Verifying Trigonometric Identities

Objective: To verify that two expressions are equivalent. That is, we want to verify that what we have is an identity.

- To do this, we generally pick the expression on one side of the given identity and manipulate that expression until we get the other side.
- In most cases, it is best to start with the more complex looking side and try to simply to match the less complex side.
- You must be very familiar with the fundamental trigonometric identities, especially the Pythagorean Identities. In some cases, a direct substitution using these fundamental identities will verify the identity you are trying to prove (Exercise 8 at the end of this document is one example).
- Some special approaches are useful for certain types of identities, which are provided below.

Identity Type	Verification	Approach
<u>Type 1</u>: Sometimes it is easier if we just rewrite everything in terms of sine and cosine to see if the expression simplifies.	Verify: $$\mathbf{\cot x + 1 = \csc x(\cos x + \sin x)}$$ $$\text{RHS} \rightarrow \mathbf{\csc x(\cos x + \sin x)} = \frac{1}{\sin x}(\cos x + \sin x) = \frac{\cos x}{\sin x} + \frac{\sin x}{\sin x} = \mathbf{\cot x + 1}$$	• Start with more complex RHS. • Rewrite $\csc x$ in terms of sine or cosine. • Remember, $\csc x = 1/\sin x$ • Also note, $\cos x/\sin x = \cot x$ • The RHS simplifies to original LHS.
<u>Type 2</u>: In some cases, the more complex side involves a fraction that can be split up. Then we rewrite everything in terms of sine and cosine.	Verify: $$\mathbf{\frac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t}$$ $$\text{LHS} \rightarrow \mathbf{\frac{\tan t - \cot t}{\sin t \cos t}} = \frac{\tan t}{\sin t \cos t} - \frac{\cot t}{\sin t \cos t} = \tan t \cdot \frac{1}{\sin t \cos t} - \cot t \cdot \frac{1}{\sin t \cos t} = \frac{\sin t}{\cos t} \cdot \frac{1}{\sin t \cos t} - \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t \cos t} = \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t} = \mathbf{\sec^2 t - \csc^2 t}$$	• Start with the more complex LHS. • Rewrite the LHS as difference of two fractions. • Split out $\tan t$ and $\cot t$ to make it easier to simplify. • Notice in the first term, the $\sin t$ cancels out; and in the second term, $\cos t$ cancels out. • The new terms are reciprocal identities • The LHS simplifies to the original RHS.

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Type 3: Using the property of conjugates is sometimes helpful. For an expression like $a + b$, the conjugate would be $a - b$. When you multiply conjugates, you often get a more useful expression, e.g., $(a + b)(a - b)$. Sometimes multiplying by the conjugate will simplify an expression and help in verifying the given identity.	Verify: $$\boldsymbol{\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}}$$ $$\text{RHS} \rightarrow \boldsymbol{\frac{1 + \sin x}{\cos x}} = \frac{1 + \sin x}{\cos x}\left(\frac{1 - \sin x}{1 - \sin x}\right)$$ $$= \frac{1 - \sin^2 x}{\cos x\,(1 - \sin x)}$$ $$= \frac{\cos^2 x}{\cos x\,(1 - \sin x)}$$ $$= \frac{\cos x \cos x}{\cos x\,(1 - \sin x)}$$ $$= \boldsymbol{\frac{\cos x}{1 - \sin x}}$$	• We could start with either side; but here we will start with the RHS. • The conjugate of the numerator $1 + \sin x$ is $1 - \sin x$. • Multiply by $\frac{1 - \sin x}{1 - \sin x} = 1$ • Remember, $1 - \sin^2 x = \cos^2 x$ • Once we reduce the fraction, we get the LHS of the original identity.
Type 4: Combining fractions before using identities may be an appropriate strategy.	Verify: $$\boldsymbol{\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} = 2\sec^2\alpha}$$ $$\text{LHS} \rightarrow \boldsymbol{\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha}} = \frac{1}{1 - \sin\alpha}\left(\frac{1 + \sin\alpha}{1 + \sin\alpha}\right) + \frac{1}{1 + \sin\alpha}\left(\frac{1 - \sin\alpha}{1 - \sin\alpha}\right)$$ $$= \frac{(1 + \sin\alpha) + (1 - \sin\alpha)}{(1 - \sin\alpha)(1 + \sin\alpha)}$$ $$= \frac{2}{1 - \sin^2\alpha}$$ $$= \frac{2}{\cos^2\alpha}$$ $$= \boldsymbol{2\sec^2\alpha}$$	• Notice that the denominators of the fractions on the LHS are conjugates. • So we will use the property of conjugates to combine the LHS fractions and simplify.

Verifying Trigonometric Identities

Verify the following trigonometric identities.

