

## Introduction to Estimation Theory Part II

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Lecture 25

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## Variance of ML Est. of Gaussian Mean

Jumping right in where we left off:

$$\begin{aligned}\text{var}_{\xi} [\hat{\mu}_{ML}(\underline{y})] &= \text{var}_{\xi} \left[ \frac{1}{L} \sum_{l=0}^{L-1} \underline{y}(l) \right] \\ &= \frac{1}{L^2} \sum_{l=0}^{L-1} \text{var}_{\xi} [\underline{y}(l)] = \frac{1}{L^2} L \sigma^2 = \frac{\sigma^2}{L}\end{aligned}$$

Recall:  $b_{\xi}(\hat{\mu}_{ML}) = 0$

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## Estimators of Gaussian Variance

$$\hat{\sigma}_{ML}^2(\underline{y}) = \text{var}_{\xi} \left\{ \frac{1}{L} \sum_{l=0}^{L-1} [y(l) - \hat{\mu}_{ML}(\underline{y})]^2 \right\}$$

$$b_{\xi}(\hat{\sigma}_{ML}^2) = -\sigma^2 / L$$

$$\sigma_{UB}^2(\underline{y}) = \text{var}_{\xi} \left\{ \frac{1}{L-1} \sum_{l=0}^{L-1} [y(l) - \hat{\mu}_{ML}(\underline{y})]^2 \right\}$$

$$b_{\xi}(\hat{\sigma}_{UB}^2) = 0$$

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## Vars. of Estimators of Gaussian Vars.

$$\text{var}_{\xi} [\hat{\sigma}_{ML}^2(\underline{y})] = 2 \frac{(\sigma^2)^2}{L} \left( \frac{L-1}{L} \right)$$

$$\text{var}_{\xi} [\hat{\sigma}_{UB}^2(\underline{y})] = 2 \frac{(\sigma^2)^2}{L} \left( \frac{L}{L-1} \right)$$

(from estimation theory notes by  
Al Hero, U. of Michigan)

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## Notion of Mean Squared Error

We will explore the idea of bias/  
variance tradeoffs in the MSE:

$$E_{\xi} [(\hat{\xi} - \xi)^2] = \text{var}_{\xi}(\hat{\xi}) + b_{\xi}^2(\hat{\xi})$$

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## Bias and Covariance Matrices

$$\begin{aligned} \mathbf{b}_{\xi}(\hat{\xi}) &= E_{\xi}[\hat{\xi}(y)] - \xi \\ &= E_{\xi} \left\{ \begin{bmatrix} \hat{\mu}(y) \\ \hat{\sigma}^2(y) \end{bmatrix} \right\} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{cov}_{\xi}[\hat{\xi}(y)] &= E_{\xi}[\hat{\xi}\hat{\xi}^T] - E_{\xi}[\hat{\xi}]E_{\xi}[\hat{\xi}^T] \\ &= \begin{bmatrix} E[\hat{\mu}^2] & E[\hat{\mu}\hat{\sigma}^2] \\ E[\hat{\mu}\hat{\sigma}^2] & E[(\hat{\sigma}^2)^2] \end{bmatrix} - \begin{bmatrix} E^2[\hat{\mu}] & E[\hat{\mu}]E[\hat{\sigma}^2] \\ E[\hat{\mu}]E[\hat{\sigma}^2] & E^2[\hat{\sigma}^2] \end{bmatrix} \end{aligned}$$

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## Decomposition of MSE Matrix

$$\begin{aligned} &E_{\xi} \{ [\hat{\xi}(y) - \xi][\hat{\xi}(y) - \xi]^T \} \\ &= E_{\xi} [ \hat{\xi}(y)\hat{\xi}^T(y) - \hat{\xi}(y)\xi^T - \xi\hat{\xi}^T(y) + \xi\xi^T ] \\ &= E[\hat{\xi}\hat{\xi}^T] - E[\hat{\xi}]\xi^T - \xi E[\hat{\xi}^T] + \xi\xi^T \\ &\quad + E[\hat{\xi}]E[\hat{\xi}^T] - E[\hat{\xi}]\xi^T - \xi E[\hat{\xi}^T] + \xi\xi^T \\ &= E[\hat{\xi}\hat{\xi}^T] - E[\hat{\xi}]E[\hat{\xi}^T] \\ &\quad + E[\hat{\xi}]E[\hat{\xi}^T] - E[\hat{\xi}]\xi^T - \xi E[\hat{\xi}^T] + \xi\xi^T \end{aligned}$$

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## Decomposition of MSE Matrix

$$\begin{aligned} &E_{\xi} \{ [\hat{\xi}(y) - \xi][\hat{\xi}(y) - \xi]^T \} \\ &= E[\hat{\xi}\hat{\xi}^T] - E[\hat{\xi}]E[\hat{\xi}^T] \\ &\quad + \underbrace{E[\hat{\xi}]E[\hat{\xi}^T] - E[\hat{\xi}]\xi^T - \xi E[\hat{\xi}^T] + \xi\xi^T}_{(E[\hat{\xi}] - \xi)(E[\hat{\xi}] - \xi)^T} \\ &= \text{cov}_{\xi}(\hat{\xi}) + \mathbf{b}_{\xi}(\hat{\xi})\mathbf{b}_{\xi}^T(\hat{\xi}) \\ &\text{In scalar case: } E_{\xi} [(\hat{\xi} - \xi)^2] = \text{var}_{\xi}(\hat{\xi}) + b_{\xi}^2(\hat{\xi}) \end{aligned}$$

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## Composite MSE

$$\begin{aligned}MSE &= \text{tr} \left\{ E_{\xi} \{ [\hat{\xi}(\underline{y}) - \xi][\hat{\xi}(\underline{y}) - \xi]^T \} \right\} \\ &= \text{tr} \{ \text{cov}_{\xi}(\hat{\xi}) + \mathbf{b}_{\xi}(\hat{\xi})\mathbf{b}_{\xi}^T(\hat{\xi}) \} \\ &= \text{tr} \{ \text{cov}_{\xi}(\hat{\xi}) \} + \text{tr} \{ \mathbf{b}_{\xi}(\hat{\xi})\mathbf{b}_{\xi}^T(\hat{\xi}) \}\end{aligned}$$

