Mathematical recreation and research with magic squares and magic cubes

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ABSTRACT

This talk will give an introduction to magic squares and magic cubes including some history and highlight some methods for constructing them. We will also show that it is a topic that contains exciting problems for both fun and research at both the undergraduate and more advanced levels.

Introduction

The popular conception of mathematics is captured in the following definition of the subject:

Miss Susan: *What is algebra exactly; is it those three cornered things?*

Phoebe: *It is x minus y equals z plus y and things like that. And all the time you are saying they are equal, you feel in your heart, why should they be?*

—*J. M. Barrie, Quality Street.*

One possible way to improve the general perception of the subject is the popularization of recreational mathematics.

What is Recreational Mathematics?

Laymen are not usually interested in the technical aspects of mathematics, but rather in those topics in the study of which it matters little whether one is a professional mathematician or not. These topics are usually recreational in nature and can convert people into mathematicians without them being aware of the fact. They also perform an important task of divulgation, and the opening of the mathematical universe to the general public. However, the role that such topics should play in the instructional curriculum remains an open problem.

Many classical recreational Mathematics topics can be found in [1, 2, 7, 8, 10, 18], the 18th century text by Falkener [6], the writings of Martin Gardner [12, 13, 14, 15] – the most influential recreational mathematician of our generation, the classic text [16], or my book Uko [28], to mention but a few.

Magic squares

According to an ancient Chinese legend, when the great Emperor Yu (2200 BC) was standing by the yellow river, a divine tortoise appeared and on its back the following array of numbers – called the *lo-shu*– appeared.

4	9	2
3	5	7
8	1	6

This is one of the very first examples of a *magic square*. If you add the numbers in any row, column or diagonal, you will get the same sum.

Magic squares, together with magic cubes and magic tesseracts (their four dimensional counterparts) have for many centuries been – and continue to be – sources of mathematical recreations and challenges for large numbers of enthusiasts consisting mostly of 'non mathematicians'.

The magic constant

A magic square of order p is a $p \times p$ square array with non-repeated entries from the set $\{1, 2, \ldots, p^2\}$ such that all rows, columns and diagonals have the same sum. The sum of the elements in the entire array is

$$1 + 2 + \cdots + p^2 = p^2(p^2 + 1)/2.$$

Since these numbers are divided into p rows each of which has the same sum, that sum (the magic constant) must be $p(p^2 + 1)/2$.

The columns and diagonals will also have the same sum.
A magic square is often considered as identical to the other seven magic squares which can be obtained from it by performing rotations and/or reflections. For simplicity, we will not make this identification in this paper, so we will regard two magic squares as identical only if they are identical in the matrix sense.

From China, the magic square found its way to India and to Japan and later to Europe.

The following magic square of order 4 – which dates from ancient times – is from India:

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

The following is a famous 1514 engraving titled ‘The melancholia’, due to Albrecht Dürer.

Can you spot where the magic square below is hidden in this engraving?

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Albrecht Dürer's magic square

Observe that the two middle cells of the last row correspond to the year (1514) of the engraving.

In Villa Albani (Rome) there is an elaborate architectural decoration consisting of a magic square with 81 cells, dating from 1766.

Benjamin Franklin's magic square

Another famous magic square is the one below, is one of those discovered by the great American ttatesman and scientist Benjamin Franklin:

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

(Rhetorical) Question: Why would Benjamin Franklin use his spare time – of which he had little! – to create elaborate magic squares?

Elaborate Magic Squares . . . Bordered Magic Squares

2	23	25	7	8
4	16	9	14	22
21	11	13	15	5
20	12	17	10	6
18	3	1	19	24

1	34	33	32	9	2
29	11	18	20	25	8
30	22	23	13	16	7
6	17	12	26	19	31
10	24	21	15	14	27
35	3	4	5	28	36

16	33	18	31	20	29	28
39	7	42	9	40	27	11
12	46	2	47	26	4	38
37	5	49	25	1	45	13
14	44	24	3	48	6	36
35	23	8	41	10	43	15
22	17	32	19	30	21	34

26	45	36	27	40	23	44	19
34	14	55	50	50	54	9	31
33	17	7	62	1	60	48	32
30	47	2	59	8	61	18	35
28	49	64	5	58	3	16	37
41	12	57	4	63	6	53	24
22	56	10	15	52	11	51	43
46	20	29	38	25	42	21	39

