Discover Companion Pell Numbers

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Table of Contents

Motivation

- $\left(1+\sqrt{2}\right)+\left(1-\sqrt{2}\right)=2$ is an integer
- $\left(1+\sqrt{2}\right)^{2}+\left(1-\sqrt{2}\right)^{2}=6$ is an integer
- Question 1: Is $\left(1+\sqrt{2}\right)^{n}+\left(1-\sqrt{2}\right)^{n}$ an integer? YES.
- Question 2: Can we define the sequence using a simpler formula or a recursive relation? YES.

Answer to Question 1

Fact

$c_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ is an integer using Binomial Theorem. It was show in a paper [2] in the journal "Parabola" in 2020.

Proof.

$$\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$$
$$= 2 + 2\binom{n}{2}2 + 2\binom{n}{4}4 + \cdots + \binom{n}{n}\left(1+(-1)^n\right)\left(\sqrt{2}\right)^n$$

where

$$\left(1+(-1)^n\right)\left(\sqrt{2}\right)^n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ 2\left(\sqrt{2}\right)^n, & \text{when } n \text{ is even} \end{cases}$$

Time travel

Submission time

Re: Submission to Parabola

Thomas Britz <britz@unsw.edu.au>
Sat 3/6/2021 10:37 AM

Reply time

From: Hu, Shannon <shannon.hu@mga.edu>
Sent: Sunday, 7 March 2021 01:45
To: Thomas Britz
Subject: Re: Submission to Parabola

Recursive formula

Theorem

The sequence $c_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ *can be defined by the following recursive relation*

$$c_n = 2c_{n-1} + c_{n-2}$$

with the initial conditions $c_1 = 2$ *and* $c_2 = 6$.*[3]*

- *Sequence:* $\{c_n\} = \{2, 6, 14, 34, 82, \cdots\}$

Proof

Proof.

Let $u = \sqrt{2} + 1$ and $v = \sqrt{2} - 1$. Then

$$c_n = \begin{cases} u^n - v^n, & n \text{ is odd} \\ u^n + v^n, & n \text{ is even} \end{cases}$$

Notice that

- $uv = 1$
- $u - v = 2.$

□

Case 1: n is odd

Proof.

Case 1: n is odd and $n = 2k + 1$

$$\begin{aligned} c_n &= u^{2k+1} - v^{2k+1} \\ &= (u - v)\left(u^{2k} + u^{2k-1}v + \ldots + u^k v^k + \ldots + v^{2k}\right) \\ &= 2\left(u^{2k} + u^{2k-2} + \ldots + u^2 + 1 + v^2 + \ldots + v^{2k}\right) \end{aligned}$$

Then

$$c_{n-2} = 2\left(u^{2k-2} + \ldots + u^2 + 1 + v^2 + \ldots + v^{2k-2}\right)$$

and

$$c_n - c_{n-2} = 2\left(u^{2k} + v^{2k}\right) = 2c_{n-1}$$

□

Case 2: n is even

Proof.

Case 2: n is even

$$
\begin{aligned}
c_n - c_{n-2} &= (u^n + v^n) - \left(u^{n-2} + v^{n-2}\right) \\
&= \left(u^n - u^{n-2}\right) + \left(v^n - v^{n-2}\right) \\
&= u^{n-2}\left(2 + 2\sqrt{2}\right) + v^{n-2}\left(2 - 2\sqrt{2}\right) \\
&= \dots \\
&= 2c_{n-2} + c_{n-1} + c_{n-3} \\
&= (2c_{n-2} + c_{n-3}) + c_{n-1} \\
&= c_{n-1} + c_{n-1} \\
&= 2c_{n-1}
\end{aligned}
$$

Companion Pell numbers

- A best rational approximation to $x \in \mathbb{R}$ is a rational number $\frac{n}{d}$, $d > 0$, that is closer to x than any approximation with a smaller or equal denominator.
- The best rational approximations to $\sqrt{2}$ are $\frac{1}{2}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$

Definition

Companion Pell numbers are twice of the numerators of the sequence

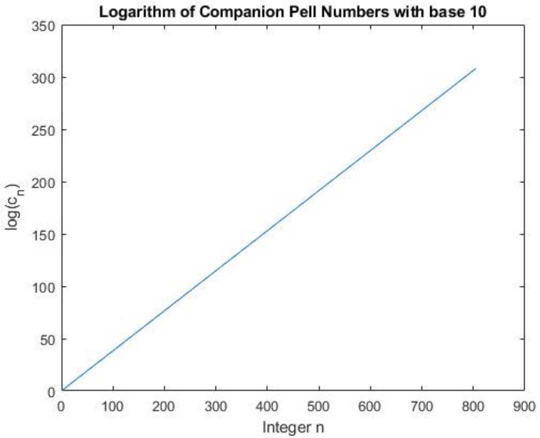
$$2, 6, 14, 34, 82, \ldots$$

Companion Pell numbers

- Companion Pell numbers: $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = a_1 = 2$. (Resource: https://oeis.org/A002203)
- $a_n = c_n$
- Increasing fast: $2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, \ldots$

Approximation

- Draw $(n, \log c_n)$ with Matlab



Approximation

Theorem

The n^{th} companion Pell number is approaching to $\left(1+\sqrt{2}\right)^n$ and $\lim_{n\to\infty}\frac{\log c_n}{n} = \log\left(1+\sqrt{2}\right)$.

Proof: Rewrite $1-\sqrt{2}$, then

$$\log c_n = \log\left(\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right)$$

$$= \log\left(\left(1+\sqrt{2}\right)^n + \frac{(-1)^n}{\left(1+\sqrt{2}\right)^n}\right)$$

$$\lim_{n\to\infty}\frac{\log c_n}{n} = \lim_{n\to\infty}\frac{\log\left(1+\sqrt{2}\right)^n}{n} = \log\left(1+\sqrt{2}\right). \square$$

Summary

- $c_n = 2c_{n-1} + c_{n-2}$ with $c_1 = 2$ and $c_2 = 6$
- $\{c_n\}$ are Companion Pell numbers
- $\lim_{n\to\infty} \frac{\log c_n}{n} = \log\left(1+\sqrt{2}\right)$ and $\lim_{n\to\infty} c_n = \left(1+\sqrt{2}\right)^n$

References

Companion Pell numbers.
The On-Line Encyclopedia of Integer Sequences.
Link: http://oeis.org/A002203

Randel Heyman.
Strange irrational power.
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Xiaoyan Hu.
Discover Companion Pell Numbers
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