1. $\cos x + \sin x \tan x = \sec x$

2. $\dfrac{\csc x - \sin x}{\sin x \csc x} = \csc x - \sin x$

3. $\dfrac{1}{\tan \beta} + \tan \beta = \dfrac{\sec^2 \beta}{\tan \beta}$

4. $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 2\sec\theta$

5. $\sec y + \tan y = \dfrac{\cos y}{1-\sin y}$

6. $\dfrac{\cos^2 x - \sin^2 x}{1-\tan^2 x} = \cos^2 x$

7. $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x} = 0$

8. $\dfrac{\sin^2\theta + \cos^2\theta + \cot^2\theta}{1+\tan^2\theta} = \cot^2\theta$

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Solutions to Exercises

1. LHS $\rightarrow$ $\cos x + \sin x \tan x$

$$= \cos x + \sin x\left(\frac{\sin x}{\cos x}\right)$$
$$= \cos x + \frac{\sin^2 x}{\cos x}$$
$$= \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos x}$$
$$= \frac{1}{\cos x}$$
$$= \sec x$$

2. LHS $\rightarrow$ $\dfrac{\csc x - \sin x}{\sin x \csc x}$

$$= \frac{1}{\sin x \csc x}(\csc x - \sin x)$$
$$= \frac{1}{\sin x \csc x}\csc x - \frac{1}{\sin x \csc x}\sin x$$
$$= \frac{1}{\sin x} - \frac{1}{\csc x}$$
$$= \csc x - \sin x$$

3. LHS $\rightarrow$ $\dfrac{1}{\tan\beta} + \tan\beta$

$$= \frac{1}{\tan\beta} + \frac{\tan^2\beta}{\tan\beta}$$
$$= \frac{1 + \tan^2\beta}{\tan\beta}$$
$$= \frac{\sec^2\beta}{\tan\beta}$$

4. LHS $\rightarrow$ $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$

$$= \frac{1+\sin\theta}{\cos\theta}\left(\frac{1+\sin\theta}{1+\sin\theta}\right) + \frac{\cos\theta}{1+\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)$$
$$= \frac{(1+\sin\theta)^2 + \cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{1 + 2\sin\theta + 1}{\cos\theta(1+\sin\theta)}$$
$$= \frac{2 + 2\sin\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$
$$= \frac{2}{\cos\theta}$$
$$= 2\sec\theta$$

Verifying Trigonometric Identities

5. RHS $\rightarrow$ $\dfrac{\cos y}{1-\sin y}$

$$= \frac{\cos y}{1-\sin y}\left(\frac{1+\sin y}{1+\sin y}\right)$$
$$= \frac{\cos y(1+\sin y)}{1-\sin^2 y}$$
$$= \frac{\cos y(1+\sin y)}{\cos^2 y}$$
$$= \frac{1+\sin y}{\cos y}$$
$$= \frac{1}{\cos y}+\frac{\sin y}{\cos y}$$
$$= \sec y + \tan y$$

6. LHS $\rightarrow$ $\dfrac{\cos^2 x-\sin^2 x}{1-\tan^2 x}$

$$= \frac{\cos^2 x-\sin^2 x}{1-\frac{\sin^2 x}{\cos^2 x}}$$
$$= \frac{\cos^2 x-\sin^2 x}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}$$
$$= \left(\cos^2 x-\sin^2 x\right)\left(\frac{\cos^2 x}{\cos^2 x-\sin^2 x}\right)$$
$$= \cos^2 x$$

7. LHS $\rightarrow$ $\dfrac{\sin x}{\cos x+1}+\dfrac{\cos x-1}{\sin x}$

$$= \frac{\sin x}{\cos x+1}\left(\frac{\sin x}{\sin x}\right)+\frac{\cos x-1}{\sin x}\left(\frac{\cos x+1}{\cos x+1}\right)$$
$$= \frac{\sin^2 x}{\sin x(\cos x+1)}+\frac{\cos^2 x-1}{\sin x(\cos x+1)}$$
$$= \frac{\sin^2 x+\cos^2 x-1}{\sin x(\cos x+1)}$$
$$= \frac{1-1}{\sin x(\cos x+1)}$$
$$= \frac{0}{\sin x(\cos x+1)}$$
$$= 0$$

8. LHS $\rightarrow$ $\dfrac{\sin^2\theta+\cos^2\theta+\cot^2\theta}{1+\tan^2\theta}$

$$= \frac{1+\cot^2\theta}{1+\tan^2\theta}$$
$$= \frac{\csc^2\theta}{\sec^2\theta}$$
$$= \frac{\frac{1}{\sin^2\theta}}{\frac{1}{\cos^2\theta}}$$
$$= \left(\frac{1}{\sin^2\theta}\right)\left(\frac{\cos^2\theta}{1}\right)$$
$$= \frac{\cos^2\theta}{\sin^2\theta}$$
$$= \cot^2\theta$$