# Some structural results for the problem of Min-Time Coverage in Constricted Environments

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**Abstract:** In a recent work, we introduced a new set of problems in the area of networked robotic systems that concern the time-optimal execution of certain coverage tasks taking place in constricted environments. That work provided the detailed problem definitions, a complete representation of these problems in terms of Mathematical Programming (MP) formulations, and a formal analysis of their worst-case computational complexity. The current work establishes some structural results for the considered problems that are useful for the strengthening of the aforementioned MIP formulations and for the further development of pertinent heuristic solution methods for these problems. We demonstrate the first possibility in this paper, and we defer the second one to future work.

*Keywords:* Networked mobile robotic systems; coverage problems; optimal multi-robot path planning on graphs; combinatorial scheduling; valid inequalities.

## 1. INTRODUCTION

In a recent work (Reveliotis and Kim (2021)), we have introduced a new class of problems in the area of networked robotic systems that concerns the employment of robotic fleets for the support of routine inspection and service operations taking place in well-structured but constricted environments. Characteristic examples of such operations concern inspection and service tasks taking place in water supply and sewage networks, mines and other underground excavation sites, and oil and gas pipeline systems. In all these environments, the deployed robots are confined to move in narrow aisles that are naturally defined by the corresponding facilities. Furthermore, the geometry and the nature of the physical boundaries delineating these aisles restrict severely the robot communication with each other and with the command-&-control (C&C-) center that manages the entire operation.

The aisle narrowness in the aforementioned operations raises issues of collision avoidance for the traveling robots, and a need to preserve the robot safety and the traffic integrity. These issues can be addressed through the imposition of a zoning scheme that splits the underlying guidepath network into zones of unit buffering capacity, and grants access to these zones to the contesting robots through a traffic coordinator. Similar zoning schemes have been used extensively in automated unit-load industrial material handling systems (Heragu (2008)), and more recently they have provided a safety control mechanism in other mobile robotic applications, as well (c.f. Yu and LaValle (2016); Parker et al. (2016)).

But in the considered applications, the imposed zoning scheme must serve the additional objective of keeping each deployed robot connected to the C&C-center through the preservation of a multi-hop wireless communication network that relays messages among the various robots and the center (c.f. Tardioli et al. (2010); Loizou and Constantinou (2016)). In order to maintain this communication network, (i) zones must be defined so that robots located in neighboring zones are within the direct communication range of each other, and (ii) the robot advancement through the zones must be further controlled so that each robot is always connected to the C&C-center by a path of the underlying guidepath network that has all of its zones occupied by other robots (and therefore, the required message-relaying function is always possible).

At the end, an effective traffic controller for the targeted applications must coordinate the robot traffic in a way that (i) observes the imposed zoning scheme and the connectivity requirements that were described in the previous paragraph, and (ii) ensures the expedient execution of the underlying tasks.

The aforementioned work of Reveliotis and Kim (2021) (i) positioned the resulting traffic control problem in the context of the literature on networked robotic systems, (ii) provided detailed definitions for a number of variations of this problem and analytical characterizations in the form of mathematical programming (MP) formulations, and (iii) established that, in their general positioning, all these variations are strongly NP-hard combinatorial optimization problems (Papadimitriou (1995)). In this work, we establish some new structural results for the considered problems and their MP formulations presented in Reveliotis and Kim (2021), that are useful for the strengthening of these MP formulations and for the further development of pertinent heuristic solution methods for these problems. We demonstrate the first usage of the derived results in this paper, and we highlight the second possibility in the concluding discussion as part of our future work.

The rest of the paper is organized as follows: In the next section, we overview the basic characterizations of the coverage problems that are considered in Reveliotis and Kim (2021), and the MP formulations for these problems that are provided in that work. Section 3 establishes our main results, and Section 4 demonstrates their usage for the strengthening of the current MP formulations. Section 5 concludes the paper and suggests some directions for future work. Finally, following standard practice in the robotics community, in the rest of the paper we refer to the considered robotic systems as *multiple mobile robot systems*, or *MMRS*, for brevity (Parker et al. (2016)).

## 2. THE CONSIDERED COVERAGE PROBLEMS AND THEIR MP FORMULATIONS

**The considered coverage tasks and the corresponding traffic-management problems:** A fleet of mobile robots must be used to inspect a set of locations of an underground guidepath network that constitutes a *tree*. As discussed in the introductory section, some examples of such a guidepath network might be a water supply network, a sewage network, or a network of tunnels in an underground mine (Tardioli et al. (2010); Loizou and Constantinou (2016)). The robots are initially located at a C&C-center of the entire facility, which defines the root of the tree, and the targeted locations are its leaves. Furthermore, the tunnels are narrow and the robots have limited sensing and maneuvering capability. Therefore, for safety reasons, the robots must be separated through the imposition of a zoning scheme with unit buffering capacity for each zone, along the lines that were discussed in the introductory section.

The robots possess wireless communication capability, but their communication range is severely limited by their operational environment. Since these wireless communication links are the only way for each robot to communicate with its operational environment, the robot motion must be coordinated in a way that, at any time point, the active links among the robots define a multi-hop communication network connecting the robots to each other and to the C&C-center. As explained in the previous section, a natural way to ensure this connectivity is by defining the imposed zones in a way that neighboring zones guarantee robust connectivity between the robots that occupy these zones, and further stipulating that, at any time point, a zone cannot be occupied unless its parent zone in the underlying tree is also occupied.

Finally, following standard practice in the formal study of the traffic dynamics that are generated by zoning schemes similar to those considered in this work, we further assume that zones are defined in a way that they have uniform traversal time. Then, picking this traversal time as the time unit, we can study the resulting traffic dynamics in discrete time. [1]

In view of the above description, the considered MMRS can be formally represented by a tuple $\mathcal{M} = \langle \mathcal{R}, \mathcal{T} \rangle$, where $\mathcal{R}$ is the set of the robots and $\mathcal{T}$ is a rooted tree representing the tunnel system. The node set $V$ of $\mathcal{T}$ represents the zones of the tunnel system, and the edge set $E$ represents the neighboring relation among the zones.