Composite Magic Squares

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

8	1	6	107	100	105	62	55	60	125	118	123
3	5	7	102	104	106	57	59	61	120	122	124
4	9	2	103	108	101	58	63	56	121	126	119
71	64	69	116	109	114	17	10	15	98	91	96
66	68	70	111	113	115	12	14	16	93	95	97
67	72	65	112	117	110	13	18	11	94	99	92
89	82	87	26	19	24	143	136	141	44	37	42
84	86	88	21	23	25	138	140	142	39	41	43
85	90	83	22	27	20	139	144	137	40	45	38
134	127	132	53	46	51	80	73	78	35	28	33
129	131	133	48	50	52	75	77	79	30	32	34
130	135	128	49	54	47	76	81	74	31	36	29

The De La Loubère/Siamese method for making magic squares

This method – learned by De La Loubère from the locals while he was envoy of Louis XIV to Siam (Thailand) in the seventeenth century – is as follows:

[1] Draw a square and divide it into a square array of an odd number of cells. Place '1' in the middle cell of the top row and let '0' occupy the top-right corner outside the square.

[2] Proceed diagonally upwards, filling the cells with successive integers.

[3] When this takes you out of the square, shift across the square from top to bottom, or from left to right, as the case may be.

[4] If this leads you to an occupied space, write the next number immediately beneath the last cell filled.

A De La Loubère's method example

	9	2	0
8	1	6	8
3	5	7	3
4	9	2	

The symmetry method for Magic squares of doubly even order

This method generates magic squares whose orders are multiples of four. To illustrate this method, make a 4-by-4 square array of cells and highlight the 4-by-4 blocks of cells along their diagonals as shown in the figure below.

*			*
	*	*	
	*	*	
*			*

In this array, each highlighted cell is symmetrical to another highlighted cell, the complete list of symmetries being given in the figure below.

a			b
	c	d	
	d	c	
b			a

Now, proceed downwards in your figure covering each row of cells (from left to right) with successive integers. Whenever this takes you into a highlighted cell, simply write the next number in the cell symmetrical to it. This process leads to the4-by-4 magic square

16*	2	3	13*
5	11*	10*	8
9	7*	6*	12
4*	14	15	1* .

Formulas for the Siamese method and the Symmetry method

In the paper [29], I and Dr. Sinclair showed that the De La Loubère/Siamese method for making a magic square of odd order p can be condensed into the formula:

$$m_{ij} = 1 + \{(2j + i - 2) \text{ Mod } p\} + p\{(j + i + (p - 3)/2) \text{ Mod } p\}$$

where the Mod p operator indicates the remainder when p divides a given number and $M = (m_{ij})$ is the odd-order magic square matrix.

More information about this formula can be found in the Wikipedia Article [33]: `https://nl.wikipedia.org/wiki/Siamese_methode` and a more detailed history of magic squares can be found in the Wikipedia Article [34]: `https://en.wikipedia.org/wiki/Magic_square`.

The census of magic squares

It is known that there are only 8 magic squares of order three, all of which are obtainable from reflections and/or rotations of the basic magic square below.

4	9	2
3	5	7
8	1	6

It is also known that there are 7040 magic squares of order four ([11, Bernard Freńicle de Bessy, 1693] – done with manual hand calculations!!).

In 1973 Richard Schroeppel used a computer program to obtain a census figure of 2,202,441,792 magic squares of order five.

The census of magic squares of orders six and above are still open problems.

The census of uniform step magic squares

A magic square (m_{ij}) is of the uniform step if it can be written in the form:

$$m_{ij} = 1 + [(a_1 i + b_1 j + c_1) \bmod n] + n[(a_2 i + b_2 j + c_2) \bmod n]$$

Some values for the number $\kappa(p)$ of uniform step magic squares of order p:

p	$\kappa(p)$
3	8
5	1,472
7	25,272
9	3,528
11	713,000
13	2,265,408
15	11,776
21	202,176
25	21,252,800
45	5,193,216
49	2,913,193,080

.

The reason why the $\kappa(p)$ values smaller when p is a multiple of 3 is given in the following advanced combinatorics research result.

