GREAT IDEAS OF MODERN MATHEMATICS: VOTING THEORY

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This electronic textbook chapter and teaching manual was created for Engaging Mathematics with support from the National Science Foundation.

An initiative of the National Center for Science and Civic Engagement, Engaging Mathematics applies the well-established SENCER method to college level mathematics courses, with the goal of using civic issues to make math more relevant to students.

Engaging Mathematics will: (1) develop and deliver enhanced and new mathematics courses and course modules that engage students through meaningful civic applications, (2) draw upon state-of-the-art curriculum in mathematics, already developed through federal and other support programs, to complement and broaden the impact of the SENCER approach to course design, (3) create a wider community of mathematics scholars within SENCER capable of implementing and sustaining curricular reforms, (4) broaden project impacts beyond SENCER by offering national dissemination through workshops, online webinars, publications, presentations at local, regional, and national venues, and catalytic site visits, and (5) develop assessment tools to monitor students' perceptions of the usefulness of mathematics, interest and confidence in doing mathematics, growth in knowledge content, and ability to apply mathematics to better understand complex civic issues.

Updates and resources developed throughout the initiative will be available online at www.engagingmathematics.net. Follow the initiative on Twitter: @MathEngaging.

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Note: The "Your Vote Counts" image from this chapter's title page is a digital reproduction of the button that the NAACP chapter in New Orleans, LA, gave to newly registered voters in 2011.

ABOUT THE AUTHOR

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Dr. Nardo was born and raised in North Carolina. He chose a liberal arts education at Wake Forest University in Winston-Salem, NC. He double-majored in Mathematics and English with a concentration in Italian Studies. The highlight of these years was a semester at Casa Artom, Wake Forest's educational center located in a palazzo on the Grant Canal in Venice, Italy.

Dr. Nardo knew he wanted to be a college professor from childhood (not your typical career aspiration on the playground). So, immediately after receiving his B.A. degree, he enrolled in a Ph.D. program in Mathematics at Emory University in Atlanta, GA. In addition to expanding his exposure to many different fields in mathematics, Dr. Nardo was able to begin teaching. After a year of training, he was given full instructor responsibility for his own section of Calculus. Though only slightly older than the students he was teaching, he took to the classroom quickly and successfully – later earning the departmental Marshal Hall Prize for Distinguished Undergraduate Instruction and a competitive, university-wide Dean's Teaching Fellowship. Along the way, he earned both M.S. and Ph.D. degrees in Mathematics. Dr. Nardo enjoyed his beginning forays into mathematical research, and his dissertation created a new result in probabilistic knot theory. This is a hybrid research area between topology and probability which is mainly used to mathematically model macromolecules like DNA and to explore their properties. But his first love was (and still is) the classroom! He wanted an academic post at an institution which values teacher-scholars instead of having a pure research focus.

After a successful introduction to the life of a tenure-track faculty member at North Georgia College & State University (now the University of North Georgia) in Dahlonega, GA, Dr. Nardo found his academic home at Oglethorpe University in Atlanta, GA. During his continuing career at Oglethorpe, he received tenure and later promotion to Professor of Mathematics. He is currently serving his third term as Division Chair for Mathematics and Computer Science and even had a year-long stint as Acting Co-Provost of the university. Along the way, he earned the university-wide Lu Thomasson Garrett Award for Meritorious Teaching. Go Stormy Petrels!

Dr. Nardo has two voting memories to share. First, he was lucky that his eighteenth birthday lined up with a Presidential election year and still remembers the thrill of his first vote. Second, his study abroad was during a fall semester, and he had his absentee ballot mailed to Venice. Knowing the vagaries of the Italian postal system, he paid an exorbitant amount to return his ballot via FedEx and ensure its arrival in time for election day. This self-described political junkie spent his recent sabbatical researching the mathematics of voting and writing this chapter.

TABLE OF CONTENTS

PREFACE

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Before starting a mathematical study of voting theory and what “fairness” means in a mathematical context, we must begin with historical and political contexts. We must first see “fairness” through the lens of problems with voting that are all too recent in our country.

A little more than fifty years ago, former President Lyndon Baines Johnson signed one of the most comprehensive pieces of Civil Rights legislation into law. The Voting Rights Act (VRA), signed into law on August 6th, 1965, was designed to provide both corrective as well as proactive measures of political redress and relief for African Americans.

In particular, the VRA was designed to prohibit racial discrimination in voting as well as provide relief from the oppressive resistance and subversive tactics used by state and local government municipalities to usurp, curtail, and deny the Black vote.

Moreover, the VRA was designed to provide widespread and all inclusive "rights" as a means of eliminating the well-known institutional, historical, man-made, and extra-legal measures used to paralyze the effective political participation and incorporation of African Americans within the U.S. political system. Some of the practices the passage of the VRA made illegal were:

- the use of literacy tests
- the use of poll taxes
- the use of the "white" primary
- the use of fear and intimidation practices
- the use of arbitrary and restrictive registration.

Given the longstanding history of Black voter suppression - especially in the South - the relatively quick passage of the Act was momentous. As President Johnson stated in his address to joint members of Congress, "Many of the issues of civil rights are very complex and most difficult. But about this there can and should be no argument. Every American citizen must have an equal right to vote. There is no reason which can excuse the denial of that right. There is no duty which weighs more heavily on us than the duty we have to ensure that right."

President Johnson continues to make his case for the Voting Rights Act of 1965 by reminding Congressional leaders of the disgraceful tactics used for almost 100 years to deny the vote solely on the basis of skin. He remarked,

> *"Every device of which human ingenuity is capable has been used to deny this right. The Negro citizen may go to register only to be told that the day is wrong, or the hour is late, or the official in charge is absent. And if he persists, and if he manages to present himself to the registrar, he may be disqualified because he did not spell out his middle name or because he abbreviated a word on the application ... The Constitution says that no person shall be kept from voting because of his race or his color. We have all sworn an oath before God to support and to defend that Constitution. We must now act in obedience to that oath."*

As we reflect on the inception and evolution of the Voting Rights Act of 1965, I believe there are some lessons we can learn for 21st Century voting and political participation. Moreover, as we prepare for the 2016 election and the possible candidacy of Mr. Jesus Walks himself - Kanye West, we must ask ourselves what, if anything, has the past fifty years taught us and what can we do individually and collectively to help eliminate "man's inhumanity towards man?" - Robert Burns

Lesson 1: The Interconnectedness of the Vote

As was the case fifty years ago, our active participation in voting is inextricably linked to our active understanding and participation with a variety of human, social, and civil rights issues currently before our lawmakers and leaders. For example, I do not believe a majority of American women lobbied for the right to vote without lobbying for equal pay (The Lilly Ledbetter Fair Pay Act of 2009). As Dr. King beautifully expressed,

> "In a real sense all life is inter-related. All men are caught in an inescapable network of mutuality, tied in a single garment of destiny. Whatever affects one directly, affects all indirectly. I can never be what I ought to be until you are what you ought to be, and you can never be what you ought to be until I am what I ought to be...
> This is the inter-related structure of reality."

Lesson 2: Justice, Equality, and the Vote

> "We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness.--That to secure these rights, Governments are instituted among Men, deriving their just powers from the consent of the governed, --That whenever any Form of Government becomes destructive of these ends, it is the Right of the People to alter or to abolish it, and to institute new Government, laying its foundation on such principles and organizing its powers in such form, as to them shall seem most likely to effect their Safety and Happiness."

If Washington Post writer Dylan Matthews is correct in his assertion that the rulings of the U.S. Supreme Court have "gutted the Voting Rights Act" by stripping away some of its most powerful protections and provisions, perhaps "unalienable Rights" are nothing more than rose colored theoretical notions that will never be experienced by certain groups. While I personally do not ascribe to such defeatist sentiments, I believe we must continue to exercise our right to vote as there are too many horrendous examples of sliding scales of justice and equality in America that beg the question of who truly governs in America. Just the other day, George Zimmerman who was found not guilty of killing an unarmed teen retweeted a picture of the dead youth's corpse with the caption " George Zimmerman .... A one man army!"

Kanye West and the Remix of the Voting: "No One Man Should Have All That Power"

During his August 30, 2015, acceptance speech for an MTV Video Vanguard award, Kanye West announced his 2016 Presidential Bid. The king of sampling and remixes stated that "we must listen to the children" prior to declaring his run. Without a doubt, there are a variety of reasons why Kanye should not run for President of the United States of America. Nonetheless, Kanye has a right to exercise the full extension of his political rights which includes declaring his candidacy for President as he meets the three main requirements. He is a natural born US citizen, he is 35 years of age, and he has been a resident of the US for 14 years. If he does run for President, there is potential to see a flip and remix of the vote that could be unprecedented in that he may be able to galvanize young voters to create cross cultural political coalitions more powerful and politically boisterous than the "Change ... Yes We Can" campaign of eight years ago.

As Kanye so passionately spits on POWER:

> "Colin Powell, Austin Powers
> Lost in translation with a whole f'in nation
> They say I was the abomination of Obama's nation
> Well that's a pretty bad way to start a conversation."

INTRODUCTORY WRITING ASSIGNMENT: THE PROMPT

We will start this chapter of our course with a writing assignment, and it will serve as the foundation for our in-class discussions on the first day of this new material.

1. Read the two articles below.

- "Relax, Nader Tells Democrats, but the Math Says Otherwise" (*New York Times,* 2-24-2004)
- "A Tale of Two Campaigns: Ralph Nader's Strategy in the 2004 Presidential Election" (*PS: Political Science and Politics,* October 2006)

Using these articles and/or other research, **write a paragraph** where you argue whether third-party candidates (candidates not from the Democratic and Republican parties) are good or bad for the political process in the USA.

NOTE: Of course, you could write long essays or dissertations on this topic. That is not the point of this exercise. Simply take a stand and offer some evidence to support it.

2. Read the article "Supreme Court Allows Texas to Use Strict Voter ID Law in Coming Election" (*New York Times,* 10-19-2014). Using this article and/or other research, **write a paragraph** where you argue whether strict voter ID laws are good or bad for the political process in the USA. The same note from #1 applies to #2.

3. **List the top five issues** about which you are concerned when thinking about elections occurring in Georgia over the next two years. You can pick from national, state, county, or city issues.

- List any of your issues that science could help address. (If you feel none, say so.)
- List any of your issues that mathematics could help address. (If you feel none, say so.)

4. What is necessary for an election to be <u>fair</u> (in whatever way you choose to interpret that word)? **List at least three items.**

- You can answer as a series of (at least three) questions that need to be answered.
- You can answer as a series of (at least three) important bullet points.
- You can answer as a series of (at least three) problems that need to be fixed or avoided.

What really matters is that you spend a little time thinking and writing about elections … and that you bring this work to share with the class. To get you started, I'll give you a few easy items – which you **CANNOT** use in your response to #4.

- Which persons are eligible to vote?
- Who decides the eligibility criteria?
- Who enforces the eligibility criteria?

INTRODUCTORY WRITING ASSINGMENT: FOLLOW UP

Mathematicians love to categorize objects, and we will do so with the issues from your writing assignments. Some are the purview of mathematics, but others are the purview of political science.

We could analyze mathematically a number of issues related to the three sample questions given as starting examples for #4 in your writing assignment. For example, we could calculate the percentage of eligible voters who actually vote in any given election; we could calculate how many people were turned away from voting because of not meeting the eligibility requirements. But these three starting questions from # 4 in your writing assignment are the purview of political science and not mathematics.

Let's consider a national election in the United States. In order to vote, a person must be a citizen of this country who is at least 18 years old and who has previously registered to vote in his/her state of legal residence. The details and deadlines for the registration process vary from state to state. Most states also prevent persons convicted of felonies or declared mentally incompetent from voting; again details vary by the state. If we wish to change those eligibility criteria, then we must engage in a political process. This is not a mathematical process that we will study in this e-chapter. Of course, many people believe that mathematics/statistics can offer evidence to be used in the political arena and are, thus, helpful and valuable to the political process. In this way and many others, mathematics can be useful in the real world. We hope you agree!

Discuss with your classmates all four questions from your writing assignment. Your professor will guide you in facilitating this discussion. You may have a full-class discussion, or you may discuss in small groups reporting back to the full class.

Here are some items that are important now and as building blocks for this chapter.

- For #1, the differences between two-candidate races and multi-candidate races will hopefully start to form. Though two-candidate races are easy to decide, issues arise when there are more than two candidates.

- For #2, this is a hot-button political issue! Students in mathematics classes (and even their professors) may be uncomfortable having this kind of classroom discussion. Be respectful of your peers (especially those with differing opinions). Try to argue as fairly and objectively as possible.

- For #3, create a "Top 5" list of most popular issues for your discussion group or for the whole class. For each issue, note whether science or mathematics could be helpful in exploring the issue.

- For #4, categorize the fairness items into political items and mathematical items.

SECTION 1: BEGINNINGS

1.1: Basic Terms

Mathematicians are extremely precise with their terms – at least we try to be. We must hammer out what we mean by the words we use in order to prevent confusion or misunderstanding. Thus, we will start our journey with a few basic definitions.

An **election** is a process by which a group makes a collective decision.

A **voter** is a person participating in an election.

A **ballot** is an instrument through which a voter marks his/her preference(s) in an election.

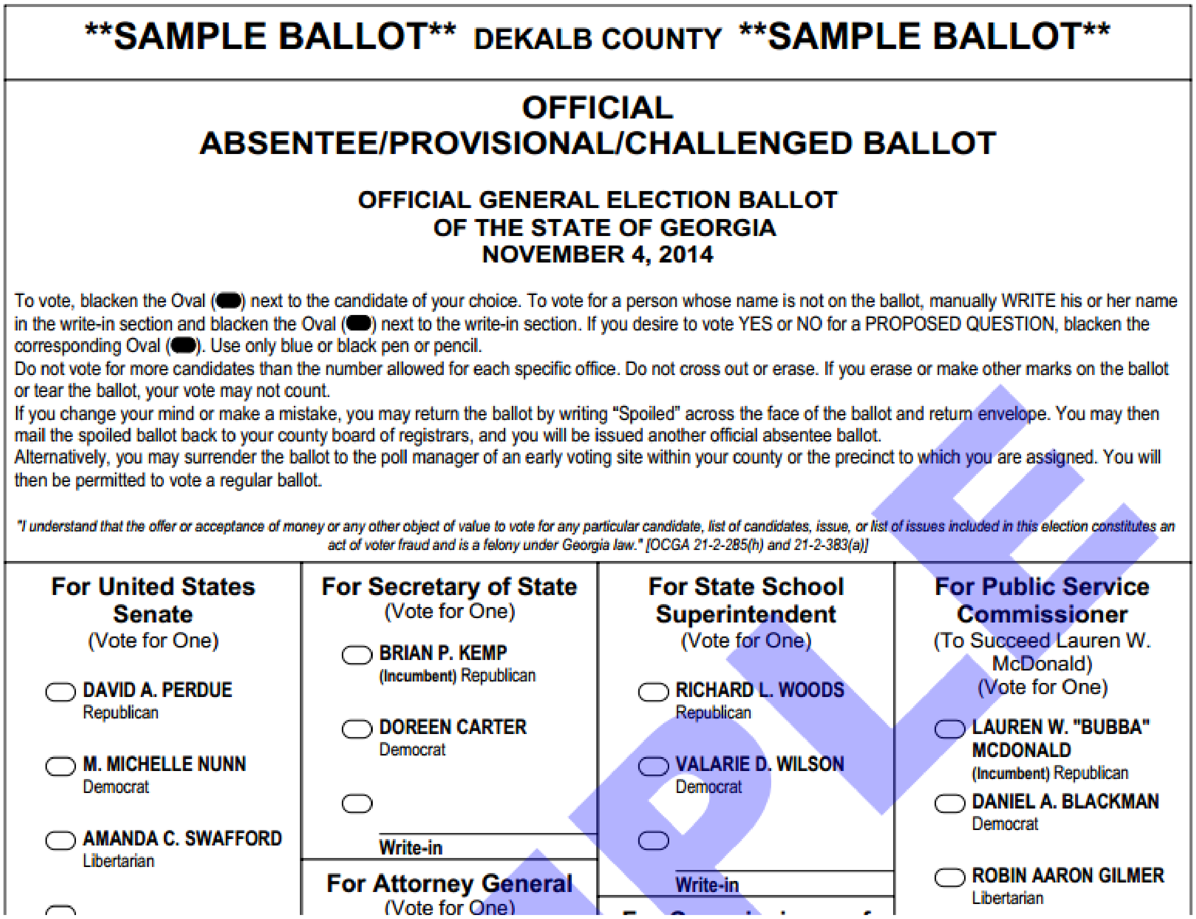
A **candidate** is one option on a ballot. If the options are not comprised of people, then an option on a ballot is called a **choice** or **alternative** instead of a candidate.

The wishes or rankings expressed by a voter on a ballot are called the **voter's preferences**. This may be simply giving his/her favorite choice, i.e. picking one candidate only. This may be giving a partial ranking of some of the candidates, for example giving the top three choices. This may be giving a full ranking of all candidates, i.e. listing the candidates from most liked to least liked.

A **voting system** is a method or algorithm for converting individual voter preferences on the ballots into an **overall societal preference**, i.e. the result of the election or the collective decision. Again, this may simply be giving a winner, or it may involve giving a partial ranking of a certain number of candidates or even a total ranking of all candidates.

We are most familiar with elections and ballots that instruct us to pick our favorite, i.e. most preferred, choice in each race. See a portion of the sample ballot for the general election for Georgia in 2014 on the next page. Each box is for a separate race, and it ends with the option for a "Write-In" candidate. The voter is directed to make one choice per box/race. Of course, the voter can leave a box/race without a choice and give up the right to vote in that race. But if a voter chooses more than one option in a box/race, then these votes will not be counted.

But we are not always restricted to just one choice per race on a ballot. You may have the opportunity to specify not just your "favorite" one choice but your lesser preferences as well. Here are all six of the full-time members of Oglethorpe's Division of Mathematics and Computer Science: Dr. Gieger, Dr. Merkel, Mr. McBride, Dr. Nardo, Dr. Patterson, and Dr. Tiu. On the next page, we have examples of partial ordering and full ordering ballots involving these professors.



SAMPLE BALLOT DEKALB COUNTY **SAMPLE BALLOT**

OFFICIAL
ABSENTEE/PROVISIONAL/CHALLENGED BALLOT

OFFICIAL GENERAL ELECTION BALLOT
OF THE STATE OF GEORGIA
NOVEMBER 4, 2014

To vote, blacken the Oval (⬬) next to the candidate of your choice. To vote for a person whose name is not on the ballot, manually WRITE his or her name in the write-in section and blacken the Oval (⬬) next to the write-in section. If you desire to vote YES or NO for a PROPOSED QUESTION, blacken the corresponding Oval (⬬). Use only blue or black pen or pencil.
Do not vote for more candidates than the number allowed for each specific office. Do not cross out or erase. If you erase or make other marks on the ballot or tear the ballot, your vote may not count.
If you change your mind or make a mistake, you may return the ballot by writing "Spoiled" across the face of the ballot and return envelope. You may then mail the spoiled ballot back to your county board of registrars, and you will be issued another official absentee ballot.
Alternatively, you may surrender the ballot to the poll manager of an early voting site within your county or the precinct to which you are assigned. You will then be permitted to vote a regular ballot.