The root node of $\mathcal{T}$ – i.e., the initial location of all robots and the point of command and control for the entire system – is denoted by $o$. The set of the leaf nodes of $\mathcal{T}$ is denoted by $L$. As already stated, each zone $v \in L$ must be visited by some robot for inspection purposes, and the inspection of a leaf zone can be carried out by the visiting robot in the time interval corresponding to a discrete period.

The set of neighbors of a zone $v \in V$ is denoted by $\mathcal{N}(v)$, and for any zone $v \neq o$, $p(v)$ denotes the parent of $v$ in $\mathcal{T}$. Let $z(r,t)$ denote the zone $v \in V$ occupied by robot $r$ at period $t$. Then, $z(r,t+1) \in \{z(r,t)\} \cup \mathcal{N}(z(r,t))$; i.e., robot $r$ can either remain in the same zone at period $t+1$, or advance to a neighboring zone $v' \in \mathcal{N}(v)$. Furthermore, at any period $t$, a zone $v \neq o$ cannot contain more than one robot.

On the other hand, at any period t, a group of robots can coordinate their advancement over a path of neighboring zones; i.e., for a group of robots $r_1, r_2, \ldots, r_n$ with $z(r_i,t) \in \mathcal{N}(z(r_{i-1},t))$, for $i = 2,\ldots,n$, we allow $z(r_i, t+1) = z(r_{i+1}, t)$, $i = 1, \ldots, n-1$, provided that robot $r_n$ moves itself to a free zone or to the root zone $o$ at period $t+1$. We characterize such a string of robot moves as a robot *flow* occurring at time $t$, and we shall denote it by $f(z(r_1), z(r_n); t)$. The net effect of this flow is the transfer of a robot from zone $z(r_1,t)$ to zone $z(r_n, t+1)$. Also, the traffic dynamics that were described in the previous paragraphs imply that two flows $f(v_o, v_d; t)$ and $f(v'_o, v'_d; t)$ are conflicting if the supporting paths of these two flows have a common internal node $v \neq o$.

Robots can reverse the direction of their motion within their zone. This assumption is reasonable in the context of the considered applications, and furthermore, it is necessary due to the tree structure of the underlying tunnel system.

Finally, as observed in the opening part of this section, the communication connectivity among the robots and the system controller is established by stipulating that, for every zone $v \neq o$ and every period $t$,

$$\exists\, r \in \mathcal{R} : z(r,t) = v \implies \exists\, r' \in \mathcal{R} : z(r',t) = p(v) \quad (1)$$

The above requirement implies that for every zone $v$ occupied by a robot in period $t$, all the zones in the path connecting zone $v$ to the root zone $o$ in tree $\mathcal{T}$ are also occupied by a robot in period $t$. Furthermore, the root zone $o$ is always occupied by at least one robot.

We want to determine a plan that will advance the robots $r \in \mathcal{R}$ in a way that is consistent with the above assumptions regarding the robot capabilities and the zone allocation protocol, and at the end of its execution, each

[1] Nonuniform traversal times for the system zones can be easily introduced into our model. But this feature would overload the employed notation and complicate some details in the pursued analysis, without adding anything substantial to the expository value of this discussion.

leaf zone $v \in L$ will have been visited by some robot.[2] Let $\mathcal{P}$ denote the set of feasible plans, and for every $P \in \mathcal{P}$ and $v \in L$, let $C(v;P)$ denote the first period that plan $P$ places a robot in zone $v$. We are especially interested in plans $P^*$ such that

$$P^* = \arg\min_{P\in\mathcal{P}} \max_{v\in L} C(v;P) \tag{2}$$

or

$$P^* = \arg\min_{P\in\mathcal{P}} \sum_{v\in L} C(v;P) \tag{3}$$

Each of Eqs 2 and 3 defines a combinatorial optimization – or (traffic-)scheduling – problem. The traffic-scheduling problem defined by Eq. 2 is characterized as the *Makespan-minimization* problem, or the *M-problem*, and the traffic-scheduling problem defined by Eq. 3 is characterized as the *Total Visitation Time-minimization* problem, or the *TVT-problem*. Also, in Reveliotis and Kim (2021) it is shown that these two problems are in a Pareto optimal relationship (Rardin (1998); Yu and LaValle (2013)); i.e., there are MMRS where the sets of optimal plans for these two problems have no common element. Hence, these two problems require separate treatments.

**MP formulation of the M- and TVT-problems:** Next, we consider the MP formulations for the M- and TVT-problems of Eqs 2 and 3 that were developed in Reveliotis and Kim (2021). In the subsequent discussion, $\bar{T}$ denotes an upper bound for the completion time of an optimal plan $P^*$ for each problem; one way to obtain such an upper bound is by considering the completion time of the plan $P$ that tries to reach one leaf zone at a time, while scanning the tree $\mathcal{T}$ in a depth-first sense.

The decision variables employed by the MP formulations of Reveliotis and Kim (2021) are as follows:

- State variables
  - $x_{v,t},\ v \in V,\ t \in \{0,1,\ldots,\bar{T}\}$: a nonnegative integer variable indicating the number of robots in zone $v$ at period $t$.
- Control variables
  - $u_{v,v',t},\ v \in V,\ v' \in \mathcal{N}(v),\ t \in \{1,\ldots,\bar{T}\}$: a nonnegative integer variable representing the number of robots moving from zone $v$ to neighboring zone $v'$ at period $t$.
- Auxiliary variables
  - $y_{v,t},\ v \in L,\ t \in \{1,\ldots,\bar{T}\}$: a binary variable for testing whether leaf zone $v$ has been visited by period $t$.[3]
  - $s_t,\ t \in \{1,\ldots,\bar{T}\}$: a binary variable for testing whether the entire leaf-node visitation task has been completed by period $t$.

