Conjectures on the Product of Matrix Exponentials

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Abstract

The matrix exponential is a matrix function on square matrices analogous to the ordinary natural exponential function $f(x) = e^x$. It was originally used to solve systems of ordinary linear differential equations. In the theory of Lie groups, the matrix exponential gives the connection between a matrix Lie algebra and the corresponding Lie group. We give various properties of matrix exponentials and use them to investigate conjectures on the product of matrix exponentials.

Notations

- $\mathbb{C}_{n\times n}$ = the vector space of $n \times n$ matrices over $\mathbb{C}$. Sometimes we use the notation $\mathfrak{gl}(n,\mathbb{C})$ which is equipped with the Lie bracket $[\cdot,\cdot]$: $[A,B] = AB - BA$, $A, B \in \mathfrak{gl}(n,\mathbb{C}) = \mathbb{C}_{n\times n}$.
- $\mathrm{GL}(n,\mathbb{C})$ = the general linear group of $n \times n$ invertible matrices.
- For a complex matrix A, $A^* = (\overline{A})^T = \overline{A^T}$. (conjugate transpose)
- $\mathrm{O}(n,\mathbb{C})$ = the complex orthogonal group.$(O^{-1} = O^T)$
- $\mathbb{H}_n$ = the set of $n \times n$ Hermitian matrices. $(H = H^*)$
  If $H \in \mathbb{H}_n$, then all eigenvalues of H are real numbers.
- $\mathbb{P}_n$ = the space of $n \times n$ positive definite matrices.
  $\mathbb{P}_n = \{P \in \mathbb{H}_n | x^* P x > 0 \, \forall x \in \mathbb{C}^n\}$, where all eigenvalues of $P \in \mathbb{P}_n$ are real and positive.
- $\mathrm{U}(n)$ = the group of $n \times n$ unitary matrices. $(U^{-1} = U^*)$
- $\mathrm{O}(n)$ = the group of $n \times n$ (real) orthogonal matrices. $(O^{-1} = O^T)$

Definition

Let X be an $n \times n$ real or complex matrix. The exponential of X, denoted by e^X or $exp(X)$, is the $n \times n$ matrix given by the power (Maclaurin) series:

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

where X^0 is defined to be the $n \times n$ identity matrix I. The above series always converges to the value of the function since e^X is analytic, so the exponential of X is well defined. If X is a 1×1 matrix, then the matrix exponential reduces to evaluating $f(x) = e^x$ where x is a single number.

Properties

Every $g \in \mathrm{GL}(n, \mathbb{C})$ sufficiently close to the identity can be expressed as $g = e^X$ for a small $X \in \mathbb{C}_{n \times n}$.

Properties

- $e^O = I$, where O is the zero matrix and I is the identity matrix.
- $\det e^X = e^{\mathrm{tr}\, X}$.
- $e^{(X^\top)} = (e^X)^\top$, $e^{(X^*)} = (e^X)^*$
- If $Y \in \mathrm{GL}(n, \mathbb{C})$, then $e^{YXY^{-1}} = Ye^XY^{-1}$.
- If $XY = YX$, then $e^X e^Y = e^{X+Y}$.

However, if $XY \neq YX$, then $e^X e^Y \neq e^{X+Y}$ in general, when $n \geq 2$.

Linear Differential Equations

Matrix exponentials were first developed to solve systems of ordinary linear differential equations. The solution of

$$\frac{d}{dt}y(t) = Ay(t),\ y(0) = y_0,$$

where A is a constant matrix, is given by

$$y(t) = e^{At}y_0.$$

Jacobi's Formula

In matrix calculus, Jacobi's formula expresses the derivative of the determinant of a matrix A in terms of the adjoint of A and the derivative of A. If A is a differentiable map from the real numbers to the $n \times n$ matrices, then

$$\frac{d}{dt} \det A(t) = \operatorname{tr} \left(\operatorname{adj}(A(t)) \frac{dA(t)}{dt} \right) = (\det A(t)) \cdot \operatorname{tr} \left(A(t)^{-1} \cdot \frac{dA(t)}{dt} \right)$$

where $tr(A)$ is the trace of the matrix A (the sum of the diagonal entries.) If dA stands for the differential of A, the general formula is

$$d \det(A) = \operatorname{tr} (\operatorname{adj}(A) dA).$$

It is named after the German mathematician Carl Gustav Jacob Jacobi.