"I understand that the offer or acceptance of money or any other object of value to vote for any particular candidate, list of candidates, issue, or list of issues included in this election constitutes an act of voter fraud and is a felony under Georgia law." [OCGA 21-2-285(h) and 21-2-383(a)]

For United States Senate
(Vote for One)
DAVID A. PERDUE
Republican
M. MICHELLE NUNN
Democrat
AMANDA C. SWAFFORD
Libertarian

For Secretary of State
(Vote for One)
BRIAN P. KEMP
(Incumbent) Republican
DOREEN CARTER
Democrat
Write-in

For Attorney General
(Vote for One)

For State School Superintendent
(Vote for One)
RICHARD L. WOODS
Republican
VALARIE D. WILSON
Democrat
Write-in

For Public Service Commissioner
(To Succeed Lauren W. McDonald)
(Vote for One)
LAUREN W. "BUBBA" MCDONALD
(Incumbent) Republican
DANIEL A. BLACKMAN
Democrat
ROBIN AARON GILMER
Libertarian

Sample Ballot for Georgia's General Election 2014

Rank the professors in this division regarding who is the most challenging in the classroom.

First: ______________________
Second: ______________________
Third: ______________________

A Partial Ordering Ballot

Rank all professors in this division from most challenging to least challenging in the classroom.

First: ______________________
Second: ______________________
Third: ______________________
Fourth: ______________________
Fifth: ______________________
Sixth: ______________________

A Full Ordering Ballot

In the most basic sense, mathematicians study collections of objects (sets) by deciding whether those objects possess or do not possess characteristics of interest (properties).

Let's connect with familiar mathematics before attempting to apply this methodology to the new voting terms just defined. We will start by taking numbers as our "objects."

1.2: A Brief Excursion into Number Systems

The **natural numbers** are simply those with which we first learned to count:

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

When these numbers are augmented with the number zero, we have the **whole numbers**:

$$W = \{0,1,2,3,\ldots\} .$$

When these numbers are augmented with the negatives of the natural numbers, we have the **integers**:

$$\mathbb{Z} = \{0,1,-1,2,-2,3,-3,\ldots\} .$$

When these numbers are augmented with all the fractions that can be formed of the integers (except for illegally dividing by zero), we have the **rational numbers**:

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0 \right\}.$$

Recall that rational numbers can be written as fractions or as decimals that either terminate or repeat with a repeat bar: $\frac{1}{4} = 0.25$ or $\frac{1}{3} = 0.\bar{3} = 0.333\ldots$.

There are numbers whose decimal expansions neither terminate nor repeat like this. We call them **irrational numbers**. Some examples from the deep recesses of your mathematical history are: $\pi = 3.14159\ldots$, $e = 2.71828\ldots$, and $\sqrt{2} = 1.41421\ldots$.
Note: There is no easy one-letter symbol for irrational numbers.

When you put all of these various types of numbers together into one new collection, it is called the set of **real numbers**, denoted $\mathbb{R}$.

Now that we have various collections of objects, we can operate on them and study properties. The easiest operations are those from arithmetic: addition, subtraction, multiplication, and division. Let's start with natural numbers.

<u>Motivating Question:</u>
Is the set of integers commutative under the operation of subtraction?

- **If yes, then the order in which we subtract natural numbers does not matter.**
- **If no, then different orders for subtraction can result in different results.**

No! Each of us has known for a very long time that integers do not commute under subtraction. The order of subtraction matters quite a bit. For example, $9-7 \neq 7-9$. Direct calculation gives $9-7=2$, but we get a different result when we switch the order: $7-9=-2$.

Just in case your memory of this mathematical term is a little fuzzy, here is the definition.

> If the order does not matter for an operation, i.e. we obtain the same result despite the order of objects, then the set is **commutative** under the operation.
>
> If the order does matter for an operation, i.e. we can obtain different results with different orderings, then the set is **not commutative** under the operation.
>
> Note: Yes, it is inconvenient that order **not** mattering is connected with **commutative** instead of **not commutative**. It might have been nicer to keep "not mattering" with "not commutative," but unfortunately, that is not how the definition evolved.

Follow-Up Question:
Is the set of natural numbers commutative under the operation of addition?

Yes! Each of us has known for a very long time that natural numbers commute under addition. Intuitively, since addition is simply "totaling up," it does not matter into what order we place the addends (the numbers in the sum). For example, 1 + 8 = 8 + 1 = 9. But proving this simple "fact" is not easy, and we cannot prove mathematical results by giving examples.

It usually takes more work to write an explanation of why a set **possesses** a particular property than it is to show that it does not. To show that a set possesses a property, you must show that for **all** objects under consideration the property holds. You cannot usually give a single example of the property "working." The word "all" is very important here.

I'm sorry to say that I have not proven that natural numbers are commutative under addition; instead, I am relying on your familiarity with basic arithmetic to believe this "truth." After all, I have only given one example. Since there are infinitely-many natural numbers, one example won't cut it: there are infinitely-many more pairs of natural numbers to check. I could even show a million examples, and there would still be infinitely-many pairs left unchecked.

This doesn't mean that examples are useless or a waste of time. Examples are a good start in most mathematical explorations. Examples are simply not sufficient – unless you have provided every possible example. Usually sets are too big for examples to be an effective technique. Again, we will vindicate this basic "truth" when we discuss induction later in this section. We will have to postpone our satisfaction on addition until that optional section.

To show that a set does **not** possess a property, you must only give one example for which it fails. That illuminating example is called a **counter-example**. The numbers seven and nine from our work above give the needed counter-example, though of course you could have used other numbers. We have conclusively shown that subtraction of integers is not commutative. We are done with this result.

1.3: Voting Properties

Now, let's move from basic arithmetic into our subject matter of voting. We will define properties that a voting system may or may not possess in an analogous way to which we defined the commutative property and explored when it was possessed and when it was not.

Each of these properties is desirable, and one could argue is needed for the election to be "fair." The "great idea" to which we are building is called Arrow's Theorem, and these properties were laid out in Kenneth Arrow's 1951 economics book *Social Choice and Individual Values.*

Before we define the properties themselves, we must give two "building block" definitions first.

A candidate who earns strictly more than 50% of the votes cast is called the **majority candidate**.

Take every pair of candidates on the ballot and construct all pairwise comparisons. If a candidate is the majority candidate in each of his/her pairwise races, then he/she is the **Condorcet candidate**.

Of course, an election may have neither of these types of candidates.

Some elections require that the winner be a majority candidate. For example, in any race for the US Senate in Georgia, if no candidate wins more than 50% of the votes cast, then there is a run-off election consisting of only the two candidates with the highest number of votes. The winner of the run-off election must be the majority candidate. (If there is a tie in the run-off election, then another run-off election must be held.)

Even a very well-liked candidate can fail to be a Condorcet candidate. To have that particular honor, a candidate must earn more than 50% of the votes in every pairwise race. If he/she falls below 50% in even one pairwise match-up, then he/she is not a Condorcet candidate.

Many times people misunderstand how to decide whether there is a majority candidate. Often, someone will colloquially say "50% plus one" or, if careful about what group is used to make the percentage, "50% of the votes plus one additional vote." Whether this common sense approach is mathematically correct depends on the total number of votes cast. If this number is even, then common sense works, but if this number is odd, then common sense does not work. Let's make a definition to clarify matters.

Let the total number of votes cast be denoted by n. The **majority threshold** is the lowest number of votes with which a candidate will have strictly more than 50% of the total votes cast. A candidate whose vote total equals or exceeds the majority threshold is said to **meet** or **satisfy** the majority threshold; thus, he/she is the majority candidate.

If n is even, then the majority threshold is: (50% of n) + 1 or $\frac{n}{2}+1$.

If n is odd, then the majority threshold is: (50% of n) + $\frac{1}{2}$ or $\frac{n}{2}+\frac{1}{2}$.

For example, suppose that n = 10,086 votes are cast. Basic arithmetic gives the result below.

$$\text{majority threshold} = (50\% \text{ of } n) + 1 = \frac{10,086}{2}+1=5,043+1=5,044$$

If a candidate earns 5,043 votes, then he/she has exactly 50% of the votes but has not surpassed 50%. We can phrase this in many equivalent ways. This candidate has not met the majority threshold; he/she does not have a strict majority; he/she is not the majority candidate.

If a candidate earns 5,044 votes (or more), then he/she will have more than 50% of the votes. This candidate has met the majority threshold; he/she does have a strict majority; he/she is the majority candidate.

Now we can move to voting properties: we may define the desirable "fairness" properties that a voting system may or may not possess.

A voting system possesses the **majority property** if the majority candidate (whenever there is one) is always the winner of the election.

A voting system possesses the **Condorcet property** if the Condorcet candidate (whenever there is one) is always the winner of the election.

A voting system possesses the **monotonicity property** if the winner of the election would still be the winner had any voter moved the candidate higher in his/her preferences on the ballot. (The upshot is that the winner cannot be "hurt" by being looked on more favorably by the voters.)

A voting system possesses the **independence of irrelevant alternatives property** if the winner of the election would still be the winner had one or more losing candidates (irrelevant alternatives) been removed from the ballot. Since this is a mouthful, it is usually abbreviated the **IIA property**.

1.4: Dictatorship Example

To take these properties out for a test drive, let's consider a voting system we would definitely consider unfair in the extreme: a dictatorship. For each fairness property, let's explore whether this dictatorship voting system possesses the fairness property or does not possess it.

The totalitarian state of "Discriminatia" is ruled by Dictator D. Though elections are a pointless display, he allows them to be held regularly. Every citizen who is 18 years of age or older is allowed to vote, including the dictator … er … ruler. However, only the dictator's ballot matters. Whichever candidate or choice the dictator picks is the one that wins the election. If partial or total rankings are used, then only the dictator's rankings matter and decide the election. This describes the unsavory voting system in this country.

Let's assume that there are 100 voters in "Discriminatia," including Dictator D. Further, let's assume that everyone exercises his/her right to vote. A brave candidate decides to run against the dictator to be ruler, and let's call her F for freedom fighter. There are only two choices on the ballot: D or F. Our freedom fighter receives every vote but the dictator's vote, for himself.

Majority Property

In order to build a majority, a candidate will need 51 votes, i.e. he/she must satisfy the majority threshold (50% of 100 + 1 vote, since *n* is even). Our freedom fighter has amassed 99 votes, and thus, she is the majority candidate. But the only ballot that matters is the Dictator's ballot. Since he voted for himself, he wins the election. Since our freedom fighter is the majority candidate but she does not win the election, this dictatorship voting system does **NOT** satisfy the majority property.

Condorcet Property

There are only two candidates on the ballot; hence, there is no need to construct theoretical pairwise comparisons. There is only one pair to consider, and the original ballot explores that pair. As shown above, the freedom fighter is the majority candidate in this lone pairwise race; thus, she is the Condorcet candidate. Since she does not win the election, this dictatorship voting system does **NOT** satisfy the Condorcet property.

Monotonicity Property

Since the election does not use partial or total rankings, we have no list of preferences. There is only one preference – the person for whom you vote.

Since the dictator will vote for himself, he cannot move himself higher in his own preferences. But the other ordinary voters could move the dictator higher in their preferences: doing so would change their votes from the freedom fighter to the dictator.

If there is such a shift in preferences, the dictator would receive more votes than his own (not that it matters). He wins the election no matter what, and getting those extra votes neither hurts nor helps him. This dictatorship **DOES** satisfy the monotonicity property.

Independence of Irrelevant Alternatives Property

Removing the losing freedom fighter candidate does not affect the race. Since only the dictator's vote for himself counts, he will win whether he is the only candidate (removing the irrelevant alternative F) or he is running against candidate F. This dictatorship **DOES** satisfy IIA property.

Discussion Questions

Our democratic values would charge that dictatorship is the least fair voting system ever devised, yet dictatorship still possesses half of our fairness properties.

- Do you believe that the monotonicity and IIA properties do not truly measure fairness?
- Are you alarmed that dictatorship satisfies these two properties?

Notation

We can use the "**greater than**" operator > to indicate a voter's preferences.

If a voter ranks A as first, B as second, and C as third, then we can simply describe that voter's preferences by writing: A > B > C.

The "greater than" symbol has the open end with the more preferred choice, and the sharp point is pointing towards the less preferred choice.

1.5: A Brief Excursion into Mathematical Induction

OPTIONAL SECTION

At your discretion, you may skip this section and go directly to the exercises (Section 1.6, pp. 20-22).

We will end our section by filling in a huge gap we left earlier. You were asked to "believe" that addition of natural numbers is commutative, and since you have "known" this for a very long time, you more than likely had no issue with that request. But mathematicians do not rely on such blind faith in our discipline: we must prove the results we use.

It would take us too far afield of voting theory to cover the subject of mathematical induction. Some of you may be familiar with this technique from high school or from other college-level mathematics classes. Others may have never heard this phrase, and that's OK.

Go to your favorite Internet search engine or YouTube® and type in the phrase "mathematical induction." You will find more than enough hits to get you started. Read any page or watch any video from your search. It is highly likely that beginning/introductory videos will showcase a formula for adding up the first *n* integers: it is a standard, familiar example of mathematical induction. The video below gives a brief bit of mathematical history to motivate the problem. Watch it in addition to your search results.

https://www.youtube.com/watch?v=Sr4O7cpV2Ss

Now that you have a beginning understanding of mathematical induction, we can finally fill in the hole in your understanding from earlier in this section.

***

Conjecture: Addition of natural numbers is commutative.

Proof:

Let m be any natural number. Define the proposition P(n) for natural number n to be:

$$m + n = n + m.$$

Basis Step:

We must prove P(1) to be true: $m + 1 = 1 + m$.

By the definition of addition of quantities in the *Harper-Collins Dictionary of Mathematics*, we are calculating the total number of units contained in these quantities. Whether we start with m units in the set and augment by one additional unit or we start with one unit in the set and augment m more units, we have a total of one more unit than the natural number m represents. Thus, our basis step is proven.

Induction Hypothesis:

We will assume P(k) to be true: $m + k = k + m$.

Induction Step:

We must show that P($k + 1$) is true: $m + (k + 1) = (k + 1) + m$.

$m + (k + 1)$	$= (m + k) + 1$	by the associative property of addition
	$= (k + m) + 1$	by the induction hypothesis
	$= k + (m + 1)$	by the associative property of addition
	$= k + (1 + m)$	by the basis step
	$= (k + 1) + m$	by the associative property of addition

Thus, we have shown P($k + 1$) to be true.

By the principle of mathematical induction, we have shown P(n) to be true for every natural number n. Ergo, the conjecture is true.

Q.E.D.

Note on Homework Assistance:

Selected answers and hints may be downloaded by any student from the same website as this textbook.

A full solutions manual is available to verified professors and high school teachers and will be sent to the linked educational/.edu e-mail address.

1.6: Exercises

1. Consider the six ballots cast below in an election with three candidates (A, B, C); the candidate with the highest number of first-place votes wins.

Voter One		Voter Two		Voter Three	
1st:	A	1st:	C	1st:	C
2nd:	B	2nd:	A	2nd:	B
3rd:	C	3rd:	B	3rd:	A
Voter Four		**Voter Five**		**Voter Six**	
1st:	C	1st:	A	1st:	B
2nd:	A	2nd:	C	2nd:	C
3rd:	B	3rd:	B	3rd:	A

A. Convert each ballot into a mathematical statement of each voter's preferences using the "greater than" operator.

B. Suppose we wish to streamline this ordering information into only one recorded piece of information per voter. Thus, we will only record a voter's top choice, i.e. each person's official vote will be his/her first choice. How many votes does each candidate receive? Who wins the election?

Suppose that Candidate C drops out of the race on election night due to a family health crisis. This would change the voter preferences as below.

Voter One		Voter Two		Voter Three	
1st:	A	1st:	A	1st:	B
2nd:	B	2nd:	B	2nd:	A
Voter Four		**Voter Five**		**Voter Six**	
1st:	A	1st:	A	1st:	B
2nd:	B	2nd:	B	2nd:	A

C. Redo Parts A and B using the modified ballots without Candidate C.

NOTE:

To accomplish Part C, you can simply delete Candidate C and one "greater than" operator from either side of Candidate C from your work in Part A. It is unnecessary to create the streamlined ballots above Part C.

2. Consider the fifteen ballots cast in an election with four candidates (A, B, C, D) below; again, the candidate with the highest number of first-place votes wins.

Voter One		Voter Two		Voter Three	
1st:	A	1st:	C	1st:	C
2nd:	B	2nd:	B	2nd:	D
3rd:	C	3rd:	A	3rd:	B
4th:	D	4th:	D	4th:	A
Voter Four		Voter Five		Voter Six	
1st:	C	1st:	A	1st:	C
2nd:	B	2nd:	B	2nd:	B
3rd:	A	3rd:	C	3rd:	A
4th:	D	4th:	D	4th:	D
Voter Seven		Voter Eight		Voter Nine	
1st:	A	1st:	C	1st:	D
2nd:	D	2nd:	D	2nd:	C
3rd:	C	3rd:	B	3rd:	B
4th:	B	4th:	A	4th:	A
Voter Ten		Voter Eleven		Voter Twelve	
1st:	C	1st:	C	1st:	A
2nd:	B	2nd:	D	2nd:	B
3rd:	A	3rd:	B	3rd:	C
4th:	D	4th:	A	4th:	D
Voter Thirteen		Voter Fourteen		Voter Fifteen	
1st:	C	1st:	A	1st:	A
2nd:	B	2nd:	B	2nd:	D
3rd:	A	3rd:	C	3rd:	C
4th:	D	4th:	D	4th:	B

A. Convert each ballot into a mathematical statement of each voter's preferences using the "greater than" operator.

B. So that you are not repeating the same mathematical statements over and over, tally the voter preferences in the table below.

Voter Preference	Number of Voters with that Preference

C. As in #1, suppose we discard full voter preference information and only record each voter's top choice as their official vote. How many votes does each candidate receive? Who wins the election?

D. As in #1, suppose that unlucky Candidate C drops out. Make a new table of ballots using this new information. Redo Parts A-C using these new ballots and the reduced field of candidates.

3. Let's consider how many different voter preferences are theoretically possible in various elections. In Parts A-D, give the total number of distinct voter preferences that are theoretically possible AND list them all using the "greater than" operator.

A. Two Candidates: A and B
B. Three Candidates: A, B, and C
C. Four Candidates: A, B, C, and D
D. Is the total number of distinct voter preferences a combination or a permutation? Justify your choice.
E. Give a formula for the total number of distinct voter preferences if there are *n* candidates.

NOTE: Of course, just because a voter preference is theoretically possible does not mean that it will be picked by any voters in an election.

4. Definition: A set is **closed** under an operation if it contains all members of the set produced by that operation acting on members of that set.

Upshot:

- If performing the operation on any two members of the set produces an "answer" that is also a member of that set, then the set is closed under that operation.
- If there is a pair of objects from the set which produces an "answer" using the operation that is not an object in the set, then the set is not closed under that operation.

Let's explore closure for our basic arithmetic operations for different sets of numbers.

A. Is the set of rational numbers closed under addition or not?
B. Is the set of natural numbers closed under subtraction or not?
C. Is the set of integers closed under subtraction or not?
D. Is the set of whole numbers closed under multiplication or not?
E. Is the set of rational numbers closed under division or not?

Of course, you could explore closure for each set under each operation. Feel free to pick up where this homework stops if it intrigues you!

5. Suppose that an election is held where the candidate with the **lowest** number of votes wins the election. For each of our four voting properties, decide whether or not this voting system possesses that property. Justify each decision.

6. An election has 3,575 votes, and there are two candidates on the ballot.
A. What is the majority threshold for this election?
B. Interpret what this number means.
C. Candidate X receives 1,787 votes. Is X the majority candidate for this election?
D. Candidate Y receives the remainder of the votes. Is Y the majority candidate for this election?
E. Who is the Condorcet candidate for this election? If there is none, explain.

SECTION 2: PLURALITY VOTING SYSTEMS

Now that we have established our basic definitions of desirable properties for voting systems to have, we need to explore the mechanics of those different voting systems. We need to see how to convert individual voter preferences from the ballots into the overall societal preference.

