Computer Aided Geometric Design 10 (1993) 1–24
North-Holland

COMAID 299

A new intersection algorithm for cyclides and swept surfaces using circle decomposition *

John K. Johnstone

Department of Computer Science, The Johns Hopkins University, Baltimore, MD 21218, USA

Received November 1991
Revised June 1992

Abstract

Johnstone, J.K., A new intersection algorithm for cyclides and swept surfaces using circle decomposition, Computer Aided Geometric Design 10 (1993) 1–24.

The present vocabulary of a solid modeler is canonically the plane, (some subset of) the quadrics, and the torus. The class of cyclides is also becoming important. Quadrics and cyclides lie in the more general class of ringed surfaces: surfaces that can be swept out by a circle. This class also contains the important class of revolute surfaces. We will present a method for the exact intersection of any ringed surface with any quadric or cyclide. This algorithm shows that it is feasible to expand the vocabulary of solid modeling primitives to include all ringed surfaces. In solid modeling, surface intersection is crucial to the design of solids and their subsequent analysis.

Our intersection algorithm is exact: that is, the intersection is computed symbolically rather than numerically. For exact intersection, we must reduce to degree-4 computations. We do this by concentrating on the decomposition of a surface into simpler components. Previous algorithmic development has centered around the degree of an algebraic surface. Two keys to our algorithm are circle decomposition and inversion. Solutions are provided for the inversion of a cyclide to a torus, a torus, a torus to a cyclide, and the inversion of any circle.

Keywords. Intersection; cyclides; circles; inversion; ringed surfaces.

1. Introduction

In this paper, we develop a new intersection method that can be used to find an exact parameterization of the intersection curve of higher degree surfaces. The surfaces that we concentrate on are swept surfaces called *ringed* surfaces, which include any surface that can be swept out by a circle. This class contains all revolute surfaces and all quadrics, surfaces that are very important for design. Note that a ringed surface can be of arbitrarily high degree.

We also concentrate on cyclides, which are special ringed surfaces and a natural extension of the quadric and torus. Cyclides have received considerable attention by the solid modeling community as a blending surface, a patch, and a tractable surface.

It is desirable to add both ringed surfaces and cyclides to a solid modeler, which in its

Correspondence to: J.K. Johnstone, Department of Computer Science, The Johns Hopkins University, Baltimore, MD 21218, USA. Email: jj@cs.jhu.edu.
* This work supported by National Science Foundation grant IRI-8910366.

rudimentary form only includes planes, natural quadrics, and tori. A key step in the integration of any primitive into the vocabulary of a solid modeler is the development of an intersection algorithm. This paper addresses this problem.

We develop a method for the intersection of any quadric or cyclide surface with a ringed surface. The new method uses circle decomposition and inversion to solve the intersection problem, rather than the traditional elimination, tracing, or implicit/parametric methods. This new intersection technique is powerful: we can find an exact parameterization of the intersection curve of a complicated surface such as the quartic cyclide with a surface of arbitrary degree, without resorting to approximate methods. This defies the common conception that exact intersection is restricted to low degree surfaces.

In the next section, we establish that cyclides and ringed surfaces are the natural extensions of quadric surfaces and the natural primitives to include in the vocabulary of a solid modeler, after the quadric surface. Section 3 reviews previous work on intersection and cyclides. The next two sections formally introduce cyclides and the inversion map, respectively. We present our new algorithm for the intersection of a ringed surface and a quadric or cyclide in Section 6. Section 7 shows how to compute the inverse of a torus, while the more difficult inversion from a cyclide to a torus is developed in Section 8. We discuss the mapping of a cyclide to a quadric in Section 9. The precise effect of inversion on a circle is examined in Section 10. Methods for the circle decomposition of a cyclide are presented in Section 11, while the advantages of this decomposition are analyzed in Section 12. Section 13 ends with conclusions and ideas for future work.

2. Cyclides and ringed surfaces are the natural extensions of quadrics

In this section, we would like to argue that the cyclide is the most natural extension of the present solid modeling CSG (constructive solid geometry) vocabulary. The plane, the quadric surfaces, and the torus are the most fundamental primitives for design. Revolute surfaces are also very popular, because they arise naturally in manmade objects. All of these surfaces (except the plane) are examples of ringed surfaces, as we shall see below.

Definition 2.1. A *ringed* [1] *surface* is a surface generated by a circle of variable radius sweeping through space. The curve that the center of the circles sweeps along is a *directrix curve*. A *radius function* and *orientation function* give the radius and orientation of the circle at each point of the directrix curve.