The Determinant of the Matrix Exponential

By Jacobi's formula, for any complex square matrix B, the following trace identity holds:

$$\det(e^B) = e^{\mathrm{tr}\,(B)}.$$

To see this, let $A(t) = e^{tB}$ in Jacobi's formula. We get:

$$\frac{d}{dt} \det\, e^{tB} = \mathrm{tr}\,(B) \det\, e^{tB}.$$

The result follows as a solution to this ordinary differential equation. In addition to providing a computational tool, this formula demonstrates that a matrix exponential is always an invertible matrix. This follows from the fact that the right hand side of the above equation is always non-zero, and so $\det(e^B) \neq 0$, which implies that e^B must be invertible. The inverse of e^B is e^{-B}.

The Matrix Logarithm

Given a matrix $B \in \mathrm{GL}(n, \mathbb{C})$, another matrix A is said to be the matrix logarithm of B if $e^A = B$. Because the exponential function is not one-to-one for complex numbers (e.g. $e^{\pi i} = e^{3\pi i} = -1$), numbers can have multiple logarithms, and as a consequence of this, some matrices have more than one logarithm. If $||B - I|| < 1$, then a logarithm of B may be computed by means of the following power series:

$$\log(B) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(B - I)^k}{k}$$

and

$$e^A = e^{\log(B)} = B.$$

The Logarithm of B when B is diagonalizable

If $B = UDU^{-1}$, where the columns of U are the eigenvectors of B corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, counting multiplicities, then

$$\log B = U(\log D)U^{-1},$$

where

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

$$\log D = \operatorname{diag}(\log \lambda_1, \log \lambda_2, \ldots, \log \lambda_n),$$

and

$$e^A = e^{\log B} = B.$$

Matrix Norms

There are many norms that can be defined on matrices, but the one used most often is the matrix 2-norm or the spectral norm. The spectral norm of a matrix A is the largest singular value of A, which is the square root of the largest eigenvalue of the matrix A^*A, where A^* denotes the conjugate transpose of A. We have:

$$||A||_2 = \sqrt{\lambda_{max}(A^*A)} = \sigma_{max}(A),$$

where $\sigma_{max}(A)$ denotes the largest singular value of A.

Matrix Norms

We also have:

$$||A^*A||_2 = ||AA^*||_2 = ||A||_2^2$$

since

$$||A^*A||_2 = \sigma_{max}(A^*A) = \sigma_{max}(A)^2 = ||A||_2^2.$$

In addition, we have an equivalent definition for $||A||_2$:

$$||A||_2 = \sup\{x^T Ay : x, y \in K^n with ||x||_2 = ||y||_2 = 1\}.$$

Real Symmetric Matrices

The matrix exponential of a real symmetric matrix is positive definite. To see this, let S be an $n \times n$ real symmetric matrix and let $x \in \mathbb{R}^n$ be a column vector. Using the elementary properties of matrix exponentials and of symmetric matrices, we have:

$$x^T e^S x = x^T e^{S/2} e^{S/2} x = x^T (e^{S/2})^T e^{S/2} x = (e^{S/2} x)^T e^{S/2} x = ||e^{S/2} x||_2^2,$$

which is non-negative for all x and positive for all $x \neq O$, since $e^{S/2}$ is invertible.

The Exponential of Sums

If matrices X and Y commute (meaning that $XY = YX$, then $e^{X+Y} = e^X e^Y$. This is not necessarily true when X and Y do not commute. When $XY \neq YX$, e^{X+Y} can be computed by the Lie product formula:

$$e^{X+Y} = \lim_{n \mapsto \infty} (e^{\frac{1}{n}X} e^{\frac{1}{n}Y})^n.$$

Computing the matrix exponential

Finding reliable and accurate methods to compute the matrix exponential is difficult, and this is still a topic of considerable current research. It is very straightforward, however, for diagonalizable matrices. Indeed, if

$$A = UDU^{-1},$$

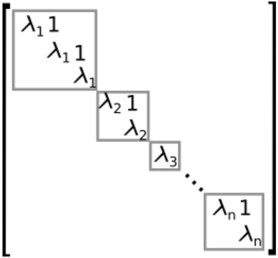
where D is diagonal, then

$$e^A = Ue^DU^{-1},$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and $e^D = \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$.

Jordan Canonical Form

In linear algebra, a Jordan normal form, also known as a Jordan canonical form, is an upper triangular matrix of a particular form called a Jordan matrix representing a linear operator on a finite-dimensional vector space with respect to some basis. Such a matrix has each non-zero off-diagonal entry equal to 1, immediately above the main diagonal (on the superdiagonal), and with identical diagonal entries to the left and below them.