2.1: Plurality

We will start with the simplest voting system: plurality. Basically, we will discard any ranking information that we might have and just count the **first**-place votes. In a crowded field of candidates, a voter might have a second choice or a third most preferred candidate, but those preferences are irrelevant for the plurality vote-counting process. You focus only on the **top** preference of voters. This matches the way that many ballots are cast in this country: you don't get to register your opinion on the full suite of candidates but instead make one choice per race.

> In the **plurality voting system**, voters are allowed one choice per race on the ballot, and whichever candidate receives the most votes is declared the winner.
>
> If a partial or total ranking of candidates is needed for a result instead of a simple winner, then the candidate with the second highest number of votes is declared second in the rankings, the candidate with the third highest number of votes is declared third in the rankings, etc.

Discussion Questions

- If a candidate is the majority candidate, then must he/she be the plurality winner?
- If a candidate is the plurality winner, then must he/she be the majority candidate?

Let's explore two U.S. Senate races from 2014: North Carolina and Virginia. Both results raise questions about the plurality voting system. Republican Thom Tillis was the victor in North Carolina, and Democrat Mark Warner was the victor in Virginia. See the official results from each state's Board of Elections below.

North Carolina's 2014 US Senate Race

Candidate	Political Party	Number of Votes	Percentage of Votes
Thom Tillis	Republican	1,423,259	48.82%
Kay Hagan	Democrat	1,377,651	47.26%
Sean Haugh	Libertarian	109,100	3.74%
Write-In Candidates		5,271	0.18%
	TOTAL	2,915,281	100.00%

Source: NC State Board of Elections (enr.ncsbe.gov, accessed 11-9-2015)

Virginia's 2014 US Senate Race

Candidate	Political Party	Number of Votes	Percentage of Votes
Mark Warner	Democrat	1,073,667	49.15%
Ed Gillespie	Republican	1,055,940	48.34%
Robert Sarvis	Libertarian	53,102	2.43%
Write-In Candidates		1,811	0.08%
	TOTAL	2,184,520	100.00%

Source: VA State Board of Elections (elections.virginia.gov, accessed 11-9-2015)

These states use the plurality voting method. Thom Tillis won with a plurality of 48.82% in North Carolina, and Mark Warner won with a plurality of 49.15% in Virginia.

Opponents of the plurality voting method would use these results to call for a different voting method!

- How can it be fair for Tillis to become North Carolina's Senator when more than half of the voters (51.18%) in the state picked a different candidate?

- How can it be fair for Warner to become Virginia's Senator when more than half of the voters (50.85%) in the state picked a different candidate?

Though these men are plurality winners, they are not majority candidates, according to our definition from the last section.

Let's contrast this voting system with the one used in Georgia. In our state, we do not use the plurality system. In Georgia, the winner must be the majority candidate. If no majority candidate emerges in the election, then a run-off election occurs between the two candidates with the highest number of votes. Such run-offs are not without issues, though. There are costs associated with run-off elections – both for the government and for the citizenry. Also, the number of voters who return weeks later to cast ballots in run-off elections is notoriously low.

<u>Discussion on the North Carolina/Virginia vs Georgia Election Systems</u>

> Compare and contrast the two election systems with pros and cons for each.
> Which one do you feel is "more fair?"

As mentioned, though run-off elections will produce a majority candidate as the winner, there are costs associated with that system. Opponents could easily argue that if we simply asked the voters to register their full preferences by ranking candidates at the original election then only that one election would be needed. Any run-off election could be decided by looking at the full preferences of voters without forcing the voters to return to the polls for a separate run-off election. This modification is called the plurality with elimination voting system.

2.2: Plurality with Elimination

In the **plurality with elimination voting system**, voters give a full ranking of their preferences in each race on the ballot.

If there is a majority candidate, then he/she wins.

If there is no majority candidate, then these full voter preferences are used to eliminate candidates until a winner results.

To start the eliminations, the candidate with the lowest number of first-place votes is eliminated. Each of his/her votes is transferred to the candidate with the next highest preference on that voter's ballot. A recount is executed with that lowest candidate deleted and his/her votes reallocated. If a majority candidate now emerges, then he/she is the winner of the election. If not, then the elimination process repeats until there is a winner.

If a partial or total ranking of candidates is needed for a result instead of a simple winner, then the candidates are ranked in the reverse order of their elimination. The last candidate to be eliminated takes second place; the second to last to be eliminated takes third place; etc.

NOTE: If two or more candidates have the same lowest number of votes in an elimination procedure, then they are all eliminated. If doing so would remove all candidates, then clearly this voting method fails and another must be chosen.

Plurality with Elimination Example

Suppose we have an election with four candidates (A, B, C, D) using the plurality with elimination as the voting system. There are 16 voters, and rather than work with their individual ballots, their preferences have been converted into the mathematical order notation, as in your previous homework, and tallied.

A > B > C > D:	4 voters
A > C > B > D:	3 voters
B > A > D > C:	3 voters
B > C > A > D:	2 voters
B > D > C > A:	2 voters
C > D > A > B:	1 voter
D > A > B > C:	1 voter

With an even number of 16 voters, a candidate would need nine votes to win a majority:

Majority Threshold = (50% of n) + 1 = (50% of 16) + 1 = 8 + 1 = 9.

No candidate has met the majority threshold: A and B have seven votes each, and C and D have one vote each. Note that under a regular plurality system, A and B would tie in the election.

For our first elimination round, we remove the candidate(s) with the lowest number of first place votes. Candidates C and D have only one first place vote each, and thus are tied at the bottom. They are both eliminated, and their votes will be reallocated.

For the voter who preferred Candidate C most, we have this voter preference: C > D > A > B. This one vote will be reallocated to Candidate A. Since C and D have been eliminated, the next most preferred candidate is A. In other words, we streamline the original voter preference into a new one by deleting the eliminated candidates. So, C > D > A > B becomes A > B, and Candidate A gets the reallocated vote.

For the voter who preferred Candidate D most, we have this voter preference: D > A > B > C. When the choice of D is eliminated, this vote is reallocated to Candidate A as well. Again, eliminating Candidate D at the top of the preference order and Candidate C at the bottom will result in a new streamlined voter preference of A > B again.

Thus, after the elimination of Candidates C and D, we convert the original voter preferences on the left into new ones on the right.

ORIGINAL VOTER PREFERENCES		NEW VOTER PREFERENCES
A > B > C > D:	4 voters	A > B
A > C > B > D:	3 voters	A > B
B > A > D > C:	3 voters	B > A
B > C > A > D:	2 voters	B > A
B > D > C > A:	2 voters	B > A
C > D > A > B:	1 voter	A > B
D > A > B > C:	1 voter	A > B

We re-calculate the totals. Candidate A has 4 + 3 + 1 + 1 = 9 votes (meeting the majority threshold), and unfortunately for Candidate B, he/she only has 3 + 2 + 2 = 7 votes.

While A and B would tie under the plurality voting system, A would win outright in the plurality with elimination voting system.

<u>Discussion on Actual Run-Off Elections vs. Plurality with Elimination</u>

- What are the pros and cons of having actual run-off elections to arrive at a majority candidate (hence winner)?
- What are the pros and cons of the plurality with elimination system?
- Which do you prefer?

2.3: Exercises

1. For this series of problems, you will be asked to create a voting scenario to accomplish the stated objective. The voting method is simple plurality. You must decide upon a total number of voters and allocate explicit votes to each candidate.

   A. With two candidates, construct a scenario for a tie between them.
   B. With three candidates, construct a scenario for a tie for the three of them.
   C. With three candidates, construct a scenario for a tie between two of them.
   D. With four candidates, construct a scenario for a tie for the four of them.
   E. With four candidates, construct a scenario for a tie between three of them.
   F. With four candidates, construct a scenario for a tie between two of them.

2. Which of our four voting fairness properties does plurality possess, if any?

3. Which of our four voting fairness properties does plurality with elimination possess, if any?

4. The Mathematics Club needs to decide on a foreign destination for its Spring Break trip. The members ranked the given trip destinations from most favorite to least favorite.

Voters:	Twenty-four student members of the Mathematics Club			
Candidates:	Argentina (A), Italy (I), and Morocco (M)			
Voter Preferences:	A > I > M:	5	A > M > I:	3
	I > A > M:	5	I > M > A:	4
	M > A > I:	4	M > I > A:	3

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?

5. The Dining Hall wants to know what fresh fruit is the most popular student snack.

Voters:	583 students surveyed in the Dining Hall			
Candidates:	Apples (A), Bananas (B), and Oranges (O)			
Voter Preferences:	A > B > O:	152	A > O > B:	47
	B > A > O:	156	B > O > A:	52
	O > A > B:	72	O > B > A:	104

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?

6. A local children's hospital explores the popularity of various ice cream flavors.

Voters:	Seventy-five patients at the local children's hospital	
Candidates:	Birthday Cake (B), Chocolate (C), and Cookie Dough (D)	
Voter Preferences:	B > C > D: 9	B > D > C: 7
	C > B > D: 19	C > D > B: 25
	D > B > C: 5	D > C > B: 10

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?

7. The Student Government asked a random sample of students where they liked to study.

Voters:	400 students from the random sample	
Candidates:	Empty Classroom (C), Library (L), and Student Center (S)	
Voter Preferences:	C > L > S: 126	C > S > L: 118
	L > C > S: 99	L > S > C: 30
	S > C > L: 16	S > L > C: 11

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?

8. A neighborhood committee has raised enough money to add a new feature to this small community. All homeowners were asked to vote on their building preferences.

Voters:	Twenty-seven homeowners in this community	
Candidates:	Pool (O), Playground (P), Tennis Court (T), and Paved Walking Paths (W)	
Voter Preferences:	O > P > T > W: 7	T > P > O > W: 7
	W > T > P > O: 8	P > O > T > W: 5

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?

9. An up-and-coming designer is trying to choose a signature color for her new clothing line. In making her decision, she surveys 200 potential customers who fit her marketing profile at a local shopping center.

Voters: 200 potential customers
Candidates: Green (G), Pink (P), Red (R), and Purple (U).
Voter Preferences:

U > R > G > P: 55	G > U > P > R: 27
U > G > R > P: 45	R > P > U > G: 42
R > U > G > P: 30	P > U > G > R: 1

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?

10. During an annual dog show, a team of reporters asked those in attendance to rank their dog breed preferences from a small selection of toy breeds.

Voters: 1,374 people in attendance at the dog show
Candidates: Havanese (H), Cavalier King Charles Spaniel (K), Maltese (M), Papillon (P), and Shih Tzu (S)
Voter Preferences:

H > M > P > S > K: 241	M > H > K > P > S: 212
H > P > S > M > K: 189	M > P > H > S > K: 219
K > H > M > S > P: 123	P > H > K > M > S: 187
S > K > H > M > P: 104	P > K > S > H > M: 99

A. Who is the Condorcet candidate? If there is none, then explain why not.
B. Who is the winner under plurality?
C. Who is the winner under plurality with elimination?

11. The "Angry Mathematicians for Social Change" political party is trying to decide the right adjective for its party name. The group can either stay "angry" or use one of several synonyms instead. All party members at the most recent convention were surveyed.

Voters: 12,008 party members at the most recent convention
Candidates: Angry (A), Enraged (E), Heated (H), Irate (I), and Outraged (O)
Voter Preferences:

A > E > I > O > H: 1,567	E > H > I > O > A: 1,141
A > I > H > O > E: 337	H > A > I > O > E: 1,531
H > I > E > A > O: 3,003	H > O > A > I > E: 1,789
I > H > E > O > A: 2,640	

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the winner under plurality?
C. Who is the winner under plurality with elimination?

12. Every Mother's Day, a marketing group researches which flowers are most popular as floral gifts for mothers in Georgia.

Voters:	53,496 surveyed Georgians who bought flowers for Mother's Day	
Candidates:	Carnations (C), Lilacs (L), Orchids (O), Roses (R), and Tulips (T)	
Voter Preferences:	R > L > T > O > C: 8,531	R > T > O > C > L: 9,785
	L > R > O > T > C: 10,832	O > T > C > R > L: 11,312
	O > L > R > T > C: 8,096	C > T > R > L > O: 4,940

A. Who is the winner under plurality?
B. Who is the winner under plurality with elimination?

The author extends his profound appreciation to the colleague and students who suggested these homework questions.

For #8:
Dr. Paul Koester, recently a Lecturer for the Department of Mathematics at the University of Kentucky, created the example which became this problem.

For #4-7 and #9-12:
The cheerful, hard-working students in the author's two sections of the course "Great Ideas in Modern Mathematics" at Oglethorpe University in Fall 2015 were the second group of students to pilot using these materials. They suggested the voting scenarios explored in these problems.

Thank you, Paul and my dear students!

SECTION 3: BORDA COUNT VOTING SYSTEMS

3.1: Extremely Brief Historical Context

The two voting systems we have explored so far focus on **first**-place votes.

In a plurality system, only the **first**-place votes are counted; in a plurality with elimination system, though preferences lower than first-place are used to re-allocate votes after an elimination, only the **first**-place votes determine the winner.

In this section, we will begin a shift to voting systems that use a voter's **full** preferences. Such voting systems were championed by two French mathematicians: Borda and Condorcet. We will start with Borda in this section and cover Condorcet in the next section.

> The eighteenth century was a period of enlightenment throughout the Old and New World. France, the United States, and Poland granted themselves constitutions. Nations were in upheaval as their citizens started demanding equal justice for all, showing concern for human rights, and calling for a regulation of the social order. At the same time, demands for quality government arose and the question of how officials were to be elected to high positions became important again.
>
> In this atmosphere two eminent French thinkers appeared on the scene. One was a military officer with numerous distinctions in land and sea battles. His name was Chevalier Jean-Charles de Borda. The other was the nobleman Marquis de Condorcet.
>
> The two men, outstanding scientists in Paris during the time of the French Revolution, did something amazing: they reinvented the election methods that Llull and Cusanus had proposed a few hundred years earlier. Actually, they did more than that: they provided the appropriate mathematical underpinnings. At odds with each other on many subjects, they also engaged in a lively debate on the theory of voting and elections.

Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present,
George G. Szpiro, Princeton University Press, 2010, p. 60

(Pronunciation Note: Borda = Bore-Dah Condorcet = Con-Dore-Say)

3.2: The Borda Count Voting System

In the Borda count voting system, a candidate is awarded points based on the full preferences of each voter. A candidate receives points from a ballot whether he/she is the first-place choice or not. Of course, the higher the preference by the voter, the more points the candidate earns.

In the **Borda count voting system**, voters give a full ranking of their preferences for each race on the ballot. Borda points are awarded to candidates from each voter's ballot based on the candidate's ranking on that ballot. In an election with *n* candidates, a first-place ranking on a ballot earns the candidate *n* Borda points, a second-place ranking earns the candidate $n - 1$ Borda points, and so forth. The candidate who is ranked last on the ballot earns just one Borda point for that ballot. Each candidate's Borda points are totaled over all ballots, and the candidate with the highest overall number of Borda points wins.

If a partial or total ranking of candidates is needed for a result instead of a simple winner, then the candidates are ranked from the highest number of total Borda points to the lowest number of total Borda points.

3.3: Example

Let's suppose that the Student Government has the three options below for the theme of the Homecoming dance.

- An athletic theme (Option A)
- A romantic theme (Option B),
- A "Game of Thrones©" theme (Option C)

Rather than use the "winner takes all" approach of plurality, the students decide to experiment with the Borda count voting system.

Here are the tallies of the voter preferences.

A > B > C 175 students	A > C > B 200 students	B > A > C 200 students
B > C > A 225 students	C > A > B 150 students	C > B > A 50 students

As a review, using a plurality system, the romantic theme (Option B) would win with 200 + 225 = 425 first-place votes. Under plurality, the athletic theme (Option A) would come in second with 175 + 200 = 375 first-place votes, and the "Game of Thrones" theme (Option C) would come in last with 150 + 50 = 200 first-place votes.

But like the senatorial races discussed in the previous section, one could dispute the fairness of declaring the romantic theme (Option B) the winner: a majority of 575 students prefer another option! What do you think? Now, let's turn to the Borda count voting system.

In this **three**-option race, a first-place ranking earns **three** Borda points, a second-place ranking earns two Borda points, and a third-place ranking (last place) earns just one Borda point.

- Option A would receive three Borda points from **each** of the 375 ballots where it was ranked **first**-place: **A** > B > C (175 students) and **A** > C > B (200 students).

- Option A would receive two Borda points from **each** of the 350 ballots where it was ranked **second**-place: B > **A** > C (200 students) and C > **A** > B (150 students).

- Option A would receive one Borda point from **each** of the 275 ballots where it was ranked **third**-place: B > C > **A** (225 students) and C > B > **A** (50 students).

Tallying these points over **all** ballots gives the result below.

$$\begin{aligned}&\text{Total Borda points for Option A}\\&=175\cdot 3+200\cdot 3+200\cdot 2+150\cdot 2+225\cdot 1+50\cdot 1\\&=525+600+400+300+225+50\\&=2{,}100\end{aligned}$$

We can calculate the Borda totals for the other two options similarly.

Total Borda points for Option B $=200\cdot 3+225\cdot 3+175\cdot 2+50\cdot 2+200\cdot 1+150\cdot 1$ $=600+675+350+100+200+150$ $=2,075$	Total Borda points for Option C $=150\cdot 3+50\cdot 3+200\cdot 2+225\cdot 2+175\cdot 1+200\cdot 1$ $=450+150+400+450+175+200$ $=1,825$

The winner via the Borda count voting system would be the athletic theme (Option A) with the plurality winner of the romantic theme (Option B) coming in second. In both voting systems, the "Game of Thrones" theme (Option C) comes in last place.

3.4: Underlying Assumptions with Borda Points and Potential Modifications

As you move up the steps of the Borda point system, each ranking is only separated by the previous ranking by one point. Last place earns one point, next to last place earns two points, etc. This is a lovely linear pattern with the same separation of values whether one is at the top of the rankings, middle of the rankings, or bottom of the rankings. After all, one of the guiding principles of the French revolution was equality: Liberté, Egalité, and Fraternité!

But, are a voter's preferences evenly spaced at all preference levels? Should this one-point difference carry through the entire voter preference calculation?

In the classical Borda point system, we make that explicit assumption: there is always a one-point separation in the rankings. However, we are free to modify the Borda point system as we please. We can tailor the calculations to whatever assumptions that the given group of voters is willing to make.

Let's explore a very crowded election with 10 candidates through a classical Borda point system and a modified one.

Ranking	Classical Borda Point System	One Potential Modified Borda Count System
1st Place	10	16
2nd Place	9	12
3rd Place	8	9
4th Place	7	7
5th Place	6	6
6th Place	5	5
7th Place	4	4
8th Place	3	3
9th Place	2	2
10th Place	1	1

In our modified system, among the "lesser" preferences, there is the usual one-point separation, but bigger separations distinguish the top three and give those candidates additional "power."

- Third-place is separated from its next lowest ranking by two points.
- Second-place is separated from its next lowest ranking by three points.
- First-place is separated from its next lowest ranking by four points.

3.5: Exercises

1. Which of our four voting fairness properties does the Borda count voting system possess, if any?

2. Write a short biographical sketch of Borda.

3. Research the differences and similarities between Borda's method and that of medieval Spanish theologian Ramon Llull (Raimundo Lulio). Should we retroactively charge Borda with plagiarism of Llull's ideas?

4. Do you agree or disagree with the underlying equality assumption in the classical Borda count system? In other words, should the one-point separation in point values extend uniformly at <u>every</u> level of the rankings?