The technological constraints employed in the MP formulations for the M- and TVT-problems in Reveliotis and Kim (2021) are as follows:

$$x_{o,0} = |\mathcal{R}| \tag{4}$$

$$\forall v \in V \setminus \{o\},\quad x_{v,0} = 0 \tag{5}$$

$$\forall v \in V,\ \forall t \in \{1,\ldots,\bar{T}\},$$
$$x_{v,t} = x_{v,t-1} + \sum_{v'\in\mathcal{N}(v)} \big(u_{v',v,t} - u_{v,v',t}\big) \tag{6}$$

$$\forall v \in V,\ \forall t \in \{1,\ldots,\bar{T}\},\quad \sum_{v'\in\mathcal{N}(v)} u_{v,v',t} \le x_{v,t-1} \tag{7}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{1,\ldots,\bar{T}\},\quad x_{v,t} \le 1 \tag{8}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{1,\ldots,\bar{T}\},\quad x_{v,t} \le x_{p(v),t} \tag{9}$$

$$\forall v \in L,\ \forall t \in \{1,\ldots,\bar{T}\},\quad y_{v,t} \le \sum_{q\in\{1,\ldots,t\}} x_{v,q} \tag{10}$$

$$\forall v \in L,\ \forall t \in \{1,\ldots,\bar{T}\},\quad s_t \le y_{v,t} \tag{11}$$

Constraints 4 and 5 define the initial distribution of the robots by means of the state variables $x_{v,0},\ v \in V$. Constraint 6 expresses the evolution of the robot distribution to the system zones at period $t$, based on the control decisions that are expressed by the variables $u_{v,v',t}$. Constraint 7 stipulates that the control decisions at period $t$ must be feasible with respect to the robot distribution over the system zones at period $t-1$. Constraint 8 enforces the buffering capacity of the zones $v \neq o$. Constraint 9 enforces the condition of Eq. 1. Constraint 10 forces the binary variable $y_{v,t}$ to zero if leaf zone $v$ has not been visited by period $t$. Finally, Constraint 11 forces the binary variable $s_t$ to zero if there is a leaf zone $v$ that has not been visited by period $t$.

The M-problem can be expressed by the following formulation:

$$\max \sum_{t\in\{1,\ldots,\bar{T}\}} s_t \tag{12}$$

s.t. Constraints 4 – 11 plus the sign restrictions for the problem variables specified during the introduction of these variables.

The TVT-problem can be expressed by the following formulation:

$$\max \sum_{v\in L} \sum_{t\in\{1,\ldots,\bar{T}\}} y_{v,t} \tag{13}$$

s.t. Constraints 4 – 10 plus the sign restrictions for the problem variables specified during the introduction of these variables.

The above two formulations are Integer Programming (IP) formulations (Wolsey (1998)). An optimal solution for each of these two formulations determines an optimal plan $P^*$ for the corresponding scheduling problem through the quantities $[u_{v,v',t} - u_{v',v,t}]^+$ for every pair $(v,v')$ of neighboring zones and period $t$.[4]

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[2] In more technical terms, a plan is a sequence of distributions, $\mathcal{D}_t, t = 0,1\ldots$, of the system robots to the various zones of the underlying tunnel system. The distribution $\mathcal{D}_{t+1}$, for period $t+1$, is obtained from the distribution $\mathcal{D}_t$ by relocating a number of robots from their zones at period $t$ to some neighboring zone, while abiding to the introduced assumptions about the maneuvering capabilities of the robots and the zone allocation protocol. This characterization is specified further through the MP formulations of the considered problems that are provided in the second part of this section.

[3] Actually, the pricing of the variables $y_{v,t}$ is part of the "informational state" that drives the underlying decision making process at period $t$. We have included these variables in the set of the auxiliary variables since their values are determined by the values of the corresponding variable sets $\{x_{v,q}, q = 1,\ldots,t\}$.

[4] We remind the reader that $[x]^+ = \max\{x, 0\}$.

Furthermore, we can relax the integrality requirements of the variables $x_{v,t}$, $y_{v,t}$ and $s_t$ to the following constraints, turning the above IP formulations into Mixed Integer Programs (MIPs):

$$\forall v \in V,\ \forall t \in \{1,\dots,\bar{T}\},\quad x_{v,t} \geq 0 \tag{14}$$

$$\forall v \in L,\ \forall t \in \{1,\dots,\bar{T}\},\quad 0 \leq y_{v,t} \leq 1 \tag{15}$$

$$\forall t \in \{1,\dots,\bar{T}\},\quad 0 \leq s_t \leq 1 \tag{16}$$

Indeed, as long as we retain the integrality requirement for the variables $u_{v,v',t}$, Constraints 4–6 will ensure the integrality of the variables $x_{v,t}$, and this fact subsequently preserves the mechanism that establishes the correct pricing of the variables $y_{v,t}$ and $s_t$ in any optimal solution of the resulting formulation.

## 3. THE NEW STRUCTURAL RESULTS FOR THE CONSIDERED TRAFFIC SCHEDULING PROBLEMS

**Preamble:** We start the presentation of the main results of this paper by introducing some further notation and terminology that are necessary for the formal statement and establishment of these results. Hence, in the subsequent discussion, we shall denote the unique path connecting any nodal pair $\{v_1, v_2\}$ of tree $\mathcal{T}$ by $\pi(v_1, v_2)$. Also, the length of path $\pi(v_1, v_2)$ is defined by the number of edges in it, and it will be denoted by $l(v_1, v_2)$. Since tree $\mathcal{T}$ is undirected, $\pi(v_1, v_2) \equiv \pi(v_2, v_1)$ and $l(v_1, v_2) = l(v_2, v_1)$. Finally, a single node can be considered as a path of zero length.

Also, for any pair of nodes $\{v_1, v_2\}$, a node $v$ of $\mathcal{T}$ that belongs on both paths $\pi(o, v_1)$ and $\pi(o, v_2)$ is a *common ancestor* of these two nodes. Let $\mathcal{CA}(v) \equiv \{v' : \exists\ v''\ s.t.\ v'$ is a common ancestor of $v$ and $v''\}$ be the set collecting all the common ancestors of node $v$. Similarly, for a set of nodes $\hat{V} \subset V$, we define $\mathcal{CA}(\hat{V}) = \bigcap_{v \in \hat{V}} \mathcal{CA}(v)$. Finally, for any node $v \in V$, we define its *closest common ancestor (CCA)* by

$$CCA(v) = \arg \max_{v' \in \mathcal{CA}(v)} l(o, v') \tag{17}$$

**An implication of the imposed zoning scheme and the induced timing of the traffic dynamics:** We start with a proposition that is an immediate implication of (i) the unit buffering capacity of the zoning scheme employed by the considered MMRS, and (ii) the traffic dynamics that determine the robot transitioning among these zones; c.f. Eqs 6–8.