An example of a matrix in Jordan normal form. The grey blocks are called Jordan blocks. Note that the λ_i in different blocks can be equal.

By the Jordan canonical form, $\exp : \mathbb{C}_{n\times n} \to \mathrm{GL}(n, \mathbb{C})$, or

$$\exp : \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$

is surjective. However, it is not injective. For example, $n = 2$, $e^0 = I_2$ and

$$J = \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix}, \quad e^J = Ue^D U^{-1} = UI_2U^{-1} = I_2.$$

$$where\, U = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, D = \mathrm{diag}\,(2i\pi, -2i\pi), U^{-1} = \begin{bmatrix} (-1/2)i & 1/2 \\ (1/2)i & 1/2 \end{bmatrix}$$

So the images of 0 and J are both I_2.

$\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$

The real exponential map

$$\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

is not surjective. (and not injective - same example as complex case)

Example: There is no 2×2 matrix X such that

$$e^X = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Reason: If X has eigenvalues α and β, then the eigenvalues of e^X are $e^\alpha, e^\beta = -1, -2$. So α and β are non-real complex numbers. But X is a real matrix, so α and β must be complex conjugate. This implies that $|e^\alpha| = |e^\beta|$, a contradiction!

Baker-Campbell-Hausdorff formula

For any $X, Y \in \mathfrak{gl}(n, \mathbb{C})$,

$$e^X e^Y = e^Z,$$

where

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots ,$$

i.e., $\log(e^X e^Y)$ can be expressed as a series in repeated commutators of X and Y.

In general

$$e^X e^Y \neq e^{X+Y}, \quad X, Y \in \mathfrak{gl}(n, \mathbb{C}).$$

Is it true that

$$e^X e^Y = e^{UXU^{-1}+VYV^{-1}}?$$

The answer is NO.

- The map $\exp : \mathbb{H}_n \to \mathbb{P}_n$ is a diffeomorphism, in particular, bijection.
- $\mathbb{P}_n^2 := \{AB : A, B \in \mathbb{P}_n\}$ is the set of all diagonalizable matrices with positive eigenvalues. So $\mathbb{P}_n \subsetneq \mathbb{P}_n^2$
- So we can pick $X, Y \in \mathbb{H}_n$ such that

$$e^X e^Y \in \mathbb{P}_n^2 \setminus \mathbb{P}_n,$$

However,

$$e^{UXU^{-1}+VYV^{-1}} \in \mathbb{P}_n.$$

Thompson's exponential formula for $\mathrm{U}(n)$

The exponential map

$$\exp : i\mathbb{H}_n \to \mathrm{U}(n)$$

sends skew Hermitian matrices to unitary matrices.

Theorem (R.C. Thompson, 1986)

Let $A, B \in \mathbb{H}_n$. Then

$$e^{iA}e^{iB} = e^{i(UAU^{-1}+VBV^{-1})}$$

for some $U, V \in \mathrm{U}(n)$.

Remark: Thompson mentioned the conjectured formula in the 1980 Auburn Matrix Theory Conference.

Thompson's proof makes use of the result announced by B.V. Lidskiǐ for the eigenvalues of a sum of Hermitian matrices (without detailed proof), a theorem of A.A. Nudel'man and P.A. Švarcman on the eigenvalues of a product of unitary matrices and perturbation theory.

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- R.C. Thompson, On the eigenvalues of a product of unitary matrices I, Linear and Multilinear Algebra **2** (1974), 13–25.
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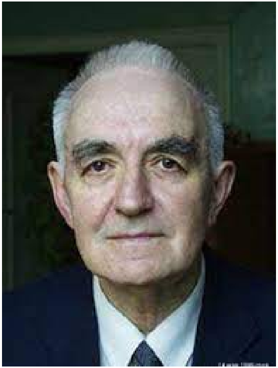
Victor B. Lidskiǐ (1924-2008)

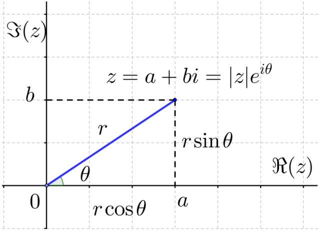
"Victor Borisovich Lidskiǐ (1924–2008)", p.1–5 in Operator Theory and Its Applications: In Memory of V. B. Lidskii (1924–2008), American Mathematical Society Translations - Series 2. 231, American Mathematical Society, 2010, edited by M. Levitin and D.Vassiliev.
https://arxiv.org/pdf/1008.2669.pdf



https://en.wikipedia.org/wiki/Victor_Lidskii

Polar decomposition of $\mathbb{C} \setminus \{0\}$

Polar decomposition for $\mathbb{C}$: $z = e^{i\theta}|z| = |z|e^{i\theta}$.