5. We have presented just one alternative Borda count system in Section 3.4. Create a different one of your own and justify it.

For #6-14, recall the winners under our previous two voting systems from the homework for Section 2 and our work there involving majority and Condorcet candidates. This information should be helpful in your explorations of the fairness properties in #1.

Again, the author extends grateful thanks to colleague Dr. Paul Koester from the University of Kentucky and also to his students at Oglethorpe University for suggesting the voting scenarios in these homework problems!

6. The Mathematics Club needs to decide on a foreign destination for its Spring Break trip. The members ranked the given trip destinations from most favorite to least favorite.

Voters:	Twenty-four student members of the Mathematics Club	
Candidates:	Argentina (A), Italy (I), and Morocco (M)	
Voter Preferences:	A > I > M: 5	A > M > I: 3
	I > A > M: 5	I > M > A: 4
	M > A > I: 4	M > I > A: 3

A. Who is the winner under the Borda count voting system?
B. Give the full rankings for the results of this election.

7. The Dining Hall wants to know what fresh fruit is the most popular student snack.

Voters:	583 students surveyed in the Dining Hall	
Candidates:	Apples (A), Bananas (B), and Oranges (O)	
Voter Preferences:	A > B > O: 152	A > O > B: 47
	B > A > O: 156	B > O > A: 52
	O > A > B: 72	O > B > A: 104

Who is the winner under the Borda count voting system?

8. A local children's hospital explores the popularity of various ice cream flavors.

Voters:	Seventy-five patients at the local children's hospital	
Candidates:	Birthday Cake (B), Chocolate (C), and Cookie Dough (D)	
Voter Preferences:	B > C > D: 9	B > D > C: 7
	C > B > D: 19	C > D > B: 25
	D > B > C: 5	D > C > B: 10

A. Who is the winner under the Borda count voting system?
B. Give the full rankings for the results of this election.

9. The Student Government asked a random sample of students where they liked to study.

Voters:	400 students from the random sample	
Candidates:	Empty Classroom (C), Library (L), and Student Center (S)	
Voter Preferences:	C > L > S: 126	C > S > L: 118
	L > C > S: 99	L > S > C: 30
	S > C > L: 16	S > L > C: 11

Who is the winner under the Borda count voting system?

10. A neighborhood committee has raised enough money to add a new feature to this small community. All homeowners were asked to vote on their building preferences.

Voters: Twenty-seven homeowners in this community
Candidates: Pool (O), Playground (P), Tennis Court (T), and Paved Walking Paths (W)
Voter Preferences: O > P > T > W: 7 T > P > O > W: 7
W > T > P > O: 8 P > O > T > W: 5

A. Who is the winner under the Borda count voting system?
B. Give the full rankings for the results of this election.

11. An up-and-coming designer is trying to choose a signature color for her new clothing line. In making her decision, she surveys 200 potential customers who fit her marketing profile at a local shopping center.

Voters: 200 potential customers
Candidates: Green (G), Pink (P), Red (R), and Purple (U).
Voter Preferences: U > R > G > P: 55 G > U > P > R: 27
U > G > R > P: 45 R > P > U > G: 42
R > U > G > P: 30 P > U > G > R: 1

Who is the winner under the Borda count voting system?

12. During an annual dog show, a team of reporters asked those in attendance to rank their dog breed preferences from a small selection of toy breeds.

Voters: 1,374 people in attendance at the dog show
Candidates: Havanese (H), Cavalier King Charles Spaniel (K), Maltese (M), Papillon (P), and Shih Tzu (S)
Voter Preferences:
H > M > P > S > K: 241 M > H > K > P > S: 212
H > P > S > M > K: 189 M > P > H > S > K: 219
K > H > M > S > P: 123 P > H > K > M > S: 187
S > K > H > M > P: 104 P > K > S > H > M: 99

A. Who is the winner under the Borda count voting system?
B. Give the full rankings for the results of this election.

13. The “Angry Mathematicians for Social Change” political party is trying to decide the right adjective for its party name. The group can either stay “angry” or use one of several synonyms instead. All party members at the most recent convention were surveyed.

Voters: 12,008 party members at the most recent convention
Candidates: Angry (A), Enraged (E), Heated (H), Irate (I), and Outraged (O)
Voter Preferences:

A > E > I > O > H: 1,567	E > H > I > O > A: 1,141
A > I > H > O > E: 337	H > A > I > O > E: 1,531
H > I > E > A > O: 3,003	H > O > A > I > E: 1,789
I > H > E > O > A: 2,640	

Who is the winner under the Borda count voting system?

14. Every Mother’s Day, a marketing group researches which flowers are most popular as floral gifts for mothers in Georgia.

Voters: 53,496 surveyed Georgians who bought flowers for Mother’s Day
Candidates: Carnations (C), Lilacs (L), Orchids (O), Roses (R), and Tulips (T)
Voter Preferences:

R > L > T > O > C: 8,531	R > T > O > C > L: 9,785
L > R > O > T > C: 10,832	O > T > C > R > L: 11,312
O > L > R > T > C: 8,096	C > T > R > L > O: 4,940

A. Who is the winner under the Borda count voting system?
B. Give the full rankings for the results of this election.

SECTION 4: PAIRWISE COMPARISONS VOTING SYSTEMS

In this section, we will finish with the voting systems introduced by the two prominent voting theorists from the French Revolution. We have already considered Borda's system of counts which utilizes the full preferences of voters, and in this section, we will consider a contemporary's rival system, which also utilizes the full preferences of voters but in a different way. This voting system was championed by the Marquis de Condorcet.

4.1: The Pairwise Comparisons Voting System

With a two-person election, there will either be a majority winner or a tie. So, with only two candidates in a race, a plurality winner is the same as a majority winner, and life is simple. But, elections get more complicated with more than two candidates. A plurality winner may not be the majority candidate, as seen with the senatorial results discussed earlier. Condorcet's voting system re-imagines an election with more than two candidates as the culmination of a series of simpler two-candidate races. Each candidate will "face off" against every other candidate using the full preferences of the voters to decide these pairwise contests. Then aggregating over all such races gives a winner. The method builds a larger result out of simpler, smaller pieces.

In the **pairwise comparisons voting system**, voters give a full ranking of their preferences for each race on the ballot. A series of pairwise races is imagined among all candidates.

- The winner of each two-candidate race is the majority candidate, and he/she earns one Condorcet point for such a victory.
- The loser of each two-candidate race earns zero Condorcet points for such a loss.
- In the case of a tie, each of the two candidates receives one-half of a Condorcet point.

Each candidate's Condorcet points are totaled over all pairwise contests, and the candidate with the highest overall number of Condorcet points wins.

If a partial or total ranking of candidates is needed for a result instead of a simple winner, then the candidates are ranked from the highest number of total Condorcet points to the lowest number of total Condorcet points.

This voting system is also known as the **head-to-head voting system** or the **Condorcet voting system**.

4.2: The Homecoming Example Revisited

Recall the Student Government example from the previous section involving the three options for the theme of the Homecoming dance: an athletic theme (Option A), a romantic theme (Option B), and a "Game of Thrones" theme (Option C).

Let's imagine that, again, the students decide not to use the "winner takes all" approach of plurality. Instead, they decide to experiment with the pairwise comparisons voting system. Recall the tallies of the voter preferences.

A > B > C 175 students	A > C > B 200 students	B > A > C 200 students
B > C > A 225 students	C > A > B 150 students	C > B > A 50 students

With these three candidates (A, B, and C), there are three pairwise comparisons, shown below.

- A vs. B
- A vs. C
- B vs. C

We much convert the given three-candidate voter preferences into theoretical two-candidate preferences. In the homework for Section 1, we explored simplifying voter preferences when a candidate "drops out," and we will operate in a similar fashion to run our theoretical pairwise races. Take the original three-candidate voter preference and delete the letter for the eliminated candidate and one "greater than" operator from either side of the deleted letter. This gives the new theoretical two-candidate or pairwise voter preference.

Pairwise Race 1: A vs. B		
Original Three-Candidate Preferences	Number of Students	Pairwise Preferences
A > B > C	175	A > B
A > C > B	200	A > B
B > A > C	200	B > A
B > C > A	225	B > A
C > A > B	150	A > B
C > B > A	50	B > A

In this hypothetical pairwise race, Option A would receive 525 votes (175 + 200 + 150 = 525), and Option B would receive 475 votes (200 + 225 + 50 = 475).

Thus, Option A is the majority candidate (explain why) and thus wins the pairwise race.

Option A receives one Condorcet point for this face-off.

Pairwise Race 2: A vs. C		
Original Three-Candidate Preferences	Number of Students	Pairwise Preferences
A > B > C	175	A > C
A > C > B	200	A > C
B > A > C	200	A > C
B > C > A	225	C > A
C > A > B	150	C > A
C > B > A	50	C > A

In this hypothetical pairwise race, Option A would receive 575 votes (175 + 200 + 200 = 575), and Option C would receive 425 votes (225 + 150 + 50 = 425).

Again, Option A is the majority candidate (explain why) and thus wins the pairwise race.

Option A receives one Condorcet point for this face-off.

Pairwise Race 3: B vs. C		
Original Three-Candidate Preferences	Number of Students	Pairwise Preferences
A > B > C	175	B > C
A > C > B	200	C > B
B > A > C	200	B > C
B > C > A	225	B > C
C > A > B	150	C > B
C > B > A	50	C > B

In this hypothetical pairwise race, Option B would receive 600 votes (175 + 200 + 225 = 600), and Option C would receive 400 votes (200 + 150 + 50 = 400).

Option B is the majority candidate (explain why) and thus wins the pairwise race.

Option B receives one Condorcet point for this face-off.

To conclude, Option A has two Condorcet points, Option B has one Condorcet point, and Option C has zero Condorcet points. Under the pairwise comparison voting method, Option A wins since it has the most total Condorcet points.

Summary of Results over Different Voting Systems

		Voting System		
		Plurality	Borda Count	Pairwise Comparison
Ranking	First Place	B	A	A
	Second Place	A	B	B
	Third Place	C	C	C

4.3: A Practical Shortcoming of the Pairwise Comparisons Voting Method

One practical difficulty with this voting method is the amount of work to be done. For a small number of candidates, the work is manageable, but for a large number of candidates, the work becomes unreasonable. Let's explore using basic combinatorics. Suppose that the total number of candidates in our election is *n*. To complete our hypothetical pairwise races, we choose candidates two at a time.

Of course, a candidate does not run against himself/herself; we are assuming that we pick two different candidates. Thus, repeats are not allowed in our selection, and we disregard illegitimate match-ups like A vs. A, B vs. B, etc. We only care which two candidates are picked and not the order in which they are listed: A vs. B is the same pairwise comparison as B vs. A. Thus, order is not important in our selection.

Ergo, we are counting via a combination and can use the combination formula from Probability.

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

Since we are taking candidates two at a time, we choose *r* to be two, and the formula simplifies.

$$ \text{Number of Pairwise Comparisons} = {}_nC_2 = \frac{n!}{(n-2)!2!} = \frac{n\cdot(n-1)\cdot\cancel{(n-2)!}}{\cancel{(n-2)!}\cdot 2\cdot 1} = \frac{n\cdot(n-1)}{2} $$

Direct calculations reveal how the number of pairwise comparisons grows with the number of candidates.

Total Number of Candidates	Number of Pairwise Comparisons
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55

Though the work involved in each pairwise comparison is not difficult (as seen in our example above), the amount of work to be done becomes distasteful as the number of those pairwise comparisons increases. This increasing amount of work would certainly argue against choosing this voting method when faced with a large field of candidates.

4.4: Exercises

1. Which of our four voting fairness properties does the pairwise comparisons voting system possess, if any?

2. Let's refresh your plurality voting skills. Decide the winning theme for the homecoming dance if the Student Government chose to use **plurality with elimination** as the voting system. See Sections 3.3 and 4.2 above for reminders on the voting options and totals.

For #3-9, recall the winners under our previous two voting systems from the homework for Section 2/3 and our work there involving majority and Condorcet candidates in Section 2. This information should be helpful in your explorations of the fairness properties in #1.

Again, the author extends grateful thanks to colleague Dr. Paul Koester from the University of Kentucky and also to his students at Oglethorpe University for suggesting the voting scenarios in these homework problems!

3. The Mathematics Club needs to decide on a foreign destination for its Spring Break trip. The members ranked the given trip destinations from most favorite to least favorite.

Voters:	Twenty-four student members of the Mathematics Club			
Candidates:	Argentina (A), Italy (I), and Morocco (M)			
Voter Preferences:	A > I > M:	5	A > M > I:	3
	I > A > M:	5	I > M > A:	4
	M > A > I:	4	M > I > A:	3

A. Who is the winner under the pairwise comparisons voting system?
B. Give the full rankings for the results of this election.

4. The Dining Hall wants to know what fresh fruit is the most popular student snack.

Voters:	583 students surveyed in the Dining Hall			
Candidates:	Apples (A), Bananas (B), and Oranges (O)			
Voter Preferences:	A > B > O:	152	A > O > B:	47
	B > A > O:	156	B > O > A:	52
	O > A > B:	72	O > B > A:	104

Who is the winner under the pairwise comparisons voting system?

5. A local children's hospital explores the popularity of various ice cream flavors.

Voters:	Seventy-five patients at the local children's hospital	
Candidates:	Birthday Cake (B), Chocolate (C), and Cookie Dough (D)	
Voter Preferences:	B > C > D: 9	B > D > C: 7
	C > B > D: 19	C > D > B: 25
	D > B > C: 5	D > C > B: 10

A. Who is the winner under the pairwise comparisons voting system?
B. Give the full rankings for the results of this election.

6. The Student Government asked a random sample of students where they liked to study.

Voters:	400 students from the random sample	
Candidates:	Empty Classroom (C), Library (L), and Student Center (S)	
Voter Preferences:	C > L > S: 126	C > S > L: 118
	L > C > S: 99	L > S > C: 30
	S > C > L: 16	S > L > C: 11

Who is the winner under the pairwise comparisons voting system?

7. A neighborhood committee has raised enough money to add a new feature to this small community. All homeowners were asked to vote on their building preferences.

Voters:	Twenty-seven homeowners in this community	
Candidates:	Pool (O), Playground (P), Tennis Court (T), and Paved Walking Paths (W)	
Voter Preferences:	O > P > T > W: 7	T > P > O > W: 7
	W > T > P > O: 8	P > O > T > W: 5

A. Who is the winner under the pairwise comparisons voting system?
B. Give the full rankings for the results of this election.

8. An up-and-coming designer is trying to choose a signature color for her new clothing line. In making her decision, she surveys 200 potential customers who fit her marketing profile at a local shopping center.

Voters:	200 potential customers	
Candidates:	Green (G), Pink (P), Red (R), and Purple (U).	
Voter Preferences:	U > R > G > P: 55	G > U > P > R: 27
	U > G > R > P: 45	R > P > U > G: 42
	R > U > G > P: 30	P > U > G > R: 1

Who is the winner under the pairwise comparisons voting system?

9. Every Mother's Day, a marketing group researches which flowers are most popular as floral gifts for mothers in Georgia.

Voters:	53,496 surveyed Georgians who bought flowers for Mother's Day	
Candidates:	Carnations (C), Lilacs (L), Orchids (O), Roses (R), and Tulips (T)	
Voter Preferences:	R > L > T > O > C: 8,531	R > T > O > C > L: 9,785
	L > R > O > T > C: 10,832	O > T > C > R > L: 11,312
	O > L > R > T > C: 8,096	C > T > R > L > O: 4,940

A. Who is the winner under the pairwise comparisons voting system?
B. Give the full rankings for the results of this election.

<u>Challenge Exercises</u>

Though the mathematics involved in this voting system is not difficult, the number of calculations becomes more intense with a large number of candidates, as discussed at the end of this section.

Accordingly, you have only been asked to do one problem with five candidates.

If you desire further practice, then take the other two questions from the previous section and calculate a winner with our new voting system. You might also be tempted to explore those two problems if you are interested in toy dog breeds or the political party of "angry" mathematicians.

SECTION 5: ARROW'S IMPOSSIBILITY THEOREM

Now that we have covered all the voting methods to be considered in our course, it is time to analyze their strengths and weaknesses using our fairness properties.

5.1: Group Work Summarizing Our Voting Systems' Properties

In each homework set, you were asked to decide which properties each voting system possessed. Summarize your findings below.

- Whenever a voting system possesses the property, you should write a full explanation arguing why it does. Then write **POSSESSES** in the appropriate cell of the table.
- Whenever a voting system does not possess the property, you should create a concrete counter-example demonstrating how it violates the property's definition. Then write **VIOLATES** in the appropriate cell of the table.

		Fairness Property			
		Majority	Condorcet	Monotonicity	Independence of Irrelevant Alternatives
Voting System	Plurality				
	Plurality with Elimination				
	Borda Count				
	Pairwise Comparisons				

5.2: Arrow's Startling Result

As mentioned at the beginning of our voting theory materials, the economist Kenneth Arrow proved a startling result about voting systems in 1951.

Arrow's Impossibility Theorem:
No voting method for an election with three or more candidates can possesses all four fairness properties.

5.3: Discussion

Each of the four fairness properties seems to be a desirable characteristic for a fair election. We naturally would prefer a voting system which possesses all of these properties. But if there are more than two candidates, this wish cannot be granted. According to Arrow, it is impossible!

Make a list of the pros and cons of each voting system. Knowing that none possesses all of the fairness properties, which do you prefer? Which do you feel is "most" fair?

5.4: Proof Examples

Recall that we have two different strategies for working with our fairness properties.

- If a voting system does **not** possess a fairness property, then all we must produce is **one counter-example** to demonstrate this failure.
- If a voting system **does** possess a fairness property, then one, two, or even many examples are inadequate to demonstrate this success. For these fairness properties, we cannot prove true via examples. Instead, we must show that, in every election there has ever been, is currently happening, or ever could be, the voting system has this fairness property. In short, we need a **mathematical proof**.

Beginning students sometimes struggle with this type of explanation and level of abstraction. Hang in there; you can do it! We will give two examples using different techniques (direct proof and proof by contradiction) to get you started with proving a voting system possesses a fairness property. The first homework problem in Section 5.5 below should provide a couple of counter-examples showing a voting system does not possess a fairness property. Explore #1 fully!

<u>THEOREM</u>: The plurality voting system possesses the Majority property.

<u>PROOF</u>: (By Contradiction)

We need to show that the majority candidate, if there is one, must be the winner under the plurality voting system.

Call the election's majority candidate M. (If there is no majority candidate, then there is nothing to discuss. The majority property is vacuously affirmed.) We will use a proof by contradiction to show that M must also be the winner under plurality.

Assume that a candidate different from M has the largest number of first-place votes and thus wins under plurality. Call him/her W for winner.

(If we can show that this assumption leads us to a mathematical contradiction, then it is a bad assumption and must be discarded. This means that no other candidate can have the largest number of first-place votes; so, the majority candidate M must have the largest number of first-place votes and be declared the winner under plurality.)

By the definition of majority candidate, M received more than 50% of the first-place votes.

By assumption, W is the plurality winner and received the largest number of first-place votes. This implies that W received more first-place votes than M; thus, W has also earned more than 50% of the first-place votes.

But this is impossible: we cannot have more than 100% of first-place votes. We have reached a contradiction and must discard our erroneous assumption.

Ergo, the majority candidate M must also be the winner under the plurality voting system: plurality possesses the majority property!

Q.E.D.

THEOREM: The Borda Count voting system possesses the monotonicity property.

PROOF: (By Direct Proof)

We need to show that the winner under the Borda Count voting system must remain the winner had any voters moved the candidate higher in their preferences on the ballot.

Call the election's winner W. By the definition of the Borda Count voting system, W received the largest total number of Borda Points.