Theorem (Uko [27])

Let $p = \prod_{i=1}^{N} q_i^{r_i}$ *be the prime factorization of the odd number* p. *Then there exist* $\kappa(n) = \prod_{i=1}^{N} \kappa(q_i^{r_i})$ *uniform step magic squares of order* p, *where* $\kappa(q_i^{r_i}) = [\tau(q_i^{r_i})]^2 - \lambda(q_i^{r_i})$,
$\lambda(q_i^{r_i}) = (q_i^{r_i} - q_i^{r_i-1})^2[2(q_i^{2r_i-1}+1)^2/(q_i+1)^2 + q_i^{3r_i-1}(q_i^{r_i} - 3q_i^{r_i-1})]$
and $\tau(q_i^{r_i}) = (q_i^{r_i} - q_i^{r_i-1})(q_i^{2r_i+1} - 2q_i^{2r_i} - q_i^{2r_i-1} + 2)/(q_i + 1)$ *for* $i = 1, \ldots, N$.

Euler's method – Latin squares

An array of order p is called a **LATIN SQUARE** if each entry (taken from the set $\mathbb{Z}_p = \{0, \ldots, p-1\}$) occurs once in each row and once in each column.

A Latin square example

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

A Latin square in which all diagonal sums coincide is called an **EULER SQUARE**.
The example above does not satisfy this extra condition which was (implicitly) used by Euler to make magic squares.

Euler squares and Orthogonality

Two Latin square arrays s $A = (a_{ij})$ and $B = (b_{ij})$ of order p are said to be **ORTHOGONAL** if

$$\{(a_{kj}, b_{kj}) \mid k, j = 1, \ldots, p\} = Z_p^2$$

where $\mathbb{Z}_p = \{0, \ldots, p-1\}$.
This means that when you superimpose the elements of A and B , every pair of the numbers occurs once and only once.

Example of orthogonal Euler squares

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

Orthogonal Latin sqaures are used extensively in the statistical design of experiments (cf.[5, 19]).

Euler's method [9]

Given a pair (A, B) of orthogonal Euler squares,

$$M = E_p + A + pB$$

is a magic square, where E_p is the $p \times p$ matrix of ones.

Examples

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}$$

Magic squares from orthogonal generalized Euler squares

Given the magic squares aboves, we can recover the orthogonal pair (A, B) of Euler squares used to generate them from the formulas:

$$a_{ij} = (m_{ij} - 1) \pmod{p}$$
$$b_{ij} = (m_{ij} - 1) \ (\mathrm{div}\ p).$$

Every magic square M has a canonical expansion of this kind, with A and B defined in this manner.

Research Question

Are the canonical components (A, B) of every given magic square $M = E_p + A + pB$ an orthogonal pair of Euler squares?

Answer:

No.

A counterexample

$$\begin{bmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} + 4 \begin{bmatrix} 0 & 3 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 3 & 0 & 0 & 3 \end{bmatrix}$$

The orthogonal components of this $p \times p$ magic square (in the case $p = 4$) are not Euler squares. However, they are **generalized Euler squares** in the sense that their rows, columns and diagonals have the same sum $p(p-1)/2$, and each entry occurs p times in each square.

Normal Magic squares

A magic square with a canonical expansion $M = E_p + A + pB$ is said to be normal if is associated array $M' = E_p + B + pA$ is also a magic square.

An example

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}$$

Theorem (Uko [27])

The $p \times p$ magic square with a canonical expansion $M = E_p + A + pB$ is normal if and only if (A, B) is a pair of orthogonal generalized Euler squares.

Research Question

Are all magic squares normal?

Answer. No.

A counter-example.

$$\begin{bmatrix} 16 & 3 & 10 & 5 \\ 1 & 12 & 7 & 14 \\ 8 & 13 & 2 & 11 \\ 9 & 6 & 15 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} + 4 \begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

In this canonical expansion $M = E + A + 4B$, the ortogonal components arrays A and B are **generalized quasi-Euler squares** (not generalized Euler squares) as defined in the following result.