Uniqueness doesn't exist for $\mathbb{C}$ at the origin,

$$\mathbb{C} \ni 0 = 0 \cdot e^{i\theta} = y \cdot 0, \quad y > 0.$$

Uniqueness exists for

$$\mathbb{C} \setminus \{0\} = \mathrm{GL}(1, \mathbb{C}).$$

Polar decomposition $\mathrm{GL}(n,\mathbb{C})$

$$\mathrm{GL}(n,\mathbb{C}) = \mathbb{P}_n \mathrm{U}(n) \quad \text{or} \quad \mathrm{U}(n)\mathbb{P}_n.$$

That is, each nonsingular complex $n \times n$ matrix A can be written as

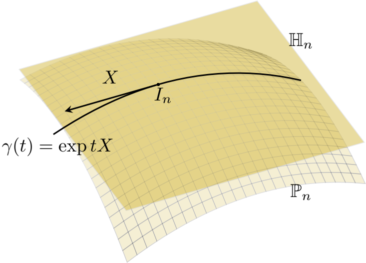
$$A = PU, \quad P \in \mathbb{P}_n,\ U \in \mathrm{U}(n).$$

Moreover, the map

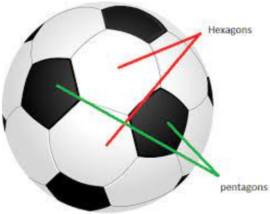
$$\begin{array}{rcl} \mathbb{H}(n) \times \mathrm{U}(n) & \to & \mathrm{GL}(n,\mathbb{C}) \\ (X, U) & \mapsto & e^X U \end{array}$$

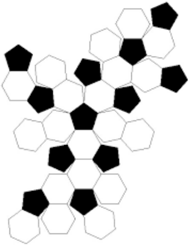
is a diffeomorphism.
Thompson's 1986 result handles the product of two elements in $\mathrm{U}(n)$.

$$e^{iA} e^{iB} = e^{i(UAU^{-1} + VBV^{-1})}, \qquad A, B \in \mathbb{H}_n$$



$$\mathbb{P}_n = \exp \mathbb{H}_n$$





Symmetric space $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$

The natural projection

$$\begin{aligned} \pi : \mathrm{GL}(n,\mathbb{C}) &\to \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n) \cong \mathbb{P}_n \\ g &\mapsto g \cdot \mathrm{U}(n) \end{aligned}$$

enables the identification $\mathbb{H}_n \cong T_o(\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n))$, the tangent space to $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ at the origin $o := I_n \cdot \mathrm{U}(n)$ via the derivative $d\pi$ of π at o.

Thus any $\mathrm{Ad}\,\mathrm{U}(n)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}_n$ induces a unique G-invariant Riemannian metric on $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$, i.e., a Riemannian metric invariant under the natural action of $\mathrm{GL}(n,\mathbb{C})$ on $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ given by

$$(g, x \cdot \mathrm{U}(n)) \mapsto (gx) \cdot \mathrm{U}(n).$$

Action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{P}_n$

Let $* : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ be the complex conjugate transpose. It is a diffeomorphism. Note $k^* = k^{-1}$ for $k \in \mathrm{U}(n)$ and $p^* = p$ for $p \in P$. The map

$$\mathrm{GL}(n, \mathbb{C}) \to \mathbb{P}_n, \quad g \mapsto gg^*$$

is surjective. Now

$$\begin{array}{rrcl} \psi : & \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n) & \to & \mathbb{P}_n \\ & g \cdot \mathrm{U}(n) & \mapsto & gg^* \end{array}$$

is a bijection. Via ψ, $\mathbb{P}_n$ may be identified with $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$. Note that for $p \in \mathbb{P}_n$, $\psi^{-1}(p) = p^{1/2} \cdot \mathrm{U}(n)$, ($pp^* = p^2$) and $\mathrm{GL}(n, \mathbb{C})$ acts on $\mathbb{P}_n$ via congruence:

$$\mathrm{GL}(n, \mathbb{C}) \times \mathbb{P}_n \ni (g, p) \mapsto gpg^* \in \mathbb{P}_n.$$

Thus $\mathbb{P}_n \subset \mathrm{GL}(n, \mathbb{C})$ acts on itself via congruence.