If any voters moved W higher in their preferences on the ballot, then those voters must have analogously moved other candidate(s) lower on their preferences. Though overall elections may result in a tie, we do not allow individual voter preferences to tie: on the ballot only one candidate may be first choice, second choice, etc.

By the definition of the Borda Count voting system, moving higher in voter preferences earns more Borda Points, and moving lower in voter preferences earns less Borda Points.

W previously had the largest total number of Borda Points, and this total has increased.

The individual total of Borda Points for each of the other candidates either decreased or remained the same.

- If a candidate did not change in voter preferences, then his/her total number of Borda Points did not change, was already less than the original Borda Point total for W, and is less than the new Borda Point total for W.

- If a candidate moved lower in voter preferences, then his/her total number of Borda Points decreased, was already less than the original Borda Point total for W, and is even less than the new Borda Point total for W.

Previously, W had the highest number of total Borda Points and was the winner; with more support, W still has the highest number of total Borda Points and is still the winner.

Thus, the Borda Count voting system possesses the monotonicity property.

Q.E.D.

5.5: Exercises

1. Recall the problem explored throughout the previous homework sets.

   A neighborhood committee has raised enough money to add a new feature to this small community. All homeowners were asked to vote on their building preferences.

   Voters: Twenty-seven homeowners in this community
   Candidates: Pool (O), Playground (P), Tennis Court (T), and Paved Walking Paths (W)
   Voter Preferences: O > P > T > W: 7 T > P > O > W: 7
   W > T > P > O: 8 P > O > T > W: 5

   Summarize your previous findings. (Or do this work if you repeatedly skipped this problem in your earlier homework.)

A. Who is the majority candidate? If there is none, then explain why not.
B. Who is the Condorcet candidate? If there is none, then explain why not.
C. Who is the winner under plurality?
D. Who is the winner under plurality with elimination?
E. Who is the winner under Borda count?
F. Who is the winner under pairwise comparisons?
G. What do these results imply about the majority property, if anything?
H. What do these results imply about the Condorcet property, if anything?

2. Fill in the table exploring our fairness properties for the four voting systems covered in this e-textbook chapter on page 45.

3. Write a short biographical sketch of the economist Kenneth Arrow. Be sure to give a brief description of the academic work which won him a Nobel Prize.

4. The problematic fairness property seems to be IIA. Would you argue to keep it as a fairness property or to delete it?

Challenge Exercise

The proof of Arrow's Impossibility Theorem is beyond the scope of our class. But if you are interested and challenged by this intriguing result, research its proof. Summarize and cite any proof whose explanation you find understandable. If you cannot find a proof which is understandable as a whole result, then write up explanations for parts/portions of any proof that you find compelling and are able to follow.

Partial Bibliography

Mathematics and Politics: Strategy, Voting, Power, and Proof (Second Edition)
Alan D. Taylor & Allison M. Pacelli
Springer
New York, NY
© 2008
ISBN 978-0-387-77643-9

The Mathematics of Voting and Elections: A Hands-On Approach
Jonathan K. Hodge & Richard E. Klima
American Mathematical Society
Providence, RI
© 2005
ISBN 978-0-8218-3798-6

Numbers Rule: The Vexing Mathematics of Democracy from Plato to the Present
George G. Szpiro
Princeton University Press
Princeton, NJ
© 2010
ISBN 978-0-691-13994-4

Chaotic Elections! A Mathematician Looks at Voting
Donald G. Saari
American Mathematical Society
Providence, RI
© 2001
ISBN 0-8218-2847-9

Basic Geometry of Voting
Donald G. Saari
Springer
Heidelberg, Germany
© 1995
ISBN 3-5406006-47

Social Choice and the Mathematics of Manipulation
Alan D. Taylor
Mathematical Association of America
Cambridge University Press
New York, NY
© 2005
ISBN 0-521-00883-2

Epilogue

There are so many people to thank for supporting this project. My family, students, and colleagues have embraced it and endured much to allow me to complete it. **Thank you all!**

The students of Oglethorpe University worked through many iterations of this project – both my own students and those of my cherished colleague, Dr. Lynn Gieger. Most of these students were not mathematics majors; instead they were a captive audience in our mathematics graduation requirement course, "Great Ideas of Modern Mathematics." They were enthusiastic about breaking in a new "great idea" for the course and suggested many additions, including the homework problems in each section. This project would have certainly been less lively without their boundless energy, and I appreciate their hard work and contributions. Also, Dr. Paul Koester, recently a Lecturer for the Department of Mathematics at the University of Kentucky, shared a particularly powerful example, which I converted into homework. Oglethorpe's Core Curriculum is quite different than "general education" at most colleges and universities, and we are quite proud of our challenging and interdisciplinary Core. Its mathematics course speaks not only to our sister disciplines in science but also to the history and writing elements infused throughout our Core. For more information, visit http://oglethorpe.edu/academics/the-core/.

It has been a joy to share my teaching and research career over the past decade and a half with Dr. Lynn Gieger, also a Professor of Mathematics at Oglethorpe University. Simply put, my frequent collaborator and dear friend has helped to make my scholarly and teaching life at Oglethorpe both successful and fun. I cherish and value her prominent place in my life.

In 2007, a group of faculty from Oglethorpe rekindled our relationship with the national SENCER movement (Science Education for New Civic Engagements and Responsibilities). The forward thinking educators at SENCER want to change how we teach science and mathematics, and their tireless efforts at rejuvenating/inspiring faculty and improving student success in STEM disciplines are awesome. Visit their website to access resources and connect with faculty around the country: http://www.SENCER.net. Recently, the SENCER group won a National Science Foundation grant to target mathematics specifically, and the "Engaging Mathematics" project was born. Oglethorpe was proud to be one of only six colleges and universities across the country named as institutional centers/partners in this work. A nice cross-section of institutions was picked; mathematics courses from introductory level to the mathematics major are targeted as part of the work for this NSF grant. Check out the wide variety of courses and projects at http://engagingmathematics.ipower.com/. Thanks to David Burns and everyone at SENCER, with special appreciation to Christine DeCarlo, our "Engaging Mathematics" coordinator.

Lastly, I offer thanks to my husband Robert and my family for putting up with me during the many hours spent researching, writing, revising, and working on this project. I love you all.

If you would like access to the student answer manual or faculty solutions manual, then please visit the "Engaging Mathematics" website listed above or get in touch with me. I am also happy to correspond with you and offer any advice I can on these materials and teaching mathematics.

Dr. John C. Nardo, Professor of Mathematics
Oglethorpe University **(404) 364-8327**
4484 Peachtree Road, NE **jnardo@oglethorpe.edu**
Atlanta, GA 30319 **http://www.oglethorpe.edu/faculty/johncnardo/**

CORE 203 – DR. NARDO – FALL 2015 – DAILY CALENDAR

MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
8-24 **First Day of OU Classes**	8-25 Syllabus/Intro	8-26	8-27 Section 3.1	8-28
8-31 **Drop/Add Ends**	9-1 Section 3.2	9-2	9-3 Section 3.3	9-4
9-7	9-8 Section 3.4	9-9	9-10 Section 3.5	9-11
9-14	9-15 Section 3.5	9-16	9-17 Lewis Carroll	9-18
9-21	9-22 Section 3.6	9-23	9-24 **REVIEW**	9-25
9-28	9-29 **TEST 1**	9-30	10-1 Sections 12.1/12.2	10-2
10-5	10-6 Section 12.3	10-7	10-8 Section 12.3	10-9
10-12	10-13 Section 12.4	10-14	10-15 Section 12.4	10-16 **Midterm Grades Due at Noon**
10-19 **Fall Break**	10-20 **Fall Break**	10-21	10-22 Section 12.5	10-23
10-26 **Last "W" Day**	10-27 Section 12.5	10-28	10-29 Probability Capstone	10-30
11-2 **Registration for SP2016 Begins**	11-3 **REVIEW**	11-4	11-5 **TEST 2**	11-6
11-9	11-10 Discussion of Writing Assignment	11-11	11-12 Basic Voting Definitions	11-13
11-16	11-17 Plurality	11-18	11-19 Borda Count	11-20
11-23	11-24 Pairwise Elimination	11-25 **Thanksgiving**	11-26 **Thanksgiving**	11-27 **Thanksgiving**
11-30	12-1 Arrow's Theorem	12-2	12-3 Arrow's Theorem & Weighted Voting	12-4
12-7	12-8 Weighted Voting	12-9	12-10 **REVIEW**	12-11 **Last Day of OU Classes**
12-14 **Finals**	12-15 **Finals**	12-16 **Finals**	12-17 **Finals**	12-18 **Finals**

CORE 203 – GREAT IDEAS OF MODERN MATHEMATICS
DR. NARDO – FALL 2015 – 8:00 AM SECTION

"Mathematics is the queen of sciences."
– Karl Friedrich Gauss

"The universe stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics."
– Galileo Galilei

WHAT IS "GREAT IDEAS OF MODERN MATHEMATICS?"

Mathematics is one of the greatest achievements of the human mind and is the cornerstone of modern scientific thought and practice. Mathematics has long been a partner to science and other academic fields; today, mathematics is used in fields as diverse as biology, business, chemistry, communications, economics, English, history, human resources, medicine, philosophy, physics, politics, and psychology. Mathematics' applications may be its strength, but its importance as a foundation of knowledge and its inherent beauty should not be ignored.

The Oglethorpe University *Bulletin* states:

> This course explores major mathematical developments and helps students to understand the unique approach to knowledge employed by mathematics. The course is organized around three major mathematical ideas that have emerged since the time of Sir Isaac Newton. These three ideas may be drawn from: game theory, graph theory, knot theory, logic, mathematics of finance, modern algebra, non-Euclidean geometry, number theory, probability, set theory and the different sizes of infinity, and topology. Students will learn how to solve basic problems in the three areas covered by the course and how to present their solutions concisely, coherently, and rigorously.

The mathematics that you will create in this course will be fundamentally different than in your previous mathematics courses. You are not simply looking for the "correct answer" by emulating steps from your textbook. The "correct answer" is **half of the point**. Our ultimate goal is for you to write and create good mathematics. A detailed, well-written solution which explains why your answer is "correct" is the **other half of the point**. This total package of answer and supporting solution is the emphasis of this class and the true measure of both your success and the success of the course as a whole.

BASIC COURSE INFORMATION

Meeting Time:	Tuesday and Thursday 8:00–9:30 a.m.
Location:	Lupton 200
Textbook:	*Mathematical Excursions* (3rd Ed.) by Aufmann, Lockwood, Nation, and Clegg
Online Quiz Website:	www.webassign.net
OU Course Website:	moodle.oglethorpe.edu

The textbook is an obvious, required part of our course: a physical copy (new or used) or an electronic copy. You must have the book. Period.

The online quiz system WebAssign® is a required part of our course: bundled with your physical book or as a stand-alone "single-term" access card. You must buy access to this software. Period.

Note: If you buy a "single-term" access card to WebAssign®, then an electronic copy of the textbook in included for free.

You may use scientific and/or graphing calculators in our course. Any recent model from Texas Instruments is recommended.

CONTACT INFORMATION

Office:	311 Lupton Hall
Telephone/Voice Mail Number:	(404) 364-8327
Electronic Mail:	jnardo@oglethorpe.edu
Webpage:	http://www.oglethorpe.edu/faculty/~j_nardo

OFFICE HOURS

Tuesday and Thursday 2:15 – 4:15 p.m. Also By Appointment

COHERENCE OF COURSE WITH GENERAL EDUCATION GOALS

This course contains instruction that is directly relevant to the following general education goals, as stated in the University *Bulletin*:

1. The ability to read critically – to evaluate arguments and the evidence and to draw appropriate conclusions,
2. The ability to convey ideas in writing and in speech – accurately, grammatically, and persuasively, and
3. Skill in reasoning logically and thinking analytically and objectively about important matters.

Meeting these goals is accomplished throughout an Oglethorpe education and is assisted in this course via its course learning objectives below.

COURSE CONTENT – OUR THREE "GREAT IDEAS"

1. Formal Logic
2. Probability
3. Mathematics of Voting

COURSE LEARNING OBJECTIVES

Mathematical Truth

- Distinguish between deductive and inductive reasoning

Formal Logic

- Identify whether a mathematical sentence is a statement
- Classify the type of compound statement
- Represent a compound statement using formal logic symbols
- Construct a truth table for a compound statement
- Construct the negation of a compound statement
- Distinguish between universal and existential quantifiers
- Decide whether two statements are equivalent
- Decide whether a statement is a tautology
- Decide whether a statement is a contradiction
- Separate a given conditional into its hypothesis and its conclusion
- For a given conditional, write its inverse, converse, and contrapositive
- Express a given conditional in its equivalent disjunctive form
- Express a given conditional in an equivalent simple English form
- Decide whether an argument is valid via truth table, via reduction to standard logic forms, and via Euler Diagram
- Demonstrate the two classical fallacies by creating real-world examples.

Probability

- Count by: listing, using a table, using a tree diagram, using a Venn Diagram, using the Fundamental and General Principles of Counting, using combinations, and using permutations
- Distinguish between counting with replacement and counting without replacement
- Calculate probabilities exactly by: sample space, counting, rules, Venn Diagram, table, and tree
- Approximate probabilities by empirical/experimental methods
- Convert between odds and probability (and vice-versa)
- Decide whether two events are disjoint
- Decide whether two events are independent
- Calculate conditional probabilities via the definition, tables, and Venn Diagrams
- Calculate and interpret mathematical expectation

Mathematics of Voting

- Determine the winner (or a ranked listing of candidates) for an election using the following voting systems: plurality, plurality with elimination, Borda count, and pairwise comparisons
- Define each of the four voting fairness properties: majority, Condorcet, monotonicity, and independence of irrelevant alternatives
- Determine if a given voting system possesses or does not possess a particular fairness property and justify via an explanation (in the affirmative) or via a counter-example (in the negative)
- State Arrow's Impossibility Theorem and explain its consequences to fairness in voting
- Decide on a personal preferred voting system and justify that choice
- Calculate and interpret the Banzhaf Power Index for weighted voting systems

ATTENDANCE

I feel strongly that regular attendance is vital for your learning and success in our course! Collegiate courses proceed at a pace that makes regular class attendance a necessity. The university cedes all control of attendance to the instructor, and I am clearly outlining my attendance expectations/policies to you. These expectations will not change and are in force for all students.

Oglethorpe's *Bulletin* states:

> Regular attendance at class sessions, laboratories, examinations and official University convocations is an obligation which all students are expected to fulfill. All instructors will make a clear statement in each course syllabus describing their policies for handling absences. Students are obligated to adhere to the requirements of each course and of each instructor.

Though I expect you to be in class when we start, official attendance will be taken at 8:05 a.m. Any student arriving after attendance has been taken will be counted absent for that class meeting. It is expected that when you come to class that you remain in class. If a student leaves class early, he/she will be counted absent for that class meeting.

As a reward for perfect attendance, any student who has **no absences** in this course will receive 3% added to his/her overall course average at the end of the semester. Properly executed excused absences (see the shaded box below) are not counted as absences in this course; thus, they do not affect the perfect attendance total and bonus.

- Students are allowed **four absences** in this course – for whatever reason. The semester is a long time; so, I encourage you to save your absences and to use them wisely.

- Upon a student's **fifth absence**, his/her course grade will be dropped by one letter grade (i.e. a 10% deduction from the overall course average at the end of the semester).

- Upon a student's **sixth absence**, he/she will earn the grade of "FA" (Failure by Absences).

There is **only one** type of excused absence in this course: officially representing Oglethorpe University.

You must give advance notification of such an absence. A letter from a faculty or staff member of the university is required, and this person must represent an office of the institution. In this letter, he/she must argue why it is vital that you miss class. Advance notification and a letter are required for **each** such absence. These absences are **not** counted in a student's tally of absences and will **not** affect his/her grade.

Since these absences are known in advance, no arrangements for missed work are necessary. The students should turn in assignments or take tests **before** the excused absence occurs. In extreme circumstances, your professor may allow a faculty or staff member to proctor an assignment during a period of extended excused absences.

Your professor has the final decision on whether absences are excused and how academic work is submitted.

CLASSROOM ETIQUETTE & USE OF ELECTRONICS IN CLASS

Every student has the right to a productive, distraction-free learning environment. Read the classroom etiquette handout to learn my expectations for your behavior in class. Violations of our classroom etiquette guidelines will result in ejection from class and an absence for that class meeting.

One of the most distracting violations of classroom etiquette is the use of electronics in class. Per the university's "Electronic Device Appropriate Use Policy," I am ruling that such devices cannot be used in our class. The only electronic device which can be used in our class is a calculator. You may bring that one electronic device to class and use it.

Laptops, phones, and other non-calculator electronic devices are **<u>not</u>** allowed in our class.
They must be powered **<u>off</u>** during our class meetings.

Violations of our class specifications of the university's "Electronic Device Appropriate Use Policy" will result in ejection from class and an absence for that class meeting.

The only exception to this policy is for students with Oglethorpe-documented learning disabilities. See the section on learning disabilities below. If a student has an official accommodation requiring electronic devices, then of course, he/she will be permitted to use them **<u>solely and exactly</u>** in accordance with the accommodation letter from the Academic Success Center. Such a student should discuss this situation privately with me in my office.

COURSE WORK & COURSE LETTER GRADE

There are three basic learning activities that I expect of you. These activities are self-directed, and they are not graded or enforced by me. But they are vital to your success in our course.

1. You are expected to read every section of the textbook that we cover in class. I prefer you read each section **<u>before</u>** we discuss it in class so that you can take a full and active part in your learning of mathematics.
2. Textbook, pencil-and-paper homework is given on your class Moodle page for every section of the textbook we cover. You are expected to do this textbook homework in its entirety.
3. You are expected to participate in class and to ask questions.

Now, we will turn our attention to graded work for our course.

Unless noted explicitly in writing on the assignment, any work submitted by a student for a grade must represent the **<u>individual</u>** effort of that student!

Course averages will be calculated using these percentages:

Quizzes (through the online WebAssign® system)	20%
Writing Assignments	20%
Highest Two Tests	50 %
Lowest Test	10%.

Letter grades will be assigned according to the University scale:

A	93-100	C+	77-79
A–	90-92	C	73-76
B+	87-89	C–	70-72
B	83-86	D+	67-69
B–	80-82	D	60-66
		F	59 and below.

Online WebAssign® Quizzes

There will be a quiz due online via WebAssign® immediately before almost every class meeting. The exact due date for each quiz will be posted clearly in the WebAssign® system.

The default will be that you have a quiz due five minutes before each class meeting which covers the material from the previous class meeting – unless posted otherwise inside WebAssign®.

These quizzes are designed to assess skill mastery and attaining the correct "answer," and there are no written explanations in the quizzes. Either you can do the desired calculations, or you cannot. These quizzes must be submitted in the WebAssign® system.

You may use **only** the following resources when completing your WebAssign® quizzes:

- Notes (notes you have personally taken in our class or those taken by an authorized note-taker in the case of a disability accommodation),
- Class handouts,
- Graphing or scientific calculator,
- These specific WebAssign® resources: the e-book, textbook videos, "Read It" tool, "Watch It" tool, and "Master It" tool
- Our textbook itself.

No other sources/tools/help are allowed. The "ask for help" tool, if it appears, is a tempting online tool, but it violates our Honor Code! As stated, these quizzes are individual work; the only other person with whom you may discuss them is your professor.

Late WebAssign® quizzes will earn a zero grade; no makeup quizzes are given. I do not grant extensions; so, do not send requests for them inside the WebAssign® system or through OU channels.