*Proposition 1.* In the considered MMRS,

$$\forall v \in L,\ \forall P \in \mathcal{P},\quad C(v; P) \geq l(o, v) \tag{18}$$

□

The result of Proposition 1 can be explicitly introduced in the MIP formulations of the M- and TVT-problems presented in the previous section, by adding the constraint:

$$\forall v \in L,\ \forall t \in \{1,\dots,l(o,v)-1\},\quad x_{v,t} = 0 \tag{19}$$

Also, similar constraints can be introduced for every internal node $v \in V$ of tree $\mathcal{T}$, based on its distance $l(o, v)$ from the origin. [5]

[5] In fact, many of the current MIP solvers may generate constraints like that of Eq. 19 during a preprocessing stage of the corresponding MIP formulations that were presented in the previous section (Wolsey (1998)).

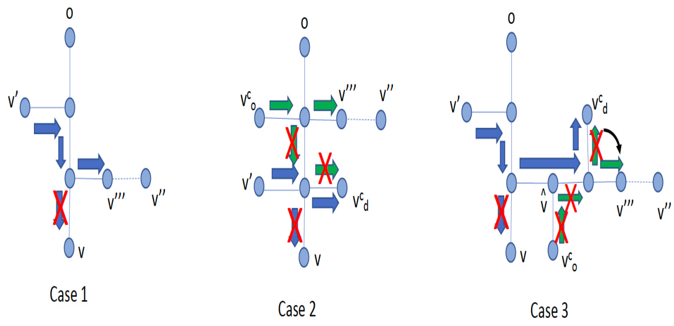


Fig. 1. A demonstration of the three cases considered in the proof of Proposition 2

**Focused plans:** The next result recognizes redundant moves in optimal plans. In order to state it formally, we need the following definition of a *focused plan*.

*Definition 1.* A plan $P \in \mathcal{P}$ for an M- or TVT-problem instance is characterized as *focused* if it satisfies the following condition: For every internal node $v$ of the corresponding tree $\mathcal{T}$, and every period $t \in \{1,\dots,\bar{T}\}$ such that $x_{v,t-1} = 0 \wedge\ x_{v,t} = 1$,

(1) $\mathcal{K} \equiv \{v' \in L : v \in \pi(o, v')\ \wedge\ C(v'; P) > t\} \neq \emptyset$;
(2) $\forall \tau \in \{t+1,\dots,\min_{v' \in \mathcal{K}} C(v'; P)\},\ x_{v,\tau} = 1$

Also, a robot move that violates the above condition will be characterized as *unfocused*. □

In plainer terms, a plan $P$ for an M- or TVT-problem instance is focused if any advancement of a robot to an internal node $v \in V \setminus L$ of tree $\mathcal{T}$ corresponding to a currently empty zone, is part of an ongoing endeavor to visit a currently unvisited leaf node $v' \in L$ located in the subtree of $\mathcal{T}$ that is rooted at node $v$. Condition (1) of Definition 1 stipulates the presence of such unvisited leaf nodes in the aforementioned subtree of $\mathcal{T}$, and Condition (2) stipulates that the considered advancement must culminate in the visit of at least one of these unvisited leaf nodes.

*Proposition 2.* For any M- or TVT-problem instance, there exists an optimal plan $P^*$ that is focused.

*Proof:* Consider an optimal plan $P^*$ that is not focused. Then, there is an internal node $v$ of the corresponding tree $\mathcal{T}$, and a period $t \in \{1,\dots,\bar{T}\}$ such that $x_{v,t-1} = 0 \wedge x_{v,t} = 1$, and either (i) $\mathcal{K} \equiv \{v' \in L : v \in \pi(o, v')\ \wedge\ C(v'; P) > t\} = \emptyset$, or (ii) $\mathcal{K} \equiv \{v' \in L : v \in \pi(o, v')\ \wedge\ C(v'; P) > t\} \neq \emptyset$ and $x_{v,\tau}$ transitions back to 0 at some period $\tau \in \{t+1,\dots,\min_{v' \in \mathcal{K}} C(v'; P) - 1\}$.

Consider the first incidence in plan $P^*$ of such an unfocused move. Then, the robot advancement to node $v$ at period $t$ is the result of a robot flow originating either at the root node $o$ or at some other node $v'$ of tree $\mathcal{T}$ located on the path $\pi(o, \tilde{v})$ of a leaf node $\tilde{v} \in L$ with $C(\tilde{v}; P^*) < t$. For further reference, let us denote this flow by $f(v', v; t)$. Also, let $v''$ denote the node that receives a robot at period $\tau$ as a result of the flow $f(v, v''; \tau)$ that will relocate the robot placed at node $v$ at that period. In addition, let $v'''$ denote the furthest ancestor of node $v''$ in tree $\mathcal{T}$ that is not occupied by a robot at period $t$. Next we show that, at period $t$, flow $f(v', v; t)$ can be redirected away from node $v$ in a way that maintains the traffic state indented by plan $P^*$ for period $\tau$.

We proceed to establish this result by distinguishing three cases that are depicted schematically in Figure 1.

In Case 1, flow $f(v', v; t)$ is redirected to node $v'''$ (i.e., it is substituted with the flow $f(v', v'''; t)$), and this redirection does not create any conflict with any other flows specified by plan $P^*$ for period $t$. Then, clearly the considered redirection is feasible at period $t$. In addition, any further flows advancing robots in the subtree emanating from node $v'''$ during the time interval $\{t+1, \ldots, \tau-1\}$ remain feasible, and by period $\tau$, the number of robots moved into this subtree will equal to the number of robots moved into the subtree by the original plan $P^*$. Similarly, this redirection does not block any flows that might place robots in the subtree emanating from node $v$ over the time interval $\{t+1, \ldots, \tau-1\}$. Furthermore, since plan $P^*$ removes the robot placed at node $v$ at period $\tau$, any robot placed on the subtree emanating from node $v$ will have been withdrawn from this subtree by period $\tau$. Hence, the state of this subtree at period $\tau$ will be consistent with the state intended by plan $P^*$. Finally, it is also clear, that at period $\tau$, the distribution of robots in every other part of tree $\mathcal{T}$ will have not been altered. Hence, our claim is proved for this case, and plan $P^*$ can continue its execution at period $\tau+1$.