So-Thompson's 1st exponential conjecture

Fact: The spectra of $e^{X/2}e^{Y}e^{X/2}$ and $e^{X}e^{Y}$ are identical.

$$\mathbb{P}_n \ni e^{X/2}e^{Y}e^{X/2} = e^{Z}, \text{ for some } Z \in \mathbb{P}_n$$

Question: What can we say about $Z \in \mathbb{H}_n$?

So-Thompson 1st exponential conjecture (1991)

For Hermitian matrices $X, Y \in \mathbb{H}_n$, there exist unitary matrices $U, V \in \mathrm{U}(n)$ such that

$$e^{X/2}e^{Y}e^{X/2} = e^{UXU^{-1}+VYV^{-1}}.$$

• W. So, and R.C. Thompson, Products of exponentials of Hermitian and complex symmetric matrices, Linear Multilinear Algebra **29** (1991), 225–233.

Some remarks

- It is obvious that the conjecture is true if $XY = YX$, $[X, Y] := XY - YX = 0$. It is because

$$e^{X/2}e^{Y}e^{X/2} = e^{X}e^{Y},$$

  that is $U = V = I_n$.

- On the other hand, if $[X, Y] \neq 0$, then

$$e^{X/2}e^{Y}e^{X/2} \neq e^{W(X+Y)W^*},$$

  for any unitary W. In fact, if $\mathrm{tr}\,(e^{X/2}e^{Y}e^{X/2}) = \mathrm{tr}\,e^{X+Y}$, then $[X, Y] = 0$.

Wasin So proved the 1st conjecture

Thompson passed away in 1995. Later Wasin So (2004) proved the So-Thompson 1st conjecture.

Theorem (So, 2004)

For $X, Y \in \mathbb{H}_n$, there exist unitary matrices U and V such that

$$e^{X/2}e^{Y}e^{X/2} = e^{UXU^{-1}+VYV^{-1}}.$$

So's proof makes use of a result of A.A. Klyachko.

• W. So, The high road to an exponential formula, Linear Algebra Appl. **379** (2004), 69–75.

• A.A. Klyachko, Random walks on symmetric spaces and inequalities for matrix spectra, Linear Algebra Appl. **319** (2000), 37–59.

Klyachko's theorem

Theorem (Klyachko, 2000)

Let $\alpha, \beta, \gamma \in \mathbb{R}^n_\downarrow$ be three n-tuples of non-increasingly ordered real numbers. Then the following statements are equivalent:

(i) There exist $X, Y \in \mathbb{H}_n$ with $\lambda(X) = \alpha$, $\lambda(Y) = \beta$, and $\lambda(X + Y) = \gamma$.

(ii) There exists $A, B \in \mathbb{P}_n$ with $\lambda(A) = e^\alpha$, $\lambda(B) = e^\beta$, and $\lambda(AB) = e^\gamma$.

Lie Theory, developed by Sophus Lie

Sophus Lie (December 17, 1842 – February 18, 1899) a Norwegian mathematician.

Wilhelm Killing and Élie Cartan

Wilhelm Killing (May 10, 1847 – February 11, 1923) a German mathematician.

Élie Cartan (April 9, 1869 – May 6, 1951) a French mathematician.

Lie Notation

- G = noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$
- Θ: $G \to G$, Cartan involution ($\Theta^{-1} = \Theta \, or \, \Theta^2 = I$) of G.
- K = the fixed point set of Θ, $K = \{g \in G | \Theta(g) = g\}$.
- $\theta = d\Theta : \mathfrak{g} \to \mathfrak{g}$ = a Cartan involution ($\theta^{-1} = \theta \, or \, \theta^2 = I$) of G.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, where $\mathfrak{k}$ is the eigenspace of θ corresponding to the eigenvalue 1 (and also the Lie algebra of K) and $\mathfrak{p}$ is the eigenspace of θ corresponding to the eigenvalue -1.
- The Killing form B on $\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, and the bilinear form $B_\theta(X,Y) := -B(X, \theta Y)$, $X, Y \in \mathfrak{g}$. is an inner product on $\mathfrak{g}$.
- Cartan decomposition: $G = KP$ or $G = PK$.

Lie Theory

The mathematician Sophus Lie initiated lines of study involving integration of differential equations, transformation groups, and contact of spheres that have come to be called Lie theory. For instance, the latter subject is Lie sphere geometry. We address his approach to transformation groups, which is an area of mathematics worked out by Wilhelm Killing and Élie Cartan.
The foundation of Lie theory is the exponential map relating Lie algebras to Lie groups which is called the Lie group–Lie algebra correspondence. The subject is part of differential geometry since Lie groups are differentiable manifolds. Lie groups evolve out of the identity (1) and the tangent vectors to one-parameter subgroups generate the Lie algebra. The structure of a Lie group is implicit in its algebra, and the structure of the Lie algebra is expressed by root systems and root data.