Though there is no place inside the WebAssign® system to put your Honor Pledge, the OU Honor Code is, of course, in effect for every one of these quizzes. I will take the fact that you have submitted your quiz officially for grading through that online system as your reaffirmation of the OU Honor Code.

You have up to four attempts on each quiz question, and you can submit a quiz in its entirety or submit individual questions or parts of questions. You can even save an answer to return to later without officially submitting that answer. Your best scores across all submitted attempts are used for grading.

In computing your quiz average, each quiz will be converted into a percentage grade, and the lowest two quiz grades will be dropped.

Writing Assignments

There will also be short writing assignments due at almost every class meeting.

The vast majority of these assignments will consist of your writing a full explanation for one problem of your choosing from the current online WebAssign® quiz. You submit the online quiz through WebAssign® before class starts, and you bring this write-up to class with you along with a print-out of the problem you are solving. Since the online quiz will already have assessed the correctness of your answer, these writing assignments do not focus on "the right answer." Instead, they focus on your being able to communicate concisely, coherently, and rigorously as a beginning mathematician. Each writing assignment should convince the reader **why** your answer is indeed correct.

I will use the scoring rubric below for grading the writing assignments based on WebAssign® quizzes.

100%	The solution is mathematically valid and well written; it is a model example for others.
95%	The solution is mathematically valid, but it is not well written in **one** way. It is lacking from an **English** standpoint. One or more of the following may be abused: capitalization, grammar, punctuation, spelling, etc.
90%	The solution is mathematically valid, but it is not well written in **one** way. It is lacking from a **mathematical** standpoint. One or more of the following may be abused: definitions, notation, symbols, terms, etc.
85%	The solution is mathematically valid, but it is not well written in **two** ways. It is lacking from **both** an English standpoint and a mathematical one.
80%	The solution is mathematically invalid. It has a sound approach, but, unfortunately, it also has **minor** missing or incorrect details.
70%	The solution is mathematically invalid. It has a sound approach, but, unfortunately, it also has **major** missing or incorrect details.
60%	The solution is mathematically invalid, but there is an **attempt** to solve the problem.
0%	Blank – There is no solution.

Occasionally, I will give you a break from writing assignments, and I will announce this in class and on Moodle. Occasionally, I will eliminate certain short, simple quiz questions from being eligible for a writing assignment. The default is that any question on a quiz is eligible. If I wish to disallow a question from being "written up," then I will announce this in class and on Moodle.

A few times in the semester, a writing assignment will be a more traditional type of writing. For example, at the beginning of the semester, you will write an essay describing the "Narrative of your Mathematical Self." Later in the semester, to jump start the first day of class for our third "Great Idea," you will write a series of short, persuasive arguments in response to journal articles. Your grade on these writing assignments will also be a percentage, but it will not use the rubric above.

Late writing assignments will earn a zero grade; no makeup writing assignments are given. I do not grant extensions; so, do not request them through OU channels.

Each writing assignment must have the full Oglethorpe Honor Pledge and your signature. If either is missing, then the writing assignment will earn a zero grade. Note that "I pledge …" is not sufficient.

If the WebAssign® print-out of your chosen problem is not included with your explanation, then that writing assignment will receive a zero grade.

In computing your writing assignment average, the lowest two grades will be dropped.

Tests & Final Examination

There will be three tests: **September 29, November 5,** and **December 17**.

Tests will cover material from class meetings, textbook sections that you have read, textbook homework, WebAssign® quizzes, and writing assignments. Tests will focus on both "correct answers" and writing a full explanation.

Missing a test is a very serious and grave matter. I do not automatically give "make-up" tests. There are several options that I may choose – depending on the circumstance. A student should discuss the circumstances of such an absence with me (preferably in advance). In order not to receive an immediate zero for a missed test, a student must:

1. inform me by OU e-mail or voice-mail within four hours of the start of the test the reason the test was missed **and**
2. provide me with sufficient documentation of the valid reason for the absence, i.e. doctor's note, hospital document, court document, etc.

To be clear, if either condition is not met, then the student will receive a zero grade for the missed test. It is solely the professor's prerogative to grant make-up options.

Allowed make-up options must be arranged and accomplished as soon as possible. Any allowed make-up option not completed within five business days of the original test will become an automatic zero.

The mandatory **final examination** is simply our last test: it is not cumulative. It occurs on: **Thursday, December 17, 8:00-11:00 a.m.** This final examination may be taken neither early nor late; no exceptions are allowed to the University's Final Examination Schedule, as published by the Registrar's Office!

Mathematics is not a spectator sport; you will not learn by simply watching me solve problems!
You must work many, many problems on your own to master the concepts of this Core class.
You must become skilled at both arriving at a "correct answer" and justifying it via a full explanation.

DISABILITIES/LEARNING DISABILITIES

Oglethorpe University is committed to equal and full access to its programs, services, and activities for people with disabilities. Any student with a disability who needs academic accommodations is welcome to meet with me privately. All such conversations will be kept confidential. Such a conversation is **required** in order to receive accommodations. Students requesting any accommodations will also need to contact the Academic Success Center (ASC) in the basement of the Library. The ASC will conduct an intake and, if appropriate, the office will provide an academic accommodation notification letter for you to bring to me. Accommodations start when I receive such a letter and are **not retroactive**.

If you are a student with a disability and feel that you may need a reasonable accommodation to fulfill the essential functions of the course that are listed in this syllabus, you are encouraged to contact Disability Services in the Academic Success Center at (404) 364-8869 or at disabilityservices@oglethorpe.edu.

SOCIAL NETWORKING WEBSITES

Though Facebook and other social networking websites are valuable ways to stay connected with friends and family in this digital age, they are **not** appropriate avenues through which to communicate about our class. While we have a professional relationship in class, I will neither communicate with you on such websites nor accept any "friend requests" from you. If you are currently "my friend," I will terminate that online relationship for the duration of our class. The appropriate venues for our communication are: the classroom, my office, OU voice-mail, and OU e-mail.

SUPPLEMENTAL READING

- Halmos, *Problems for Mathematicians, Young and Old*, MAA, Washington, D.C., 1991
- National Science Foundation, *Women, Minorities, and Persons with Disabilities in Science and Engineering*, NSF, 1997
- National Science Foundation, *Women and Science Celebrating Achievements, Charting Challenges*, NSF, 1997
- Trentacosta & Kenney, *Multicultural and Gender Equity in the Mathematics Classroom: The Gift of Diversity*, NCTM, 1997
- Parker, *She Does Math!*, The Mathematical Association of America, 1995

UNIVERSITY POLICY: THE HONOR CODE

Persons who come to Oglethorpe University for work and study join a community that is committed to high standards of academic honesty. The honor code contains the responsibilities we accept by becoming members of the community and the procedures we will follow should our commitment to honesty be questioned.

The students, faculty and staff of Oglethorpe University expect each other to act with integrity in the academic endeavor they share. Members of the faculty expect that students complete work honestly and act toward them in ways consistent with that expectation. Students are expected to behave honorably in their academic work and are expected to insist on honest behavior from their peers.

Oglethorpe welcomes all who accept our principles of honest behavior. We believe that this code will enrich our years at the University and allow us to practice living in earnest the honorable, self-governed lives required of society's respected leaders.

Our honor code is an academic one. The code proscribes cheating in general terms and also in any of its several specialized sub-forms (including but not limited to plagiarism, lying, stealing and interacting fraudulently or disingenuously with the honor council). The Code defines cheating as "the umbrella under which all academic malfeasance falls. Cheating is any willful activity involving the use of deceit or fraud in order to attempt to secure an unfair academic advantage for oneself or others or to attempt to cause an unfair academic disadvantage to others. Cheating deprives persons of the opportunity for a fair and reasonable assessment of their own work and/or a fair comparative assessment between and among the work produced by members of a group. More broadly, cheating undermines our community's confidence in the honorable state to which we aspire."

The honor code applies to all behavior related to the academic enterprise. Thus, it extends beyond the boundaries of particular courses and classrooms *per se*, and yet it does not extend out of the academic realm into the purely social one.

Examples of cheating include but are not limited to:

1. The unauthorized possession or use of notes, texts, electronic devices (including, for example, tablets, computers and smartphones), online materials or other such unauthorized materials/devices in fulfillment of course requirements.
2. Copying another person's work or participation in such an effort.
3. An attempt or participation in an attempt to fulfill the requirements of a course with work other than one's original work for that course.
4. Forging or deliberately misrepresenting data or results. Submitting results of an experiment, at which one was not present or present for less than the full time, as one's own.
5. Obtaining or offering either for profit or free of charge materials one might submit (or has submitted) for academic credit. This includes uploading course materials to online sites devoted, in whole or in part, to aiding and abetting cheating under the guise of providing "study aids." There is no prohibition concerning uploading exemplars of one's work to one's personal website or to departmental, divisional, University or professional society websites for purposes of publicity, praise, examination or review by potential employers, graduate school admissions committees, etc.

6. Violating the specific directions concerning the operation of the honor code in relation to a particular assignment.
7. Making unauthorized copies of graded work for future distribution.
8. Claiming credit for a group project to which one did not contribute.
9. Plagiarism, which includes representing someone else's words, ideas, data or original research as one's own and in general failing to footnote or otherwise acknowledge the source of such work. One has the responsibility of avoiding plagiarism by taking adequate notes on reference materials (including material taken off the internet or other electronic sources) used in the preparation of reports, papers and other coursework.
10. Submitting one's own work for a course that was previously submitted for the same course, or another course, without proper citation.
11. Lying, such as: Lying about the reason for an absence to avoid a punitive attendance penalty or to receive an extension on an exam or on a paper's due date; fraudulently obtaining Petrel Points by leaving an event soon after registering one's attendance and without offering to surrender the associated Petrel Point, or by claiming fictitious attendance for oneself or another; forging or willfully being untruthful on documents related to the academic enterprise, such as on an application for an independent study or on a registration form.
12. Stealing, such as: Stealing another's work so that he/she may not submit it or so that work can be illicitly shared; stealing reserve or other materials from the library; stealing devices and materials (such as computers, calculators, textbooks, notebooks and software) used in whole or in part to support the academic enterprise.
13. Fraudulent interaction on the part of students with the honor council, such as: Willfully refusing to testify after having been duly summoned; failing to appear to testify (barring a *bona fide* last-minute emergency) after having been duly summoned; testifying untruthfully.

Students pledge that they have completed assignments honestly by attaching the following statement to each piece of work submitted in partial fulfillment of the requirements for a course taken for academic credit: **"I pledge that I have acted honorably." (Followed by the student's signature)**

The honor code is in force for every student who is enrolled (either full- or part-time) in any of the academic programs of Oglethorpe University at any given time. All cases of suspected academic dishonesty will be handled in accordance with the provisions established in this code. The honor council has sole jurisdiction in matters of suspected academic dishonesty. Alternative ways of dealing with cases of suspected academic fraud are prohibited. In cases of alleged academic dishonesty on the part of students, the honor council is the final arbiter. Reference the current Oglethorpe University Bulletin for information concerning all aspects of the honor code.

UNIVERSITY POLICY: INCOMPLETES

The passage below from the *Bulletin* gives the University policy on incompletes.

> If a student is unable to complete the work for a course on time for reasons of health, family tragedy or other circumstances the instructor deems appropriate, the grade "I" (Incomplete) may be assigned. However, the grade "I" may not be assigned in any course for a student who is currently on academic probation.
>
> If the student completes the work within 30 days of the last day of final examinations of the semester in question, the instructor will evaluate the work and turn in a revised grade on a change of grade form. Any "I" not changed by the instructor within 45 days of the last day of final examinations will automatically be changed to a grade of "F" unless the grade of "I" is issued because an unresolved honor code case prevents the computation of the final grade. Only in that case will the "I" persist until the honor code case is fully adjudicated and the honor council secretary has informed the registrar and the instructor of record for the course as to the nature of the final disposition of the case and what impact it will have on the student's grade. The instructor or the honor council secretary will then have up to five days to file a change of grade form with the registrar.
>
> The grade "I" has no effect on the GPA, and no credit is awarded.

UNIVERSITY POLICY: WITHDRAWAL

Dropping a course at the beginning of the semester has **no** long-term effects. There is no grade for a dropped course; a dropped course does not show on the student's transcript. It is like the course never happened for the student.

But once the drop/add period ends on Monday, August 31, dropping is no longer possible, and the only way to leave a course is by **withdrawing**. This option **does** have long-term effects for the student; in particular, the following could be affected negatively: full-time status, athletic eligibility, financial aid, etc. Any student considering withdrawal should see his/her academic advisor immediately.

Withdrawn courses show on the student's transcript. One of two grades ("W" or "WF") will be received. A "W" has no effect on the GPA, but a "WF" counts like an "F" in the GPA.

- If a student completes withdrawal paperwork for individual course(s) with the Registrar's Office **by the end of business on October 26**, then he/she will receive a no-penalty "W" grade.
- If a student completes withdrawal paperwork for individual course(s) **after October 26**, then he/she will automatically receive the failing "WF" grade.
- The last day to withdraw entirely from the university and exit all courses is **December 11**.

Note that the withdrawal form requires multiple signatures. Do not leave signatures to the last minute, or you may miss this important deadline!

WORLD WIDE WEB RESOURCES (FOR LEARNING CONCEPTS AND UNGRADED WORK)

American Mathematical Society http://www.ams.org/	American Statistical Association http://www.amstat.org/
Chance Quantitative Literacy Course http://www.dartmouth.edu/~chance/	
College Math Resources http://qpr.ca/math/resources/doResults.php?topicID=102&topicLongName=	
Mathematical Association of America http://www.maa.org/	Mathematics Archives http://archives.math.utk.edu/
Mathematics Forum http://mathforum.org/	Multicultural Pavilion http://www.edchange.org/multicultural/sites/math.html
Texas Instruments http://education.ti.com/	Women in Mathematics http://www.agnesscott.edu/lriddle/women/women.htm

APPENDIX: DAILY COURSE CALENDAR

See the separate daily calendar document for important dates from the university calendar generally, important dates for our course specifically, and the textbook section to be covered every day. Being familiar with this schedule will allow you to know what reading assignment you need to complete for every day of class.

IMPORTANT NOTE:
The daily calendar and the dates given in this syllabus may be altered during the term by the professor!
Please see your course Moodle page for the most up-to-date information.

CORE 203 – GREAT IDEAS OF MODERN MATHEMATICS – DR. NARDO
FALL 2015 – TEST 3 – VOTING THEORY – TTH 8:00 SECTION

Printed Name: ____________________________ Signature: ____________________________

Your signature reaffirms your acceptance of the Oglethorpe Honor Code; it certifies that you have acted honorably on this test.

- **It is expected that you will not only give answers but also explain fully why those answers are correct! This has been the operating procedure in our class all semester.**
- If an explanation is not needed, then this will be explicitly noted in a problem.
- You must use correct mathematical symbols and correct mathematical terms/definitions.
- Problem #1 is worth 10 points, and each of the other problems is worth 18 points.

1. Define each of the terms below.

A. Majority Threshold **(I am NOT looking for a formula here!)**

B. Majority Candidate

C. Condorcet Candidate

D. Majority Property

E. Independence of Irrelevant Alternatives Property

2. One hundred students were asked to rank their preferences for bottled water brands with the results below.

Aquafina	2	3	1	2
Dasani	1	1	3	3
Evian	3	2	2	1
Number of Votes	22	17	31	30

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below, showing any sums where needed and not just final "answers."

# of first-place votes for Aquafina =	
# of first-place votes for Dasani =	
# of first-place votes for Evian =	

C. Who is the winner under the plurality voting system? (If none, then write none.)

Answer: _______________

Explanation:

D. Who is the majority candidate? (If none, then write none.)

Answer: _______________

Explanation:

E. Is this problem a valid counter-example which shows that the plurality voting system does **NOT** possess the majority property? (Yes or No)

Answer: _______________

Explanation:

3. In ramping up for the new "Star Wars" movie "The Force Awakens," seventeen fans ranked their preferences for four of the franchise's movies with the results below.

S > E > J > R: 6 **E > S > J > R: 5** **J > R > E > S: 4** **R > J > S > E: 2**

NOTE: S = "Star Wars" E = "Empire Strikes Back"
J = "Return of the Jedi" R = "Revenge of the Sith"

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below. **(No explanations are needed for Part B.)**

# of first-place votes for S =	# of first-place votes for E =
# of first-place votes for J =	# of first-place votes for R =

C. Which film is the winner under the plurality with elimination voting system? (If none, then write none.)

Answer: _______________

Explanation:

4. A student club must decide what to do with left-over funds from the fall semester. There are three choices, and the club members rank their preferences as below.

Proposal A: Refund to Students	1	3	3
Proposal B: Save the Fund for the Spring Semester	2	1	2
Proposal C: Throw a Party for Club Members	3	2	1
Number of Votes	55	50	3

A. Who is the winner under the Borda Count voting system?
(If none, then write none.) **Answer:** _______________

Explanation:

B. Who is the Condorcet candidate? (If none, then write none.)
Answer: _______________

Explanation:

C. Is this problem a valid counter-example which shows that the Borda Count voting system does **NOT** possess the Condorcet property? (Yes or No)
Answer: _______________

Explanation:

5. You ask customers at the OU Starbucks to rank three beverage choices in popularity.

E = Hot Espresso Drinks F = Frappuccino Drinks T = Tea Drinks

Your sample of customers gives the preference data below.

E > F > T: 20 **T > F > E: 25** **F > T > E: 15**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. How is the majority threshold used in the Pairwise Comparisons voting system?

C. Who is the winning drink under the Pairwise Comparisons voting system?

Winner: _______________

Explanation: Fill in the table below and then add any needed explanation afterwards.

	Pairwise Race #1	Pairwise Race #2	Pairwise Race #3
First Candidate:			
Second Candidate:			
Three Modified Preference Inequalities with Vote Totals:			
# of Votes for First Candidate =			
# of Votes for Second Candidate =			
Pairwise Winner =			
How were Condorcet Points awarded to all candidates in this pairwise race?			

D. You are disappointed that your favorite drink was not the winner and explore this voting data more closely. Show that a different voting system can give a different winner here.

New Winner: _______________

New Voting System: _______________

Explanation:

6. Consider the following weighted voting system: {19: 19, 3, 16}.

A. List all coalitions. **(No explanations are needed for Part A.)**

B. Make a table (like in class) that shows all winning coalitions and the critical voters for each winning coalition. **(No explanations are needed for Part B.)**

C. From Part B, pick a winning coalition and a critical voter from that coalition. Explain why the coalition is a winning one and why the voter is critical, by class definitions.

Critical Voter: _____________

Winning Coalition: _______________

Explanation:

(Continued)

6. **(Continued)**

D. Give the definition of the Banzhaf power index. You should use generic voter *v*.

BPI(*v*) =

E. Compute the Banzhaf power index for each voter. Give a reduced fraction "answer."

F. At first glance, it might appear that voter A is a dictator in this weighted voting system since he/she has a weight equal to the quota. Explain briefly why A is <u>not</u> a dictator.

BONUS – 8 POINTS

As seen in class, there is just one of our voting systems which has only the Monotonicity Property (i.e. not the other fairness properties). Identify that voting system, and prove it possesses the Monotonicity Property.

CORE 203 – GREAT IDEAS OF MODERN MATHEMATICS – DR. NARDO
FALL 2015 – TEST 3 – VOTING THEORY – TTH 9:45 SECTION

Printed Name: ___________________________ Signature: ___________________________

Your signature reaffirms your acceptance of the Oglethorpe Honor Code; it certifies that you have acted honorably on this test.

- **It is expected that you will not only give answers but also explain fully why those answers are correct! This has been the operating procedure in our class all semester.**
- If an explanation is not needed, then this will be explicitly noted in a problem.
- You must use correct mathematical symbols and correct mathematical terms/definitions.
- Problem #1 is worth 10 points, and each of the other problems is worth 18 points.