In Case 2, the redirection of flow $f(v', v; t)$ results in a conflict with another flow, $f(v_o^c, v_d^c; t)$, over a path of tree $\mathcal{T}$, with the two flows moving in opposite directions on this path. This problem can be easily addressed by substituting the original conflicting flows with the two new flows $f(v', v_d^c; t)$ and $f(v_o^c, v'''; t)$, as shown in Figure 1. Then, the rest of the argument for this case proceeds as in Case 1.

In Case 3, the redirection of flow $f(v', v; t)$ to node $v'''$ results in a conflict with some other flow $f(v_o^c, v_d^c; t)$ over a path of $\mathcal{T}$, and both flows have the same direction over this path. In this case, flow $f(v', v; t)$ is redirected to node $v_d^c$, while flow $f(v_o^c, v_d^c; t)$ is cancelled. Hence, node $v_d^c$ receives the robot intended for it at period $t$, but node $v'''$ does not. This deficit is addressed by trying to advance the robot that remained at node $v_o^c$ due to the cancelation of flow $f(v_o^c, v_d^c; t)$ at some subsequent period $q \in \{t+1, \ldots, \tau\}$. As long as this attempted advancement conflicts with some other flow $f(v_1, v_2; q)$ scheduled by plan $P^*$, it is deferred to the next period. On the other hand, the feasibility of this advancement by period $\tau$ can be argued as follows: The corresponding flow $f(v_o^c, v''; \tau)$ can be decomposed to the two flows $f(v_o^c, \hat{v}; \tau)$ and $f(\hat{v}, v''; \tau)$ where $\hat{v}$ is the node where the path supporting flow $f(v_o^c, v''; \tau)$ meets the path supporting flow $f(v, v''; \tau)$ (c.f. Figure 1). The feasibility of flow $f(\hat{v}, v''; \tau)$ is guaranteed by the presence of flow $f(v, v''; \tau)$ in plan $P^*$. However, flow $f(v_o^c, \hat{v}; \tau)$ may be in conflict with some other flow $f(v_1, v_2; \tau)$. In this case, first we notice that, due to the presence of flow $f(v, v''; \tau)$ in plan $P^*$, all these conflicting flows evolve on paths that do not include node $\hat{v}$. Furthermore, flow conflicts of the type considered in Case 2 above can be addressed as discussed in that case. The remaining flow conflicts are of the type discussed in the current case (i..e., Case 3). Among these conflicts, consider the flow $f(v_1, v_2; \tau)$ involving the subpath $\tilde{\pi}$ of the path $\pi(v_o^c, \hat{v})$ that is closest to node $\hat{v}$. The cancelation of the original flow $f(v_o^c, v_d^c; t)$ at period $t$ implies that it is possible to advance, in that period, the robot located at node $v_o^c$ towards the subtree that is rooted on path $\tilde{\pi}$ and contains node $v_2$. Hence, at period $\tau$, flow $f(v_o^c, \hat{v}; \tau)$ can be substituted by flow $f(v_1, \hat{v}; t)$, and this case is resolved, as well.

If the suggested redirection of flow $f(v', v; t)$ towards node $v''$ generates a number of conflicts, we can process them one at a time, advancing on the corresponding path from node $v'$ to node $v'''$. As long as the encountered conflicts are of the type described in Case 2, they can be addressed as discussed above. On the other hand, the first encounter of a conflicting flow of the type described in Case 3 will redirect the robot at node $v'$ to the destination node of this conflicting flow, the conflicting flow will be cancelled, and the robot that is released from this cancelation will be advanced towards node $v''$ in the subsequent periods, according to the corresponding discussion for Case 3.

Finally, a focused plan $\tilde{P}^*$ can be obtained from the provided plan $P^*$ by removing the unfocused moves from plan $P^*$ one move at a time, starting with the first occurrence of such a move and working as discussed above. □

From a more conceptual standpoint, the result of Proposition 2 implies that, while working towards the satisfaction of the visitation requirements of the various leaf nodes of $\mathcal{T}$, there is no need to use the zones of tree $\mathcal{T}$ as "temporary buffers" for the traveling robots.

Furthermore, for a more thorough understanding of the notion of a focused plan, we emphasize that Condition (2) of Definition 1 requires that the considered node $v$

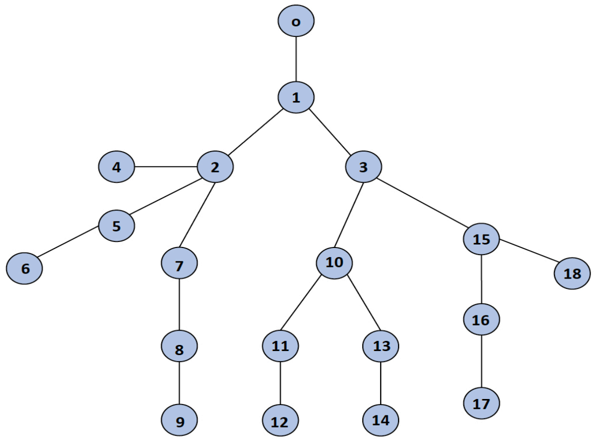


Fig. 2. The tree $\mathcal{T}$ for a TVT-problem instance where robots are withdrawn from a subtree before all the leaf nodes of this subtree have been visited, in order to be used in the remaining part of $\mathcal{T}$.