Lie Group

In mathematics, a Lie group is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space, whereas groups define the abstract, generic concept of multiplication and the taking of inverses (division). Combining these two ideas, one obtains a continuous group where points can be multiplied together, and their inverse can be taken. If, in addition, the multiplication and taking of inverses are defined to be smooth (differentiable), one obtains a Lie group.

Lie Algebra

An algebra over a field (often simply called an algebra) is a vector space equipped with a bilinear product. It is an algebraic structure consisting of a set together with operations of multiplication, addition and scalar multiplication by elements of a field and satisfying the axioms implied by a vector space and a bilinear product. The multiplication operation in an algebra may or may not be associative. A Lie algebra is an algebra equipped with an operation called the Lie bracket, an alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}\,(x, y) \mapsto [x, y]$, that satisfies the Jacobi identity: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$. For matrices, the Lie bracket is defined by $[A, B] = AB - BA$. One can easily verify that this definition satisfies the Jacobi identity. The vector space together with this operation is a non-associative algebra, meaning that the Lie bracket is not necessarily associative.

Simple and Semi-Simple

A simple Lie algebra is a Lie algebra $\mathfrak{g}$ that is non-abelian and contains no nonzero proper ideals. An ideal in $\mathfrak{g}$ is a subset I of $\mathfrak{g}$ that absorbs multiplication by elements in $\mathfrak{g}$; that is, $ix \in I$ and $xi \in I$ for all $i \in I$ and $x \in \mathfrak{g}$. It is also required that $(I, +)$ be a subgroup of $(\mathfrak{g}, +)$. The notation is $I \triangleleft \mathfrak{g}$. The classification of real simple Lie algebras is one of major achievements of Wilhelm Killing and Élie Cartan. A direct sum of simple Lie algebras is called a semisimple Lie algebra. A simple Lie group is a connected Lie group whose Lie algebra is simple. A connected Lie group is called semisimple if its Lie algebra is a semisimple Lie algebra, i.e. a direct sum of simple Lie algebras.

Adjoint Representation

The adjoint representation (or adjoint action) of a Lie group G is a way of representing the elements of the group as linear transformations of the group's Lie algebra, considered as a vector space. For example, if G is $GL(n, \mathbb{R})$, the Lie group of real $n \times n$ invertible matrices, then the adjoint representation is the group homomorphism that sends an invertible $n \times n$ matrix g to an endomorphism of the vector space of all linear transformations of $\mathbb{R}^n$ defined by: $x \mapsto gxg^{-1}$. That is, $\mathrm{Ad}_g(x) = gxg^{-1}$.

The Killing Form

Consider a Lie algebra $\mathfrak{g}$ over a field K. Every element x of $\mathfrak{g}$ defines the adjoint endomorphism $ad(x)$ (also written as ad_x) of $\mathfrak{g}$ with the help of the Lie bracket, as $\mathrm{ad}(x)(y) = [x, y]$. Now, supposing $\mathfrak{g}$ is of finite dimension, the trace of the composition of two such endomorphisms defines a symmetric bilinear form $B(x, y) = \mathrm{trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$, with values in K, called the Killing form on $\mathfrak{g}$.

Cartan Involutions and Decompositions

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $B(\cdot,\cdot)$ be its Killing form. An involution on $\mathfrak{g}$ is a Lie algebra automorphism θ of $\mathfrak{g}$ whose square is equal to the identity. Such an involution is called a Cartan involution on $\mathfrak{g}$ if $B_\theta(X,Y) := -B(X,\theta Y)$ is a positive definite bilinear form. If θ is an involution on a Lie algebra $\mathfrak{g}$, then $\theta^2 = I$, and the linear map θ has the two eigenvalues ± 1. If $\mathfrak{k}$ and $\mathfrak{p}$ denote the eigenspaces corresponding to $+1$ and -1, respectively, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The pair $(\mathfrak{k}, \mathfrak{p})$ is called the Cartan pair of $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated with a Cartan involution is called a Cartan decomposition of $\mathfrak{g}$. There is a Lie group automorphism Θ with differential θ at the identity that satisfies $\Theta^2 = I$. If G, K and P are the Lie groups associated with $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively, the Cartan decomposition of G is given by $G = KP$ or $G = PK$.