1. Define each of the terms below.

A. Majority Threshold **(I am <u>NOT</u> looking for a formula here!)**

B. Majority Candidate

C. Condorcet Candidate

D. Condorcet Property

E. Monotonicity Property

2. One hundred students were asked to rank their preferences for bottled water brands with the results below.

Aquafina	1	1	3	3
Dasani	2	3	1	2
Evian	3	2	2	1
Number of Votes	22	17	31	30

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below, showing any sums where needed and not just final "answers."

# of first-place votes for Aquafina =	
# of first-place votes for Dasani =	
# of first-place votes for Evian =	

C. Who is the winner under the plurality voting system? (If none, then write none.)

Answer: _______________

Explanation:

D. Who is the majority candidate? (If none, then write none.)

Answer: _______________

Explanation:

E. Is this problem a valid counter-example which shows that the plurality voting system does **NOT** possess the majority property? (Yes or No)

Answer: _______________

Explanation:

3. In ramping up for the new "Star Wars" movie "The Force Awakens," seventeen fans ranked their preferences for four of the franchise's movies with the results below.

S > E > J > R: 6 **E > S > J > R: 5** **J > R > E > S: 4** **R > J > S > E: 2**

NOTE: S = "Star Wars" E = "Empire Strikes Back"
J = "Return of the Jedi" R = "Revenge of the Sith"

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below. **(No explanations are needed for Part B.)**

# of first-place votes for S =	# of first-place votes for E =
# of first-place votes for J =	# of first-place votes for R =

C. Which film is the winner under the plurality with elimination voting system? (If none, then write none.)

Answer: _______________

Explanation:

4. A student club must decide what to do with left-over funds from the fall semester. There are three choices, and the club members rank their preferences as below.

Proposal A: Refund to Students	1	3	3
Proposal B: Throw a Party for Club Members	2	1	2
Proposal C: Save the Fund for the Spring Semester	3	2	1
Number of Votes	55	50	3

A. Who is the winner under the Borda Count voting system?
(If none, then write none.) **Answer:** ________________

Explanation:

B. Who is the Condorcet candidate? (If none, then write none.)
Answer: ________________

Explanation:

C. Is this problem a valid counter-example which shows that the Borda Count voting system does **NOT** possess the Condorcet property? (Yes or No)
Answer: ________________

Explanation:

5. You ask customers at the OU Starbucks to rank three beverage choices in popularity.

E = Hot Espresso Drinks F = Frappuccino Drinks T = Tea Drinks

Your sample of customers gives the preference data below.

E > F > T: 25 **T > F > E: 20** **F > T > E: 15**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. How is the majority threshold used in the Pairwise Comparisons voting system?

C. Who is the winning drink under the Pairwise Comparisons voting system?

Winner: _______________

Explanation: Fill in the table below and then add any needed explanation afterwards.

	Pairwise Race #1	Pairwise Race #2	Pairwise Race #3
First Candidate:			
Second Candidate:			
Three Modified Preference Inequalities with Vote Totals:			
# of Votes for First Candidate =			
# of Votes for Second Candidate =			
Pairwise Winner =			
How were Condorcet Points awarded to all candidates in this pairwise race?			

D. You are disappointed that your favorite drink was not the winner and explore this voting data more closely. Show that a different voting system can give a different winner here.

New Winner: _______________

New Voting System: _______________

Explanation:

6. Consider the following weighted voting system: {19: 19, 17, 2}.

A. List all coalitions. **(No explanations are needed for Part A.)**

B. Make a table (like in class) that shows all winning coalitions and the critical voters for each winning coalition. **(No explanations are needed for Part B.)**

C. From Part B, pick a winning coalition and a critical voter from that coalition. Explain why the coalition is a winning one and why the voter is critical, by class definitions.

Critical Voter: ______________

Winning Coalition: ________________

Explanation:

(Continued)

6. **(Continued)**

D. Give the definition of the Banzhaf power index. You should use generic voter v.

BPI(v) =

E. Compute the Banzhaf power index for each voter. Give a reduced fraction "answer."

F. At first glance, it might appear that voter A is a dictator in this weighted voting system since he/she has a weight equal to the quota. Explain briefly why A is <u>not</u> a dictator.

BONUS – 8 POINTS

As seen in class, only one of our four voting systems possesses the Condorcet Property. Identify that voting system, and give a proof of this claim.

CORE 203 – GREAT IDEAS OF MODERN MATHEMATICS – DR. NARDO
SPRING 2016 – TEST 3 – VOTING THEORY – MW 2:15 SECTION

Printed Name: ______________________ Signature: ______________________

Your signature reaffirms your acceptance of the Oglethorpe Honor Code; it certifies that you have acted honorably on this test.

- **It is expected that you will not only give answers but also explain fully why those answers are correct! This has been the operating procedure in our class all semester.**
- If an explanation is not needed, then this will be explicitly noted in a problem.
- You must use correct mathematical symbols and correct mathematical terms/definitions.

1. Define each of the terms below. **(15 pts)**

A. Majority Threshold **(I am NOT looking for a formula here!)**

B. Majority Candidate

C. Condorcet Candidate

D. Majority Property

E. Independence of Irrelevant Alternatives Property

2. One hundred students were asked to rank their preferences for bottled water brands with the voter preferences below. **(20 pts)**

Note:
A = Aquafina
D = Dasani
E = Evian

D > A > E: 22 votes **D > E > A: 17 votes**
A > E > D: 31 votes **E > A > D: 30 votes**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below, showing any sums where needed and not just final "answers."

# of first-place votes for Aquafina =	
# of first-place votes for Dasani =	
# of first-place votes for Evian =	

C. Who is the winner under the plurality voting system? (If none, then write none.)

Answer: _______________

Explanation:

D. Who is the majority candidate? (If none, then write none.)

Answer: _______________

Explanation:

E. Is this problem a valid counter-example which shows that the plurality voting system does **NOT** possess the majority property? (Yes or No)

Answer: _______________

Explanation:

3. To honor the Blu-Ray release of the "Star Wars" movie "The Force Awakens," seventeen fans ranked their preferences for four of the franchise's movies with the results below.

S > E > J > R: 6 **E > S > J > R: 5** **J > R > E > S: 4** **R > J > S > E: 2**

<u>Note</u>: S = "Star Wars" E = "Empire Strikes Back"
J = "Return of the Jedi" R = "Revenge of the Sith" **(15 pts)**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below. **(No explanations are needed for Part B.)**

# of first-place votes for S =	# of first-place votes for E =
# of first-place votes for J =	# of first-place votes for R =

C. Which film is the winner under the plurality with elimination voting system? (If none, then write none.)

Answer: _______________

Explanation:

4. A student club must decide what to do with left-over funds from this year. There are three choices, and the club members rank their preferences as below. **(20 pts)**

Proposal	A: Refund to Students	**Voter**	A > B > C: 55 votes
Notes:	B: Save the Fund for the Spring Semester	**Preferences:**	B > C > A: 50 votes
	C: Throw a Party for Club Members		C > B > A: 3 votes

A. Who is the winner under the Borda Count voting system?
(If none, then write none.) **Answer:** ______________

Explanation:

B. Who is the Condorcet candidate? (If none, then write none.)
Answer: ______________

Explanation:

C. Is this problem a valid counter-example which shows that the Borda Count voting system does **NOT** possess the Condorcet property? (Yes or No)
Answer: ______________

Explanation:

5. You ask customers at the OU Starbucks to rank three beverage choices in popularity.

E = Hot Espresso Drinks F = Frappuccino Drinks T = Tea Drinks

Your sample of customers gives the preference data below. **(20 pts)**

E > F > T: 20 **T > F > E: 25** **F > T > E: 15**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. How is the majority threshold used in the Pairwise Comparisons voting system?

C. Who is the winning drink under the Pairwise Comparisons voting system?

Winner: _______________

Explanation: Fill in the table below and then add any needed explanation afterwards.

	Pairwise Race #1	Pairwise Race #2	Pairwise Race #3
First Candidate:			
Second Candidate:			
Three Modified Preference Inequalities with Vote Totals:			
# of First-Place Votes for the First Candidate =			
# of First-Place Votes for the Second Candidate =			
Pairwise Winner =			
How were Condorcet Points awarded to all candidates in this pairwise race?			

D. You are disappointed that your favorite drink was not the winner and explore this voting data more closely. Show that a different voting system can give a different winner here.

New Winner: _______________

New Voting System: _______________

Explanation:

6. Consider the following weighted voting system: {19: 19, 3, 16}. **(10 pts)**

A. List all coalitions. **(No explanations are needed for Part A.)**

B. Make a table (like in class) that shows all winning coalitions and the critical voters for each winning coalition. **(No explanations are needed for Part B.)**

C. From Part B, pick a winning coalition and a critical voter from that coalition. Explain why the coalition is a winning one and why the voter is critical, by class definitions.

Critical Voter: _____________

Winning Coalition: _______________

Explanation:

(Continued)

6. (Continued)

D. Give the definition of the Banzhaf power index. You should use generic voter *v*.

BPI(*v*) =

E. Compute the Banzhaf power index for each voter. Give a reduced fraction "answer."

F. At first glance, it might appear that voter A is a dictator in this weighted voting system since he/she has a weight equal to the quota. Explain briefly why A is <u>not</u> a dictator.

BONUS – 10 POINTS

As seen in class, there is just one of our voting systems which has only the Monotonicity Property (i.e. not the other fairness properties). Identify that voting system, and prove it possesses the Monotonicity Property.

CORE 203 – GREAT IDEAS OF MODERN MATHEMATICS – DR. NARDO
SPRING 2016 – TEST 3 – VOTING THEORY – MW 4:00 SECTION

Printed Name: ___________________________ Signature: ___________________________
Your signature reaffirms your acceptance of the Oglethorpe Honor Code; it certifies that you have acted honorably on this test.

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- You must use correct mathematical symbols and correct mathematical terms/definitions.

1. Define each of the terms below. **(15 pts)**

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C. Condorcet Candidate

D. Condorcet Property

E. Monotonicity Property

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A > D > E: 22 votes **A > E > D: 17 votes**
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A. Calculate the majority threshold. **Answer:** ______________

Explanation:

B. Give the vote totals below, showing any sums where needed and not just final "answers."

# of first-place votes for Aquafina =	
# of first-place votes for Dasani =	
# of first-place votes for Evian =	

C. Who is the winner under the plurality voting system? (If none, then write none.)

Answer: ______________

Explanation:

D. Who is the majority candidate? (If none, then write none.)

Answer: ______________

Explanation:

E. Is this problem a valid counter-example which shows that the plurality voting system does **NOT** possess the majority property? (Yes or No)

Answer: ______________

Explanation:

3. To honor the Blu-Ray release of the "Star Wars" movie "The Force Awakens," seventeen fans ranked their preferences for four of the franchise's movies with the results below.

S > E > J > R: 6 **E > S > J > R: 5** **J > R > E > S: 4** **R > J > S > E: 2**

Note: S = "Star Wars" E = "Empire Strikes Back"
J = "Return of the Jedi" R = "Revenge of the Sith" **(15 pts)**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. Give the vote totals below. **(No explanations are needed for Part B.)**

# of first-place votes for S =	# of first-place votes for E =
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C. Which film is the winner under the plurality with elimination voting system? (If none, then write none.)

Answer: _______________

Explanation:

4. A student club must decide what to do with left-over funds from this year. There are three choices, and the club members rank their preferences as below. **(20 pts)**

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A. Who is the winner under the Borda Count voting system? (If none, then write none.) **Answer:** _______________

Explanation:

B. Who is the Condorcet candidate? (If none, then write none.) **Answer:** _______________

Explanation:

C. Is this problem a valid counter-example which shows that the Borda Count voting system does **NOT** possess the Condorcet property? (Yes or No) **Answer:** _______________

Explanation:

5. You ask customers at the OU Starbucks to rank three beverage choices in popularity.

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Your sample of customers gives the preference data below. **(20 pts)**

E > F > T: 25 **T > F > E: 20** **F > T > E: 15**

A. Calculate the majority threshold. **Answer:** _______________

Explanation:

B. How is the majority threshold used in the Pairwise Comparisons voting system?

C. Who is the winning drink under the Pairwise Comparisons voting system?

Winner: _______________

Explanation: Fill in the table below and then add any needed explanation afterwards.

	Pairwise Race #1	Pairwise Race #2	Pairwise Race #3
First Candidate:			
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# of First-Place Votes for the First Candidate =			
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Pairwise Winner =			
How were Condorcet Points awarded to all candidates in this pairwise race?			

D. You are disappointed that your favorite drink was not the winner and explore this voting data more closely. Show that a different voting system can give a different winner here.

New Winner: _______________

New Voting System: _______________

Explanation:

6. Consider the following weighted voting system: {19: 19, 17, 2}. **(10 pts)**

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C. From Part B, pick a winning coalition and a critical voter from that coalition. Explain why the coalition is a winning one and why the voter is critical, by class definitions.

Critical Voter: _____________

Winning Coalition: _______________

Explanation:

(Continued)

6. (Continued)

D. Give the definition of the Banzhaf power index. You should use generic voter *v*.

BPI(v) =

E. Compute the Banzhaf power index for each voter. Give a reduced fraction "answer."

F. At first glance, it might appear that voter A is a dictator in this weighted voting system since he/she has a weight equal to the quota. Explain briefly why A is <u>not</u> a dictator.

BONUS – 10 POINTS

As seen in class, only one of our four voting systems possesses the Condorcet Property. Identify that voting system, and give a proof of this claim.

GREAT IDEAS OF MODERN MATHEMATICS: VOTING THEORY

STUDENT ANSWERS MANUAL

Dr. John C. Nardo
Professor of Mathematics
Oglethorpe University
Atlanta, GA, USA

jnardo@oglethorpe.edu

GRANT AND CREATIVE COMMONS LICENSING INFORMATION

This electronic textbook chapter and teaching manual was created for Engaging Mathematics with support from the National Science Foundation.

An initiative of the National Center for Science and Civic Engagement, Engaging Mathematics applies the well-established SENCER method to college level mathematics courses, with the goal of using civic issues to make math more relevant to students.

Engaging Mathematics will: (1) develop and deliver enhanced and new mathematics courses and course modules that engage students through meaningful civic applications, (2) draw upon state-of-the-art curriculum in mathematics, already developed through federal and other support programs, to complement and broaden the impact of the SENCER approach to course design, (3) create a wider community of mathematics scholars within SENCER capable of implementing and sustaining curricular reforms, (4) broaden project impacts beyond SENCER by offering national dissemination through workshops, online webinars, publications, presentations at local, regional, and national venues, and catalytic site visits, and (5) develop assessment tools to monitor students' perceptions of the usefulness of mathematics, interest and confidence in doing mathematics, growth in knowledge content, and ability to apply mathematics to better understand complex civic issues.

Updates and resources developed throughout the initiative will be available online at www.engagingmathematics.net. Follow the initiative on Twitter: @MathEngaging.

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Note: The "Your Vote Counts" image from this chapter's title page is a digital reproduction of the button that the NAACP chapter in New Orleans, LA, gave to newly registered voters in 2011.

TABLE OF CONTENTS

SECTION ONE – BASIC DEFINITIONS
HOMEWORK ANSWERS

1A.

Voter One	A > B > C
Voter Two	C > A > B
Voter Three	C > B > A
Voter Four	C > A > B
Voter Five	A > C > B
Voter Six	B > C > A

1B.

Number of Votes for Candidate A	2
Number of Votes for Candidate B	1
Number of Votes for Candidate C	3
Total Number of Votes (*n*)	6

Candidate C has the highest number of first-place votes and thus wins the election.

1C.

Voter One	A > B
Voter Two	A > B
Voter Three	B > A
Voter Four	A > B
Voter Five	A > B
Voter Six	B > A

Number of Votes for Candidate A	4
Number of Votes for Candidate B	2
Total Number of Votes (*n*)	6

With Candidate C out of the election, Candidate A has the highest number of first-place votes and thus wins.

2A.

Voter One	A > B > C > D	Voter Nine	D > C > B > A
Voter Two	C > B > A > D	Voter Ten	C > B > A > D
Voter Three	C > D > B > A	Voter Eleven	C > D > B > A
Voter Four	C > B > A > D	Voter Twelve	A > B > C > D
Voter Five	A > B > C > D	Voter Thirteen	C > B > A > D
Voter Six	C > B > A > D	Voter Fourteen	A > B > C > D
Voter Seven	A > D > C > B	Voter Fifteen	A > D > C > B
Voter Eight	C > D > B > A		

2B.

Voter Preference	**Number of Voters with that Preference**
A > B > C > D	4
A > D > C > B	2
C > B > A > D	5
C > D > B > A	3
D > C > B > A	1
Total	**15**

2C.

Number of Votes for Candidate A	6
Number of Votes for Candidate B	0
Number of Votes for Candidate C	8
Number of Votes for Candidate D	1
Total Number of Votes (*n*)	15

Candidate C has the highest number of first-place votes and thus wins the election.

2D.

Voter One		Voter Two		Voter Three	
1st:	A	1st:	B	1st:	D
2nd:	B	2nd:	A	2nd:	B
3rd:	D	3rd:	D	3rd:	A
Voter Four		**Voter Five**		**Voter Six**	
1st:	B	1st:	A	1st:	B
2nd:	A	2nd:	B	2nd:	A
3rd:	D	3rd:	D	3rd:	D
Voter Seven		**Voter Eight**		**Voter Nine**	
1st:	A	1st:	D	1st:	D
2nd:	D	2nd:	B	2nd:	B
3rd:	B	3rd:	A	3rd:	A
Voter Ten		**Voter Eleven**		**Voter Twelve**	
1st:	B	1st:	D	1st:	A
2nd:	A	2nd:	B	2nd:	B
3rd:	D	3rd:	A	3rd:	D
Voter Thirteen		**Voter Fourteen**		**Voter Fifteen**	
1st:	B	1st:	A	1st:	A
2nd:	A	2nd:	B	2nd:	D
3rd:	D	3rd:	D	3rd:	B

Voter One	A > B > D	Voter Nine	D > B > A
Voter Two	B > A > D	Voter Ten	B > A > D
Voter Three	D > B > A	Voter Eleven	D > B > A
Voter Four	B > A > D	Voter Twelve	A > B > D
Voter Five	A > B > D	Voter Thirteen	B > A > D
Voter Six	B > A > D	Voter Fourteen	A > B > D
Voter Seven	A > D > B	Voter Fifteen	A > D > B
Voter Eight	D > B > A		

Voter Preference	Number of Voters with that Preference
A > B > D	4
A > D > B	2
B > A > D	5
D > B > A	4
Total	**15**

Number of Votes for Candidate A	6
Number of Votes for Candidate B	5
Number of Votes for Candidate D	4
Total Number of Votes (n)	15

With Candidate C out of the election, Candidate A has the highest number of first-place votes and thus wins.

3A. Two Candidates: A and B (HINT: Use the General Counting Principle.)

Total Number of Orderings = 2

Theoretical Voter Preferences: **You do it.**

3B. Three Candidates: A, B, and C

Total Number of Orderings = 6

Theoretical Voter Preferences: **You do it.**

3C. Four Candidates: A, B, C, and D

Total Number of Orderings = 24

Theoretical Voter Preferences: **You do it.** (HINT: A tree diagram may help!)

3D. This is a **permutation**. **You explain why.**

3E. Total Number of Orderings with n Candidates = $n!$

4A. Yes
4B. No
4C. Yes
4D. Yes
4E. No

NOTES:

- For each case where closure is possessed, you are **NOT** expected to use mathematical induction to prove your claim. Instead, argue intuitively using your knowledge of these number systems and arithmetic.
- For each case where closure is not possessed, you are expected to give and explain a valid counterexample.