Table 1. An optimal plan $P^*$ for the TVT-problem instance that is defined by the tree $\mathcal{T}$ of Figure 2 and $|\mathcal{R}| = 19$.

| $t$ | Occupied Zones | $t$ | Occupied Zones |
|---|---|---|---|
| 1 | 1 | 7 | 1, 2, 3, 5, 10, 11, 12 |
| 2 | 1, 3 | 8 | 1, 2, 3, 5, 6, 10, 11, 13 |
| 3 | 1, 3, 15 | 9 | 1, 2, 3, 5, 7, 10, 13, 14, 15 |
| 4 | 1, 3, 15, 18 | 10 | 1, 2, 3, 7, 8, 10, 13, 15, 16 |
| 5 | 1, 2, 3, 10, 15 | 11 | 1, 2, 3, 7, 8, 9, 10, 15, 16, 17 |
| 6 | 1, 2, 3, 4, 10, 11 | | |

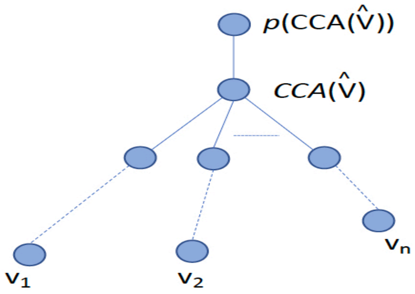


Fig. 3. A maximal set of leaf nodes having the same CCA.

in this definition must remain occupied by a robot until the first visit of some descendant leaf node of $v$ in the set $\mathcal{K}$, but not until all the leaf nodes in $\mathcal{K}$ have been visited. As an example to the potential sub-optimality of this stronger requirement, Table 1 provides an optimal plan $P^*$ for the TVT-problem instance that is defined by the tree $\mathcal{T}$ depicted in Figure 2 and $|\mathcal{R}| = 19$.[6] In plan $P^*$, node $v_{15}$ is initially occupied by a robot during the periods 3–5 for the visitation of its descendant leaf node $v_{18}$, but subsequently, the entire subtree of $\mathcal{T}$ that is rooted at node $v_{15}$ is divested of any robots during the periods 6–8. All robots are used to visit some leaf nodes in the remaining part of tree $\mathcal{T}$ during these periods, or they are idling to the root zone $o$ due to the bottlenecks that are defined by the unit capacity of the other zones. Robots are returned to node $v_{15}$ and its emanating subtree in periods 9–11, for the eventual visit of the remaining leaf node $v_{17}$ of this subtree. More importantly, it can be verified that there is no optimal plan for the considered TVT-problem instance that will keep node $v_{15}$ continuously occupied by a robot until all its descendant leaf nodes have been visited.

All the plans $P$ considered in the rest of this document are assumed to be focused. The generation of a focused plan $P^*$ by the MIP formulations of the previous section can be ensured by adding the term $-c \cdot \sum_v \sum_{v'} \sum_t u_{v,v',t}$ to their objective function. The cost $c$ associated with the variables $u_{v,v',t}$ penalizes unnecessary flows, and it must be chosen sufficiently small so that it does not compromise the generated solution with respect to the original objective of the formulation.

**The impact of common ancestors:** Next we focus on maximal subsets of leaf nodes of tree $\mathcal{T}$ that have the same CCA, and this common ancestor is a node other than the root node of $\mathcal{T}$. Such a maximal subset $\hat{V} = \{v_1, v_2, \ldots, v_n\}$ is depicted in Figure 3, where, with a slight abuse of notation, we denote the common ancestor of all nodes in $\hat{V}$ by $CCA(\hat{V})$. Proposition 2, together with the unit buffering capacity of node $CCA(\hat{V})$, imply the existence of an optimal plan $P^*$ that addresses the visitation requests that are posed by the nodes in $\hat{V}$ one at time; i.e., plan $P^*$ will order the nodes in $\hat{V}$ according to some total order $\langle v_{[1]}, v_{[2]}, \ldots, v_{[n]} \rangle$ and it will deploy the robots that are directed by the plan to the subtree of Figure 3, first on path $\pi(CCA(\hat{V}), v_{[1]})$), next on path $\pi(CCA(\hat{V}), v_{[2]})$), and so on, until all the leaf nodes of this subtree have been visited. This operation implies that for the considered plan $P^*$, and for all nodal pairs $\{v, v'\}$ in $\hat{V}$,

$$|C(v; P^*) - C(v'; P^*)| \quad \geq \\ \min\left\{ l(CCA(\hat{V}), v), l(CCA(\hat{V}), v') \right\} \qquad (20)$$

Furthermore, the next proposition establishes that, in the case of the TVT-problem, the aforementioned ordering of the nodes of the considered set $\hat{V}$ that is employed by any optimal plan $P^*$, must be in increasing distance from node $CCA(\hat{V})$.

*Proposition 3.* For the TVT-problem, every optimal plan $P^*$ visits the leaf nodes $v_1, v_2, \ldots, v_n$ of the subtree structures depicted in Figure 3,[7] in increasing distance from their common ancestor $CCA(\hat{V})$.

*Proof:* It is clear that *any* optimal plan $P^*$ must address the visitation requirements of the nodes in $\hat{V}$ one at a time; i.e., there must exist a total ordering $\langle v_{[1]}, v_{[2]}, \ldots, v_{[n]} \rangle$ of the set $\hat{V}$, and the plan $P^*$ will first deploy robots on the path $\pi(CCA(\hat{V}), v_{[1]})$) until node $v_{[1]}$ has been visited, next on path $\pi(CCA(\hat{V}), v_{[2]})$) until node $v_{[2]}$ has been visited, and so on, until all the leaf nodes in $\hat{V}$ have been visited.

Among this set of plans, consider a plan $P$ that visits the leaf nodes $v_1, v_2, \ldots, v_n$ of the subtree depicted in Figure 3 according to a sequence that is not consistent with the nodal ordering specified in Proposition 3. Also, let $v_{[[i]}, v_{[i+1]}$ be the first nodal pair with $l(CCA(\hat{V}), v_{[i]}) > l(CCA(\hat{V}), v_{[i+1]})$ in the visitation sequence that is employed by plan $P$. Next we specify a plan $\tilde{P}$ that swaps the position of nodes $v_{[[i]}, v_{[i+1]}$ in the visitation sequence that is employed by plan $P$ and has a better performance than $P$.

