{1}{3}(6y^2-4y)\ln 2.5$$

$$\therefore\ p(y) = \begin{cases} (2\ln 2.5)y^2 + (2 - \frac{4}{3}\ln 2.5)y, & 0 \le y \le 1 \\ 0, & \text{oth.} \end{cases}$$

$$w(\theta|y) = \frac{p_\theta(y)\, w(\theta)}{p(y)}$$

$$0 \le y \le 1,\ 0 \le \theta \le 1 \Rightarrow w(\theta|y) = \frac{p_\theta(y)}{p(y)}$$

<u>MMSE Estimator:</u>

$$\hat{\theta}_{MMSE}(y) = E\{\theta | Y=y\} = \int_{-\infty}^{\infty} \theta\, w(\theta|y)\, d\theta = \frac{6}{p(y)} \int_0^1 \frac{y^2\theta + y\theta^2}{2+3\theta}\, d\theta$$

$$\begin{array}{l|l} y\theta^2 + y^2\theta & 3\theta + 2 \\ \hline y\theta^2 + \frac{2}{3}y\theta & \frac{y}{3}\theta + \frac{1}{3}(y^2 - \frac{2}{3}y) \\ \hline (y^2 - \frac{2}{3}y)\theta & \\ (y^2 - \frac{2}{3}y)\theta + \frac{2}{3}(y^2 - \frac{2}{3}y) & \\ \hline -\frac{2}{3}y^2 + \frac{4}{9}y & \end{array}$$

$$\Rightarrow \frac{y^2\theta + y\theta^2}{2+3\theta} = \frac{1}{3}y\theta + \frac{1}{3}\left(y^2 - \frac{2}{3}y\right) + \frac{-2/3\,y^2 + 4/9\,y}{2+3\theta}$$

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$$\therefore\ \hat{\theta}_{MMSE}(y) = \frac{2(y^2 - 2/3\,y)}{p(y)} + \frac{2y}{p(y)}\int_0^1 \theta\, d\theta + \frac{6}{p(y)}\int_0^1 \frac{-2/3\,y^2 + 4/9\,y}{2+3\theta}\, d\theta$$

$$= \frac{2(y^2 - 2/3\,y)}{p(y)} + \frac{y}{p(y)} + \frac{2(-2/3\,y^2 + 4/9\,y)}{p(y)} \ln 2.5$$

$$= \frac{(2 - 4/3 \ln 2.5)y^2 + (8/9 \ln 2.5 - 1/3)y}{p(y)}$$

$$0 \le y \le 1 \Rightarrow \hat{\theta}_{MMSE}(y) = \frac{(2 - 4/3 \ln 2.5)y^2 + (8/9 \ln 2.5 - 1/3)y}{(2\ln 2.5)y^2 + (2 - 4/3 \ln 2.5)y}$$

$$\cong \frac{0.77828\,y + 0.48115}{1.8326\,y + 0.77828}$$

$$MSE = E[\theta^2] - E[\hat{\theta}^2_{MMSE}(Y)]$$

$$E[\theta^2] = \int_0^1 \theta^2\, d\theta = \left[\frac{1}{3}\theta^3\right]_0^1 = 1/3$$

$$E[\hat{\theta}^2_{MMSE}(Y)] =$$

$$\int_0^1 \left[\frac{(2 - 4/3 \ln 2.5)y + (8/9 \ln 2.5 - 1/3)}{(2\ln 2.5)y + (2 - 4/3 \ln 2.5)}\right]^2 y\left[(2\ln 2.5)y + (2 - \tfrac{4}{3}\ln 2.5)\right]$$

$$= \int_0^1 y\, \frac{\left[(2 - 4/3 \ln 2.5)y + (8/9 \ln 2.5 - 1/3)\right]^2}{(2\ln 2.5)y + (2 - 4/3 \ln 2.5)}\, dy$$

$$\cong 0.250344534$$

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(b) $\hat{\theta}_{MAP}(y) = \arg\max_{\theta} \{ w(\theta|y) \}$

$$= \underset{0\le\theta\le1}{\arg\max} \{ P_\theta(y) \}$$

$$\frac{dP_\theta(y)}{d\theta} = 0 \Rightarrow \frac{d}{d\theta}\left[\frac{y+\theta}{2+3\theta}\right] = 0 \Rightarrow 2+3\theta - 3(y+\theta) = 0$$

$\Rightarrow 2 - 3y = 0$? $\Rightarrow$ $P_\theta(y)$ doesn't have any local maximum

$\therefore$ $\boxed{\hat{\theta}_{MAP}(y) = 0}$

MMAE

$$\int_{-\infty}^{\hat{\theta}} w(\theta|y)\,d\theta = 0.5$$

assuming $0 \le \hat{\theta} \le 1 \Rightarrow \int_0^{\hat{\theta}} P_\theta(y)\,d\theta = 0.5\,p(y)$

$0 \le y \le 1 \Rightarrow \int_0^{\hat{\theta}} \frac{6(y^2+\theta y)}{2+3\theta}\,d\theta = 0.5\,p(y)$

$$\Rightarrow 6y^2 \int_0^{\hat{\theta}} \frac{d\theta}{2+3\theta} + 2y \int_0^{\hat{\theta}} \frac{3\theta+2-2}{2+3\theta}\,d\theta = 0.5\,p(y)$$

$$\Rightarrow 2y \ln\left(\frac{2+3\hat{\theta}}{2}\right) + 2\left[\hat{\theta} - \frac{2}{3}\ln\left(\frac{2+3\hat{\theta}}{2}\right)\right]$$

$$= (\ln 2.5)y + \left(1 - \frac{2}{3}\ln 2.5\right)$$

$$\Rightarrow \left(2y - \frac{4}{3}\right)\ln\left(\frac{2+3\hat{\theta}}{2}\right) + 2\hat{\theta} = (\ln 2.5)y + 1 - \frac{2}{3}\ln 2.5$$

Solving the above equation for $\hat{\theta}$, the MMAE estimator will be obtained.