Remark 2.2. A well-known subclass of ringed surfaces is the canal surface, which is a surface generated by sweeping a sphere along a curve or, equivalently, a ringed surface where the plane of the circle is always perpendicular to the directrix. Also, note the similarity between ringed and ruled surfaces, which are surfaces generated by a line sweeping through space.

The Dupin cyclide is a special ringed surface that has received considerable attention (see Section 3). It is very useful in blending and patching. We shall introduce it formally in Section 4. We believe that the renewed interest in the cyclide is natural: it is not only very useful, but it is the logical extension of the quadric surface and the torus, which are of course the most popular primitives for solid modeling. To see this, note that the quadric is a ringed surface

[1] Since we have not been able to find a term for this class of surfaces in the literature, we have introduced the term 'ringed'. The following definition explains our choice of this term: *Ring*: anything having a circular form; v., to move in a ring or a curving course (Random House Dictionary, 1980).

with a line directrix and a quadratic radius function [Johnstone & Shene '90, Snyder & Sisam '14], a torus is a ringed surface with a circle directrix and a constant radius function, while a Dupin cyclide is a ringed surface with a conic directrix and a quadratic radius function.

3. Previous work

Our method of intersection, using circle decomposition and inversion, contrasts with more traditional techniques of intersection, such as tracing and elimination [Hoffmann '89]. Tracing [Bajaj et al. '88] is an excellent technique when numeric computation of the intersection is necessary. The output is a collection of points on the intersection of two surfaces. Elimination theory provides a very general method for the intersection of two curves or surfaces, using the elimination of a variable by a resultant. This method is not appropriate for the intersection of a ringed surface and a cyclide, because the resultant is of degree $4n$, where n is the degree of the ringed surface (e.g., $4n = 16$ if the ringed surface is a cyclide).

Another popular method of intersection is the implicit/parametric method: (1) the parametric equation of one surface is substituted into the implicit equation of the other surface, (2) the resulting equation (in the parameters s and t) is solved for s (in terms of t), and (3) this solution is substituted back into the original parametric equation. It is important to note that de Pont and Martin [de Pont '84, Martin et al. '86] use this method to solve the intersection of cyclides with planes, quadrics, and cyclides. They observe that the implicit/parametric method is applicable to cyclide/cyclide intersection because the equation derived in step (2) is of degree 4 in each of the parameters, although it is of degree 8 overall. Thus, it is possible to solve the equation symbolically for either s or t. Our method is an extension of their work to general ringed surfaces, as well as an interesting alternative for cyclide/cyclide and cyclide/quadric intersection.

Remark 3.1. Sabin [Sabin '89] comments on the tractability of cyclide intersection, noting that the finite real part of the intersection of two cyclides is a curve of degree 8 (rather than 16 as expected) and genus 3, the latter property implying that it has a parametric equation in terms of solutions of fourth-order equations, and thus can be solved symbolically.

It is interesting that cyclide/cyclide intersection is exactly solvable, reducing to a degree-4 computation, despite the fact that computing the distance between two circles, which is related to detecting a null intersection between two tori, has no closed-form solution [Neff '90].

There is a considerable body of work on the Dupin cyclide. Two early classical mathematical studies are the papers of Maxwell [Maxwell 1868] and Cayley [Cayley 1896]. They develop the key properties of Dupin cyclides. A more recent discussion of the geometry, construction, and classification of cyclides is available in [Chandru et al. '89]. Cyclides are an object of interest for mathematicians, because they are the only surface whose lines of curvature are all circular, and because they are the only surfaces whose evolute (or surface of centers) consists of two curves, rather than two surfaces [Hilbert & Cohn-Vossen '52].

The use of cyclides in solid modeling and design was first explored at Cambridge, with a collection of work including the theses: [Martin '82], [de Pont '84], and [Sharrock '85]. They develop a surface design system based on patching together cyclides. The system uses principal patches, whose boundaries are lines of curvature, which makes the use of cyclides natural.

The use of cyclides as blending surfaces has been developed in [Hoffmann '88], then [Chandru et al. '90], and [Dutta '89]. Blending is the operation of smoothing off sharp corners and edges, a fundamental operation in solid modeling. Cyclides prove to be natural blending surfaces, because they are the envelope of a sphere rolling about a conic. The rolling of a sphere about is intersection of two surfaces is a good way of blending them; and the spine of this spherical envelope can be approximated by conic segments. Pratt [Pratt '89a,b, '90] shows how to use a cyclide to blend a plane and cylinder, a sphere and cone, two cones, a plane and cyclide, and a sphere and cyclide. He also develops the representation of a cyclide as a rational biquadratic Bezier patch and discusses the offsets of cyclides. Further discussion of cone/cone blending is available in [Boehm '90], which also analyzes the cyclide Bézier patch, and [Shene & Johnstone '92]. Other papers on cyclides are [McLean '85] and [Degen '90].