5. Voting System: The candidate with the **lowest** number of votes wins.
This voting system does NOT possess any of our four fairness properties.

You explain by constructing counterexamples!

6. **An election has 3,575 votes, and there are two candidates on the ballot.**

6A. **What is the majority threshold?** 1,788

6B. **Interpret what this number means.**

The lowest number of votes a candidate can receive but still be the majority candidate is 1,788 votes. In other words, if a candidate receives 1,788 votes (or more), then he/she will have strictly more than 50% of the votes.

6C. **Candidate X receives 1,787 votes. Is X the majority candidate for this election?**

No

6D. **Candidate Y receives the remainder of the votes. Is Y the majority candidate for this election?**

Yes

6E. **Who is the Condorcet candidate for this election? If there is none, explain.**

Candidate Y

SECTION TWO – PLURALITY & PLURALITY WITH ELIMINATION VOTING HOMEWORK ANSWERS

1A. In an election with two candidates, here is a tie scenario.

Candidate A:	50
Candidate B:	50
	n = 100

1B. In an election with three candidates, here is a scenario for a tie among all three.

Candidate A:	50
Candidate B:	50
Candidate C:	50
	n = 150

1C. In an election with three candidates, here is a scenario for a tie for two candidates.

Candidate A:	50
Candidate B:	50
Candidate C:	5
	n = 105

1D. In an election with four candidates, here is a scenario for a tie among all four.

Candidate A:	50
Candidate B:	50
Candidate C:	50
Candidate D:	50
	n = 200

1E. In an election with four candidates, here is a scenario for a tie for three candidates.

Candidate A:	50
Candidate B:	50
Candidate C:	5
Candidate D:	50
	n = 155

1F. In an election with four candidates, here is a scenario for a tie for two candidates.

Candidate A:	50
Candidate B:	50
Candidate C:	10
Candidate D:	5
	n = 115

2. I will not give any answers to this question since we will be returning to this idea in Section 5. Keep working on this as we progress through our voting theory material.

Hint: Plurality **does** possess the majority property. Write up a proof which shows that in every theoretical plurality election the majority property is satisfied.

3. The same note and hint applies for plurality with elimination.

4. The Mathematics Club (n = 24)

Voter Preferences:				
	A > I > M:	5	A > M > I:	3
	I > A > M:	5	I > M > A:	4
	M > A > I:	4	M > I > A:	3

4A. Who is the majority candidate? If there is none, then explain why not.

None

4B. Who is the Condorcet candidate? If there is none, then explain why not.

None

4C. Who is the winner under plurality?

Candidate I

4D. Who is the winner under plurality with elimination?

None

This voting system has failed, and the Mathematics Club must pick another voting method!

5. The Dining Hall (n = 583)

Voter Preferences:				
	A > B > O:	152	A > O > B:	47
	B > A > O:	156	B > O > A:	52
	O > A > B:	72	O > B > A:	104

5A. Who is the majority candidate? If there is none, then explain why not.

None

5B. Who is the Condorcet candidate? If there is none, then explain why not.

Candidate B

5C. Who is the winner under plurality?

Candidate B

5D. Who is the winner under plurality with elimination?

Candidate B

6. Ice Cream Flavors (n = 75)

Voter Preferences:				
	B > C > D:	9	B > D > C:	7
	C > B > D:	19	C > D > B:	25

D > B > C: 5	D > C > B: 10

6A. Who is the majority candidate? If there is none, then explain why not.

Candidate C

6B. Who is the Condorcet candidate? If there is none, then explain why not.

Candidate C

6C. Who is the winner under plurality?

Candidate C

6D. Who is the winner under plurality with elimination?

Candidate C

7. Study Locations (n = 400)

Voter Preferences:		
	C > L > S: 126	C > S > L: 118
	L > C > S: 99	L > S > C: 30
	S > C > L: 16	S > L > C: 11

7A. Who is the majority candidate? If there is none, then explain why not.

Candidate C

7B. Who is the Condorcet candidate? If there is none, then explain why not.

Candidate C

7C. Who is the winner under plurality?

Candidate C

7D. Who is the winner under plurality with elimination?

Candidate C

8. Building Preferences (n = 27)

Voter Preferences:		
	O > P > T > W: 7	T > P > O > W: 7
	W > T > P > O: 8	P > O > T > W: 5

8A. Who is the majority candidate? If there is none, then explain why not.

None

8B. Who is the Condorcet candidate? If there is none, then explain why not.

Candidate T

8C. Who is the winner under plurality?

Candidate W

8D. Who is the winner under plurality with elimination?

Candidate O

9. Designer's Signature Color (*n* = 200)

Voter Preferences:

U > R > G > P: 55	G > U > P > R: 27
U > G > R > P: 45	R > P > U > G: 42
R > U > G > P: 30	P > U > G > R: 1

9A. Who is the majority candidate? If there is none, then explain why not.

None

9B. Who is the Condorcet candidate? If there is none, then explain why not.

Candidate U

9C. Who is the winner under plurality?

Candidate U

9D. Who is the winner under plurality with elimination?

Candidate U

10. Toy Dog Breeds (*n* = 1,374)

Voter Preferences:

H > M > P > S > K: 241	M > H > K > P > S: 212
H > P > S > M > K: 189	M > P > H > S > K: 219
K > H > M > S > P: 123	P > H > K > M > S: 187
S > K > H > M > P: 104	P > K > S > H > M: 99

10A. Who is the Condorcet candidate? If there is none, then explain why not.

Candidate H

10B. Who is the winner under plurality?

Candidate M

10C. Who is the winner under plurality with elimination?

Candidate H

11. Political Party (n = 12,008)
Voter Preferences:

A > E > I > O > H: 1,567
A > I > H > O > E: 337
H > I > E > A > O: 3,003
I > H > E > O > A: 2,640
E > H > I > O > A: 1,141
H > A > I > O > E: 1,531
H > O > A > I > E: 1,789

11A. Who is the majority candidate? If there is none, then explain why not.

Candidate H

11B. Who is the winner under plurality?

Candidate H

11C. Who is the winner under plurality with elimination?

Candidate H

12. Mother's Day Flowers (n = 53,496)
Voter Preferences:

R > L > T > O > C: 8,531
L > R > O > T > C: 10,832
O > L > R > T > C: 8,096
R > T > O > C > L: 9,785
O > T > C > R > L: 11,312
C > T > R > L > O: 4,940

12A. Who is the winner under plurality?

Candidate O

12B. Who is the winner under plurality with elimination?

Candidate R

SECTION THREE – BORDA COUNT VOTING SYSTEM
HOMEWORK ANSWERS

1. I will not give the full answers to this question since we will be returning to this idea in Section 5. Keep working on this as we progress through our voting theory material.

Hint: Borda count **does** possess the monotonicity property. Write up a proof which shows that in every theoretical Borda count election the monotonicity property is satisfied.

2. Each student should write his/her own short biographical sketch of Borda.

3. Each student should research the differences and similarities between Borda's method and that of medieval Spanish theologian Ramon Llull (Raimundo Lulio) and decide whether we should we retroactively charge Borda with plagiarism of Llull's ideas. The book referenced in your e-textbook chapter would be an ideal place to start.

4. Each student should decide whether to agree or disagree with the underlying equality assumption in the classical Borda count system and justify that choice.

5. Your e-textbook chapter presented one alternative Borda count system in Section 3.4 and justified it. Each student should do the same.

6. The Mathematics Club needs to decide on a foreign destination for its Spring Break trip. The members ranked the given trip destinations from most favorite to least favorite.

Voters: Twenty-four student members of the Mathematics Club
Candidates: Argentina (A), Italy (I), and Morocco (M)
Voter Preferences:

A > I > M:	5	A > M > I:	3
I > A > M:	5	I > M > A:	4
M > A > I:	4	M > I > A:	3

Ranking	Borda Points
1st	3
2nd	2
3rd	1

6A. Who is the winner under the Borda count voting system?

Candidate I

6B. Give the full rankings for the results of this election.

We rank the candidates from most Borda points to least Borda points.

1st: Italy/I 2nd: Argentina/A 3rd: Morocco/M

7. The Dining Hall wants to know what fresh fruit is the most popular student snack.

Voters:	583 students surveyed in the Dining Hall	
Candidates:	Apples (A), Bananas (B), and Oranges (O)	
Voter Preferences:	A > B > O: 152	A > O > B: 47
	B > A > O: 156	B > O > A: 52
	O > A > B: 72	O > B > A: 104

Ranking	Borda Points
1^{st}	3
2^{nd}	2
3^{rd}	1

Candidate B

8. A local children's hospital explores the popularity of various ice cream flavors.

Voters:	Seventy-five patients at the local children's hospital	
Candidates:	Birthday Cake (B), Chocolate (C), and Cookie Dough (D)	
Voter Preferences:	B > C > D: 9	B > D > C: 7
	C > B > D: 19	C > D > B: 25
	D > B > C: 5	D > C > B: 10

Ranking	Borda Points
1^{st}	3
2^{nd}	2
3^{rd}	1

8A. Who is the winner under the Borda count voting system?

Candidate C

8B. Give the full rankings for the results of this election.

We rank the candidates from most Borda points to least Borda points.
1^{st}: Chocolate/C 2^{nd}: Chocolate Chip Cookie Dough/D 3^{rd}: Birthday Cake/B

9. The Student Government asked a random sample of students where they liked to study.

Voters:	400 students from the random sample	
Candidates:	Empty Classroom (C), Library (L), and Student Center (S)	
Voter Preferences:	C > L > S: 126	C > S > L: 118
	L > C > S: 99	L > S > C: 30
	S > C > L: 16	S > L > C: 11

Ranking	Borda Points
1st	3
2nd	2
3rd	1

Candidate C

10. A neighborhood committee has raised enough money to add a new feature to this small community. All homeowners were asked to vote on their building preferences.

Voters: Twenty-seven homeowners in this community
Candidates: Pool (O), Playground (P), Tennis Court (T), and Paved Walking Paths (W)
Voter Preferences: O > P > T > W: 7 T > P > O > W: 7
W > T > P > O: 8 P > O > T > W: 5

Ranking	Borda Points
1st	4
2nd	3
3rd	2
4th	1

10A. Who is the winner under the Borda count voting system?

Candidate P

10B. Give the full rankings for the results of this election.

We rank the candidates from most Borda points to least Borda points.
1st: Playground/P 2nd: Tennis Court/T 3rd: Pool/O 4th: Walking Paths/W

11. An up-and-coming designer is trying to choose a signature color for her new clothing line. In making her decision, she surveys 200 potential customers who fit her marketing profile at a local shopping center.

Voters: 200 potential customers
Candidates: Green (G), Pink (P), Red (R), and Purple (U).
Voter Preferences: U > R > G > P: 55 G > U > P > R: 27
U > G > R > P: 45 R > P > U > G: 42
R > U > G > P: 30 P > U > G > R: 1

Ranking	Borda Points
1st	4
2nd	3
3rd	2
4th	1

Candidate U

12. During an annual dog show, a team of reporters asked those in attendance to rank their dog breed preferences from a small selection of toy breeds.

Voters: 1,374 people in attendance at the dog show
Candidates: Havanese (H), Cavalier King Charles Spaniel (K), Maltese (M), Papillon (P), and Shih Tzu (S)
Voter Preferences:

H > M > P > S > K: 241
H > P > S > M > K: 189
K > H > M > S > P: 123
S > K > H > M > P: 104
M > H > K > P > S: 212
M > P > H > S > K: 219
P > H > K > M > S: 187
P > K > S > H > M: 99

Ranking	Borda Points
1st	5
2nd	4
3rd	3
4th	2
5th	1

12A. Who is the winner under the Borda count voting system?

Candidate H

12B. Give the full rankings for the results of this election.

We rank the candidates from most Borda points to least Borda points.
1st: Havanese/H 2nd: Maltese/M 3rd: Papillon/P
4th: Cavalier King Charles Spaniel/K 5th: Shih Tzu/S

13. The "Angry Mathematicians for Social Change" political party is trying to decide the right adjective for its party name. The group can either stay "angry" or use one of several synonyms instead. All party members at the most recent convention were surveyed.

Voters: 12,008 party members at the most recent convention
Candidates: Angry (A), Enraged (E), Heated (H), Irate (I), and Outraged (O)
Voter Preferences:

A > E > I > O > H: 1,567
A > I > H > O > E: 337
H > I > E > A > O: 3,003
I > H > E > O > A: 2,640
E > H > I > O > A: 1,141
H > A > I > O > E: 1,531
H > O > A > I > E: 1,789

Ranking	Borda Points
1st	5
2nd	4
3rd	3
4th	2
5th	1

Candidate H

14. Every Mother's Day, a marketing group researches which flowers are most popular as floral gifts for mothers in Georgia.

Voters:	53,496 surveyed Georgians who bought flowers for Mother's Day	
Candidates:	Carnations (C), Lilacs (L), Orchids (O), Roses (R), and Tulips (T)	
Voter Preferences:	R > L > T > O > C: 8,531	R > T > O > C > L: 9,785
	L > R > O > T > C: 10,832	O > T > C > R > L: 11,312
	O > L > R > T > C: 8,096	C > T > R > L > O: 4,940

Ranking	Borda Points
1st	5
2nd	4
3rd	3
4th	2
5th	1

14A. Who is the winner under the Borda count voting system?

Candidate R

14B. Give the full rankings for the results of this election.

We rank the candidates from most Borda points to least Borda points.

1st: Roses/R 2nd: Orchids/o 3rd: Tulips/T
4th: Lilacs/L 5th: Carnations/C

SECTION FOUR – PAIRWISE COMPARISONS VOTING SYSTEMS HOMEWORK ANSWERS

1. I will not give the full answers to this question since we will be returning to this idea in Section 5. Keep working on this as we progress through our voting theory material.

Hint: The pairwise comparisons voting system **does** possess the Condorcet property. Write up a proof which shows that in every theoretical pairwise comparison election the Condorcet property is satisfied

2. If the Student Government chose to use **plurality with elimination** as the voting system, then the winner would be Option A. The students will have an athletic theme for homecoming using this voting system.

Again, the author extends grateful thanks to colleague Dr. Paul Koester from the University of Kentucky and also to his students at Oglethorpe University for suggesting the voting scenarios in these homework problems!

3. **The Mathematics Club** ($n = 24$)

Voter Preferences:

A > I > M:	5	A > M > I:	3
I > A > M:	5	I > M > A:	4
M > A > I:	4	M > I > A:	3

3A. Who is the winner under the pairwise comparisons voting system?

None
The voting system fails, and there is a tie between A and I.

3B. Give the full rankings for the results of this election.

We rank from highest number of Condorcet points to lowest.
Tie for 1st Place: A & I   Last Place: M

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
I	None	I	None

4. **The Dining Hall** ($n = 583$)

Voter Preferences:

A > B > O: 152	A > O > B: 47
B > A > O: 156	B > O > A: 52
O > A > B: 72	O > B > A: 104

Who is the winner under pairwise comparisons?

Candidate B

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
B	B	B	B

5. **Ice Cream Flavors** (n = 75)
Voter Preferences: B > C > D: 9 B > D > C: 7
C > B > D: 19 C > D > B: 25
D > B > C: 5 D > C > B: 10

5A. Who is the winner under the pairwise comparisons voting system?

Candidate C

5B. Give the full rankings for the results of this election.

We rank from highest number of Condorcet points to lowest.
1st Place: C 2nd Place: D 3rd Place: B

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
C	C	C	C

6. **Study Locations** (n = 400)
Voter Preferences: C > L > S: 126 C > S > L: 118
L > C > S: 99 L > S > C: 30
S > C > L: 16 S > L > C: 11

Who is the winner under the pairwise comparisons voting system?

Candidate C

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
C	C	C	C

7. **Building Preferences** (n = 27)
Voter Preferences: O > P > T > W: 7 T > P > O > W: 7
W > T > P > O: 8 P > O > T > W: 5

7A. Who is the winner under the pairwise comparisons voting system?

Candidate T

7B. Give the full rankings for the results of this election.

We rank from highest number of Condorcet points to lowest.
1st Place: T 2nd Place: P 3rd Place: O 4th Place: W

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
W	O	P	T

8. **Designer's Signature Color** (*n* = 200)

Voter Preferences:

U > R > G > P: 55
U > G > R > P: 45
R > U > G > P: 30
G > U > P > R: 27
R > P > U > G: 42
P > U > G > R: 1

Who is the winner under the pairwise comparisons voting system?

Candidate U

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
U	U	U	U

9. **Mother's Day Flowers** (*n* = 53,496)

Voter Preferences:

R > L > T > O > C: 8,531
L > R > O > T > C: 10,832
O > L > R > T > C: 8,096
R > T > O > C > L: 9,785
O > T > C > R > L: 11,312
C > T > R > L > O: 4,940

9A. **Who is the winner under the pairwise comparisons voting system?**

Candidate R

9B. **Give the full rankings for the results of this election.**

We rank from highest number of Condorcet points to lowest.

1st Place: R   2nd Place: O   3rd Place: L   4th Place: T   5th Place: C

Comparisons of Winners			
Plurality	Plurality with Elimination	Borda Count	Pairwise Comparisons
O	R	R	R

SECTION FIVE – ARROW'S IMPOSSIBILITY THEOREM HOMEWORK ANSWERS

1. **Building Preferences** ($n = 27$)

Voter Preferences: O > P > T > W: 7 T > P > O > W: 7
W > T > P > O: 8 P > O > T > W: 5

1A. **Who is the majority candidate? If there is none, then explain why not.**

None

1B. **Who is the Condorcet candidate? If there is none, then explain why not.**

Candidate T

1C. **Who is the winner under plurality?**

Candidate W

1D. **Who is the winner under plurality with elimination?**

Candidate O

1E. **Who is the winner under the Borda count voting system?**

Ranking	Borda Points
1^{st}	4
2^{nd}	3
3^{rd}	2
4^{th}	1

Candidate P

1F. **Who is the winner under the pairwise comparisons voting system?**

Candidate T

Note that there is a different winner under each voting system!

1G. For a voting system to possess the majority property, the conditional statement below must be true.

If there is a majority candidate, then he/she must win.

This example tells us nothing about the majority property. It is neither a proof nor a counterexample.

1H. For a voting system to possess the Condorcet property, the conditional statement below must be true.

If there is a Condorcet candidate, then he/she must win.

Comparing the winners under these voting systems and the Condorcet candidate provides a counterexample to the Condorcet property for these three voting systems: plurality, plurality with elimination, and Borda count. **You explain clearly in a concrete, short sentence for each.**

Only under the pairwise comparisons voting system does the Condorcet candidate actually win the election. This implies that this voting system **could** possess the Condorcet property, i.e. this homework problem is **not** a counterexample.

However, do not be confused: this one homework problem does **not** prove that the pairwise comparisons voting system possesses this fairness property. You would need to show that every possible election using pairwise comparisons results in a Condorcet candidate, if there is one, winning the election. A proof is needed here, not this one example.

Thanks again to Dr. Paul Koester for providing this awesome example; what an elegant and compelling example! It provides three counterexamples in one problem. Nice!

2. Please see your professor or teacher for answers to this question! He/she may not want them revealed until your class reaches a certain point in learning this material.

		Fairness Property			
		Majority	Condorcet	Monotonicity	Independence of Irrelevant Alternatives
Voting System	Plurality				
	Plurality with Elimination				
	Borda Count				
	Pairwise Comparisons				

3. Each student will research and write his/her own biographical sketch. Perhaps a visit to the Library or work with your campus librarians would be of assistance here? Reach out to them, if you're inclined.

4. Answers vary.