Theorem (Anatomy of the generic magic square – Uko [26])

Every magic square of order p can be written in the canonical form $M = E_p + A + pB$ with the entries of the matrices A and B generated from the equations

$$a_{ij} = (m_{ij} - 1) \pmod{p}$$
$$b_{ij} = (m_{ij} - 1) \ (\mathrm{div}\ p).$$

The arrays $A = (a_{ij})$ and $B = (b_{ij})$ are orthogonal **generalized quasi-Euler squares** *in the sense that every element of the set $\mathbb{Z}_p$ occurs p times in each array, and they satisfy the row, column and diagonal sum equations*
$\sum_{j=1}^{p} a_{ij} + pr_j = \sum_{j=1}^{p} a_{ji} + pr_{p+j} = \sum_{j=1}^{p} a_{jj} + pr_{2p+1} = \sum_{j=1}^{p} a_{j,p+1-j} + pr_{2p+2} = \sum_{j=1}^{p} b_{ij} - r_j = \sum_{j=1}^{p} b_{ji} - r_{p+j} = \sum_{j=1}^{p} b_{jj} - r_{2p+1} = \sum_{j=1}^{p} b_{j,p+1-j} - r_{2p+2} = p(p-1)/2$
for some integers $r_1, \ldots, r_{2p+2}$.

Note: A magic square is normal if and only if its associated coefficient $(r_1, r_2, \ldots, r_{2p+2})$ is the zero vector.

Non-standard magic squares

In the Literature (cf. [3, 17]), one occasionally encounters 'magic squares' that lack some of the conditions of standard magic squares. We refer to such arrays as non-standard magic squares. An example is the array

101	203	2
3	102	201
202	1	103

which is magic in all aspects with the exception of the non consecutiveness of its integer entries. This example was generated from the following result.

Theorem (Uko [24])

If $M = E + A + pB$ is a $p \times p$ normal magic square, and if m and n are positive integers such that $m \geq pn$ or $n \geq mp$, then $M^* = E + mA + nB$ is a (possibly non-standard) magic square.

Magic cubes

A Magic Cube of order p is a $p \times p \times p$ cubical array with non-repeated entries from the set $\{1, 2, \ldots, p^3\}$, such that all rows, columns, pillars and space diagonals have the same sum. It is a natural extension of the concept of a magic square.
The sum of the elements in a magic cube of order p is $1 + 2 + \cdots + p^3 = p^3(p^3 + 1)/2$. Since these numbers are divided into p^2 rows each of which has the same sum, that sum (the magic constant) must be $p(p^3 + 1)/2$. The columns, pillars and space diagonals will also have the same sum.

The analogous $p \times p \times p \times p$ hypersquare arrays are called magic tesseracts.

The following is an example of a magic cube:

$M_{:\,:\,1}$			$M_{:\,:\,2}$			$M_{:\,:\,3}$		
20	6	16	15	25	2	7	11	24
18	19	5	1	14	27	23	9	10
4	17	21	26	3	13	12	22	8,

where $M_{:\,:\,k} \stackrel{\text{def}}{=} \{m_{ijk} : 1 \leq i, j \leq p\}$.

A magic cube is usually considered to be identical with the 47 magic cubes obtainable from it by performing rotations and/or reflections.

It is known that there are 192 magic cubes of order 3. However, the census of magic squares of orders four and above are still open problems.

Some magic cube generation formulas

From Trenkler [22]:

$$m_{ijk} = 1 + [(i - j + k - 1) \bmod p] + p[(i - j - k) \bmod p] + p^2[(i + j + k - 2) \bmod p], \; i, j, k \in \mathbb{N}_p.$$

From Uko & Barron [32]:

$$m_{ijk} = 1 + [(i + j + k + 1) \bmod p] + p[(-i - j + k) \bmod p] + p^2[(i - j - k) \bmod p],$$

$$m_{ijk} = 1 + [(-2i + j - 2k + 1) \bmod p] + p[(i - j + k - 1) \bmod p] + p^2[(i - j - k) \bmod p],$$

$$m_{ijk} = 1 + [(i + j + k - 2) \bmod p] + p[(i - j - k) \bmod p] + p^2[(i - j + k - 1) \bmod p].$$

Meet the recreational math community on magic squares and cubes

https://www.google.com/search?q=magic+squares&oq=magic+squares&aqs=chrome..69i57j0l5.3919j0j8&sourceid=chrome&ie=UTF-8

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