In order to define the plan $\tilde{P}$, first notice that, according to Proposition 2, the visitation of nodes $v_{[i]}$ and $v_{[i+1]}$ by $P$ involves a total outflow from node $CCA(\hat{V})$ towards the paths $\pi(CCA(\hat{V}), v_{[i]})$ and $\pi(CCA(\hat{V}), v_{[i+1]})$ of $l(CCA(\hat{V}), v_{[i]}) + l(CCA(\hat{V}), v_{[i+1]})$ robots. The first $l(CCA(\hat{V}), v_{[i]})$ of these robots will be directed to node $v_{[i]}$, and the remaining $l(CCA(\hat{V}), v_{[i+1]})$ will be directed to node $v_{[i+1]}$. Furthermore, this robot conveyance will start at some period $\tau > C(v_{[i-1]}; P)$ and will finish at period $C(v_{[i+1]}; P)$, when leaf node $v_{[i+1]}$ is visited. For a complete understanding of the involved dynamics, we also notice that after node $v_{[i]}$ has been reached, and while robots are conveyed via node $CCA(\hat{V})$ to node $v_{[i+1]}$, some of the robots located on the path $\pi(CCA(\hat{V}), v_{[i]})$ might be directed via node $CCA(\hat{V})$ towards the rest of tree $\mathcal{T}$ (i.e., towards its parent node $p(CCA(\hat{V}))$).

[6] Table 1 reports the distribution of the system robots to zones $v \neq o$, at each period $t$ of the plan makespan. Robots that are not used for the occupation of the reported zones, at any period $t$, are at node $o$.

[7] The result of Proposition 3 extends straightforwardly to leaf node sets $\hat{V} \subseteq L$ with $CCA(\hat{V}) = o$. We leave the relevant details to the reader.

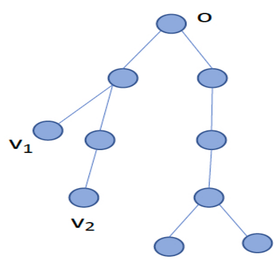


Fig. 4. For the M-problem instance defined by the depicted tree $\mathcal{T}$ and $|\mathcal{R}| = 7$, the optimal plan $P^*$ has a makespan of 5 periods, and it visits nodes $v_1$ and $v_2$ in the sequence $\langle v_2, v_1\rangle$. Furthermore, any plan visiting these two nodes in the sequence $\langle v_1, v_2\rangle$ has a makespan of at least 6 periods.

Plan $\tilde{P}$ first will redirect to the path $\pi(CCA(\hat{V}), v_{[i+1]})$ the first $l(CCA(\hat{V}), v_{[i+1]})$ robots originally destined from node $CCA(\hat{V})$ to the path $\pi(CCA(\hat{V}), v_{[i]})$. The next $l(CCA(\hat{V}), v_{[i]}) - l(CCA(\hat{V}), v_{[i+1]})$ originally destined from node $CCA(\hat{V})$ to the path $\pi(CCA(\hat{V}), v_{[i]})$ will still be sent to this path. This first phase of plan $\tilde{P}$ will finish exactly at period $C(v_{[i]}; P)$, with node $v_{[i+1]}$ having already been visited. In the remaining $C(v_{[i+1]}; P) - C(v_{[i]}; P)$ periods, plan $\tilde{P}$ will execute the corresponding moves of plan $P$ that concern the withdrawal and/or the placement of robots at the respective paths $\pi(CCA(\hat{V}), v_{[i]})$ and $\pi(CCA(\hat{V}), v_{[i+1]})$, via node $CCA(\hat{V})$, but now the role of these two paths as robot providers or recipients will be reversed. This reversal is consistent with, both, the robot availability and the timing of the corresponding moves defined by plan $P$. Also, plan $\tilde{P}$ will eventually visit node $v_{[i]}$ at period $C(v_{[i+1]}; P)$.

From the above description of plan $\tilde{P}$, it is clear that: (i) $C(v_{[i]}; \tilde{P}) = C(v_{[i+1]}; P)$ (c.f. the closing part of the previous paragraph). (ii) $C(v_{[i+1]}; \tilde{P}) < C(v_{[i]}; P)$ (since $l(CCA(\hat{V}), v_{[i]}) > l(CCA(\hat{V}), v_{[i+1]})$ and the conveyance pattern of the necessary robots for the first phase of plan $\tilde{P}$, via node $CCA(\hat{V})$, is the same as in plan $P$). (iii) $C(v; \tilde{P}) = C(v; P)\ \forall v \in L \setminus \{v_{[i]}, v_{[i+1]}\}$ (since plan $\tilde{P}$ preserves the robot inflow and outflow patterns, via node $CCA(\hat{V})$, towards the rest of tree $\mathcal{T}$). Hence, $\sum_{v\in L} C(v; \tilde{P}) < \sum_{v\in L} C(v; P)$, and Proposition 3 has been proved. □

Let $\langle v_{[1]}, v_{[2]}, \ldots, v_{[n]}\rangle$ be a total ordering of the nodes of the subtree depicted in Figure 3 in increasing length of the corresponding paths $\pi(CCA(\hat{V}), v_i)$, $i = 1, \ldots, n$; nodes with paths of equal length are ordered arbitrarily. Then, in the case of the TVT-problem, Proposition 3 enables the strengthening of Equation 20 as follows:

$$\forall i = 1, \ldots, n-1,$$
$$C(v_{[i+1]}; P^*) - C(v_{[i]}; P^*) \ \geq\ l\big(CCA(\hat{V}), v_{[i+1]}\big) \quad (21)$$

On the other hand, Figure 4 provides a counter-example to a possible extension of Proposition 3 to the M-problem. However, we notice that, from a computational standpoint, even in the case of the M-problem, it is beneficial to (i) impose an arbitrary total ordering on every subset of leaf nodes in the subtree of Figure 3 that have equal distance from node $CCA(\hat{V})$, and (ii) introduce in the corresponding MIP formulation a set of constraints similar to that of Eq. 21 for each such nodal subset and its specified ordering. This practice will break the symmetry with respect to these nodes in the generated solutions by the MIP solver, and can expedite significantly the overall computation.