Symmetric space G/K

Let

$$\pi : G \to G/K, \quad g \mapsto gK,$$

be the natural projection onto the quotient space G/K. Then $\mathfrak{p}$ may be identified with the tangent space $T_o(G/K)$ of G/K at the origin $o = eK$ via $d\pi$.

Thus any $\operatorname{Ad} K$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ induces a unique G-invariant Riemannian metric on G/K, i.e., a Riemannian metric invariant under the natural action of G on G/K given by

$$(g, xK) \mapsto gxK,$$

that is, by translation.

Action of G on P

Let $* : G \to G$ be the diffeomorphism defined by

$$g^* = \Theta(g^{-1}).$$

Then $k^* = k^{-1}$ for $k \in K$ and $p^* = p$ for $p \in P$. The map

$$G \to P, \quad g \mapsto gg^*$$

is onto. Because for any $g \in G$, it maps gK to a single point gg^*, it follows that the map

$$\psi : \ G/K \ \to P, \quad gK \mapsto gg^*,$$

is a bijection. It is in fact a diffeomorphism by the Cartan decomposition $G = PK$.

Via ψ, P may be identified with G/K, and so may be regarded as a symmetric space of noncompact type. Note that for $p \in P$, $\psi^{-1}(p) = p^{1/2}K$, $(pp^* = p^2)$ and G acts on P by

$$(g, p) \mapsto gpg^*.$$

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Extension of So-Thompson's result

Cartan decomposition: Each $g \in G$ can be uniquely written as $g = pk$ with $p = p(g) \in P$ and $k = k(g) \in K$.

The map $\mathfrak{p} \times K \to G$, $(X, k) \mapsto e^X k$, is a diffeomorphism.

Theorem (Gan, Liu, Tam, 2021)

For all $X, Y \in \mathfrak{p}$, there exist $u, v \in K$ such that

$$e^{X/2} e^Y e^{X/2} = e^{\mathrm{Ad}\,(u)X + \mathrm{Ad}\,(v)Y}.$$

• L. Gan, X. Liu, T.-Y. Tam, On two geometric means and sum of adjoint orbits, Linear Algebra Appl., **631** (2021), 156–173.

• A. Alekseev, E. Meinrenken, C. Woodward, Linearization of Poisson actions and singular values of matrix products, Ann. Inst. Fourier (Grenoble) **51** (2001), 1691–1717.

Example: $\mathrm{GL}(n, \mathbb{R})$

Product of exponentials of real symmetric matrices

For $n \times n$ real symmetric matrices X, Y there exist orthogonal matrices $U, V \in \mathrm{SO}(n)$ such that

$$e^{X/2} e^{Y} e^{X/2} = e^{UXU^\top + VYV^\top}.$$

Proof: Apply Gan-Liu-Tam's theorem to $\mathrm{SL}(n, \mathbb{R})$:

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{k} + \mathfrak{p}$$

- $\mathfrak{k}$ is the algebra of $n \times n$ real skew symmetric matrices,
- $\mathfrak{p}$ is the space of $n \times n$ real symmetric matrices of zero trace,
- $K = \mathrm{SO}(n)$.

What happens if tr X and tr Y are not zero?

Given two real symmetric matrices X, Y, write

$$X = \left(X - \frac{\operatorname{tr} X}{n} I_n\right) + \frac{\operatorname{tr} X}{n} I_n = \hat{X} + \alpha I_n,$$

$$Y = \left(Y - \frac{\operatorname{tr} Y}{n} I_n\right) + \frac{\operatorname{tr} Y}{n} I_n = \hat{Y} + \beta I_n.$$

So $\operatorname{tr} \hat{X} = \operatorname{tr} \hat{Y} = 0$. Now

$$\begin{aligned}
e^{X/2} e^{Y} e^{X/2} &= e^{\frac{\hat{X}}{2} + \frac{\alpha}{2} I_n} e^{\hat{Y} + \beta I_n} e^{\frac{\hat{X}}{2} + \frac{\alpha}{2} I_n} \\
&= e^{\frac{\hat{X}}{2}} e^{\hat{Y}} e^{\frac{\hat{X}}{2}} \cdot e^{(\alpha+\beta) I_n} \\
&= e^{U \hat{X} U^\top + V \hat{Y} V^\top} \cdot e^{(\alpha+\beta) I_n} \\
&= e^{U(\hat{X} + \alpha I_n) U^\top + V(\hat{Y} + \beta I_n) V^\top} \\
&= e^{U X U^\top + V Y V^\top},
\end{aligned}$$

for some $U, V \in \mathrm{SO}(n)$.

$\mathrm{O}(n, \mathbb{C})$

The complex orthogonal group is

$$\mathrm{O}(n, \mathbb{C}) = \{Q \in \mathrm{GL}(n, \mathbb{C}) : Q^\top Q = I_n\}$$

has two connected components. The identity component is

$$\mathrm{SO}(n, \mathbb{C}) = \{Q \in \mathrm{SL}(n, \mathbb{C}) : Q^\top Q = I_n\}.$$

Note:

$$\mathrm{SO}(n) \subset \mathrm{SO}(n, \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C}).$$

Polar decomposition of $\mathrm{O}(n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$

Each complex orthogonal matrix $Q \in \mathrm{O}(n,\mathbb{C})$ can be decomposed in the form

$$Q = Re^{iK}, \quad R \in \mathrm{O}(n),\ K \in \mathbb{R}_{n\times n},\ K^\top = -K.$$

If $Q \in \mathrm{SO}(n,\mathbb{C})$, then we can choose

$$R \in \mathrm{SO}(n),\ K \in \mathbb{R}_{n\times n},\ K^\top = -K.$$

So $\det R = 1$ and $\det e^{iK} = 1$, i.e., $R, e^{iK} \in \mathrm{SO}(n,\mathbb{C})$.

Product of exponentials associated $O(n,\mathbb{C})$

Product of exponentials of real skew symmetric matrices

For $n \times n$ real skew symmetric matrices X, Y there exist special orthogonal matrices $U, V \in SO(n)$ such that

$$e^{iX/2}e^{iY}e^{iX/2} = e^{i(UXU^\top + VYV^\top)}.$$

Proof: Apply Gan-Liu-Tam's theorem on $SO(n,\mathbb{C})$:

$$\mathfrak{so}(n,\mathbb{C}) = \mathfrak{k} + \mathfrak{p},$$

- $\mathfrak{k}$ is the algebra of $n \times n$ real skew symmetric matrices,
- $\mathfrak{p} = i\mathfrak{k}$,
- $K = SO(n)$.

2nd conjecture of So-Thompson (still open)

So-Thompson's 2nd conjecture (1991)

Let S, T be $n \times n$ complex symmetric matrices in a neighborhood of zero. Then there exist complex orthogonal matrices $U, V \in \mathrm{O}(n, \mathbb{C})$ such that

$$e^{S/2} e^{T} e^{S/2} = e^{USU^\top + VTV^\top}.$$

The conjecture is about local behavior.

2×2 case

So and Thompson gave

- a proof of the 2×2 case.
- a 2×2 example to show that the statement is false if the symmetric matrices S, T are too far from 0:

$$S = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix}.$$

• W. So, and R.C. Thompson, Products of exponentials of Hermitian and complex symmetric matrices, Linear Multilinear Algebra **29** (1991), 225–233.

A conjecture

Tam's conjecture (2021)

Let S, T be $n \times n$ complex symmetric matrices. Then there exist unitary matrices $U, V \in \mathrm{U}(n)$ such that

$$e^{S/2} e^{T} e^{S/2} = e^{USU^\top + VTV^\top}.$$

The conjecture is about global behavior.

Tool

Theorem (Alekseev, Meinrenken, Woodward, 2001)

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathfrak{k}^*$ be given co-adjoint orbits, and $\mathcal{D}_j = E(\mathcal{O}_j) \subset K^*$ the corresponding dressing orbits, where $E : \mathfrak{k}^* \to K^*$ is some diffeomorphism and K-equivariant map with respect to the coadjoint action on $\mathfrak{k}^*$ and the left dressing action on K^*. The following two moduli spaces are homeomorphic

- $\mathcal{M}_{\mathcal{O}} = \{(\xi_1, \ldots, \xi_n) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_n : \xi_1 + \cdots + \xi_n = 0\}/K.$
- $\mathcal{M}_{\mathcal{D}} = \{(g_1, \ldots, g_n) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_n : g_1 \cdots g_n = e\}/K.$

• A. Alekseev, E. Meinrenken, C. Woodward, Linearization of Poisson actions and singular values of matrix products, Ann. Inst. Fourier (Grenoble) **51** (2001), 1691–1717.

• S. Evens and J.-H. Lu, Thompson's conjecture for real semisimple Lie groups in "The Breadth of Symplectic and Poisson Geometry", 121–137, Progr. Math., 232, Birkhäuser, Boston, MA, 2005.

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