## 4. STRENGTHENING THE MIP FORMULATIONS

The quantity $C(v; P^*) - C(v'; P^*)$ appearing in the left-hand-side of Equation 21 can be expressed in the MIP formulations of Section 2 by the sum $\sum_{t=1}^{\bar{T}}(y_{v',t} - y_{v,t})$; i.e., we can introduce Eq. 21 in the MIP formulation of Section 2 for the TVT-problem, by adding the constraint:

$$\forall i = 1, \ldots, n-1,$$
$$\sum_{t=1}^{\bar{T}}(y_{v_{[i]},t} - y_{v_{[i+1]},t}) \ \geq\ l\big(CCA(\hat{V}), v_{[i+1]}\big) \quad (22)$$

In this section, we report some computational results that highlight the ability of Constraint 22 to expedite the solution of the presented MIP formulation for the TVT-problem. These results are tabulated in Table 2.

More specifically, in the experiment that underlies the results of Table 2, we chose randomly 10 instances of the TVT-problem, and formulated and solved the corresponding MIP of Section 2 without and with the addition of Constraint 22. Columns '$|V|$', '$|\mathcal{R}|$' and '$|L|$' in Table 2 report, respectively, the number of nodes of tree $\mathcal{T}$, the number of robots, and the number of leaf nodes for the corresponding problem instance. Collectively, the values reported in these three columns characterize the "size" of the corresponding problem instance, and define a degree of difficulty for it. The formulated MIPs for these problem instances were run for up to 2 hours (or 7,200 secs), and Column 'Obj. Value' reports the best objective value obtained during this computation. Column 'MIP gap (%)' reports the optimality gap assessed by the employed MIP solver for the best reported objective value; in particular, this percentage is computed through the formula: (Best Upper Bound - Best Obj. Value ) / Best Obj. Value. Finally, the column 'Comp. Time (sec)' reports the computational time involved. All the involved formulations were solved by CPLEX in Python, on a laptop with i7-8850H 2.6GHz CPU, 16 GB RAM, and running Mac OS.

It is clear from the values reported in Table 2 that, in all 10 cases, the addition of Constraint 22 to the MIP formulation led either to a faster acquisition of an optimal solution (cases 1–5), or to solutions of improved quality compared to the solutions obtained during the same computational time by the original MIP of Section 2. This quality improvement is obvious for cases 6, 7, 8 and 9 when juxtaposing the corresponding objective values. But even in case 9, where the objective values are equal, we can see that the inclusion of Constraint 22 enabled a more accurate assessment of the corresponding optimality gaps.

Table 2. The computational results that are discussed in Section 4.

| Instance | $\|V\|$ | $\|\mathcal{R}\|$ | $\|L\|$ | Constr. 22 | Obj. Value | MIP gap (%) | Comp. Time (sec) |
|---|---|---|---|---|---|---|---|
| 1 | 25 | 17 | 12 | NO | 122 | 0.00 | 35.46 |
| | | | | YES | 122 | 0.00 | 15.11 |
| 2 | 25 | 18 | 13 | NO | 135 | 0.00 | 43.00 |
| | | | | YES | 135 | 0.00 | 19.00 |
| 3 | 50 | 28 | 19 | NO | 225 | 0.00 | 1318.61 |
| | | | | YES | 225 | 0.00 | 304.54 |
| 4 | 50 | 26 | 29 | NO | 215 | 0.00 | 12.32 |
| | | | | YES | 215 | 0.00 | 9.70 |
| 5 | 75 | 36 | 39 | NO | 299 | 0.00 | 540.87 |
| | | | | YES | 299 | 0.00 | 480.47 |
| 6 | 75 | 38 | 38 | NO | 697 | 19.86 | 7200.00 |
| | | | | YES | 687 | 17.23 | 7200.00 |
| 7 | 100 | 51 | 44 | NO | 1442 | 46.98 | 7200.00 |
| | | | | YES | 1289 | 40.69 | 7200.00 |
| 8 | 100 | 47 | 49 | NO | 1108 | 47.25 | 7200.00 |
| | | | | YES | 758 | 25.15 | 7200.00 |
| 9 | 125 | 59 | 64 | NO | 1869 | 44.53 | 7200.00 |
| | | | | YES | 1869 | 40.60 | 7200.00 |
| 10 | 125 | 58 | 72 | NO | 1616 | 29.93 | 7200.00 |
| | | | | YES | 1615 | 29.89 | 7200.00 |

Closing the discussion of this section, we notice that a similar experiment has highlighted the value of symmetry-breaking constraints for the M-problem that were discussed in the closing part of Section 3. We might also consider the inclusion of the constraints of Eq. 20 in the MIP formulation for the M-problem. But the linearization of this constraint requires the introduction of a new binary variable $\zeta_{v,v'}$ with $\zeta_{v,v'}$ indicating whether $C(v;P^*) > C(v';P^*)$ or not. Hence, in this case, the inclusion of the information that is provided by Eq. 20 to the corresponding MIP formulations might not be able to expedite the solution of these formulations.

## 5. CONCLUSIONS

In this work, we have provided some structural results for the M- and TVT-problems, and their optimal plans, that were recently introduced in Reveliotis and Kim (2021). We have also demonstrated how the derived results can expedite the solution of the MIP formulations of the considered problems that were developed in Reveliotis and Kim (2021), and / or enhance the quality of any partial results that are derived through these formulations.

On the other hand, Reveliotis and Kim (2021) has also established the NP-hardness of the M- and the TVT-problems, and this fact is further manifested in the results of Table 2; as it can be seen in this table, the solution of the employed MIP formulations for larger problem instances is a challenging task, even after the inclusion of the information that is provided by the new results developed in this paper.

Hence, in our future work, we shall also seek the development of pertinent heuristic methods for the M- and the TVT-problem, adapting some related ideas and methods that are provided by combinatorial optimization theory (Papadimitriou and Steiglitz (1998); Aarts and Lenstra (2003)). The results developed in this work are expected to have a significant role in the shaping of the solution spaces and in the further structuring of the search processes to be developed in these endeavors.

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