The use of a circle decomposition of a ringed surface to solve ringed/quadric and ringed/cyclide intersection is reminiscent of the use of a line decomposition of a rule surface to solve the intersection of a ruled surface with any surface of degree four or less. The decomposition of ruled surfaces to the line (t) representation is covered in [Johnstone '89]. Levin recognized that it is easy to intersect with ruled surfaces: he reduced quadric intersection to the intersection of a quadric with a ruled [Levin '76].

4. Dupin cyclides

4.1. Definitions

There are many definitions in use for a cyclide, so it is important to make it clear what our definition is. A general definition of a cyclide is a quartic surface with a double curve at the circle at infinity [2] or a cubic surface that contains the circle at infinity (e.g., [Forsyth '12, Sommerville '34]). A Dupin cyclide is a special type of cyclide, and the first type of cyclide studied. The most general definition of a Dupin cyclide, and one that is often used in the literature (e.g., [Chandru et al. '89, Fischer '86]), is a surface whose lines of curvature are all circles or lines. Note that, under this definition, the Dupin cyclide includes the plane, sphere, cylinder, and cone: those key surfaces that are already in the vocabulary of a solid modeler. In this paper, we shall use a more restricted definition of the Dupin cyclide, which agrees with Dupin's original definition (the envelope of a sphere that is tangent to three fixed spheres in a continuous manner [Maxwell 1868]).

Definition 4.1. A *Dupin cyclide* is a surface whose lines of curvature are all circles.

This definition removes surfaces such as the plane, cylinder, cone, and parabolic cyclide, some of whose lines of curvature are lines. In the rest of the paper, 'cyclide' will refer to the quartic Dupin cyclide of Definition 4.1.

Remark 4.2. The centers of the lines of curvature of a Dupin cyclide lie on a conic [Forsyth '12, Hilbert & Cohn-Vossen '52]. That is, the Dupin cyclide is a circle sweeping along a conic.

In this sense, a Dupin cyclide is the generalization of a quadric surface, since any quadric surface is a circle sweeping along a line.

[2] The circle at infinity is the curve $x^2 + y^2 + z^2 + w^2 = 0$.

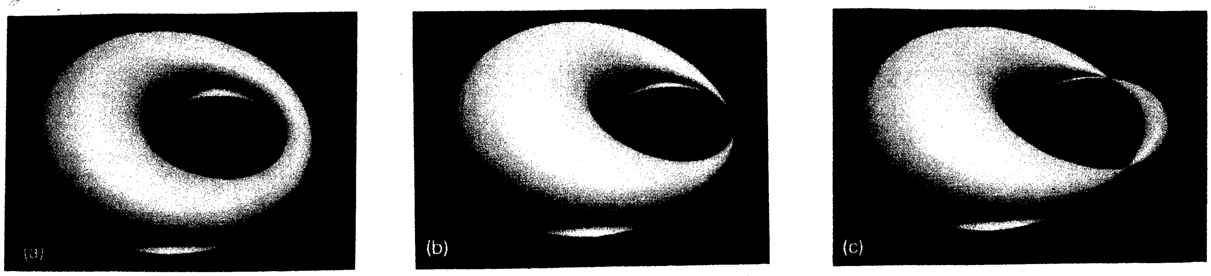


Fig. 1. Cyclides: (a) ring cyclide, (b) singly-horned cyclide, (c) doubly-horned cyclide.

4.2. Mechanisms for defining cyclides

4.2.1. Implicit equation

Forsyth [Forsyth '12] developed a normal form for the implicit equation of a Dupin cyclide:

$$\left(x^2+y^2+z^2-\mu^2+b^2\right)^2=4(ax-c\mu)^2+4b^2y^2 \tag{1}$$

where a, $b>0$, c, $\mu \geqslant 0$, and $a^2=b^2+c^2$.

4.2.2. The planes of symmetry of a cyclide

The cyclide has two planes of symmetry, and its cross-section by either of these planes is a pair of circles. These circles yield a lot of information about the cyclide.

Lemma 4.3. *The planes of symmetry of the cyclide in* (1) *are* $y=0$ *and* $z=0$ (Fig. 2). *The two circles in the plane* $y=0$ *are*

$$(x-a)^2+z^2=(\mu-c)^2 \quad \text{and} \quad (x+a)^2+z^2=(\mu+c)^2.$$

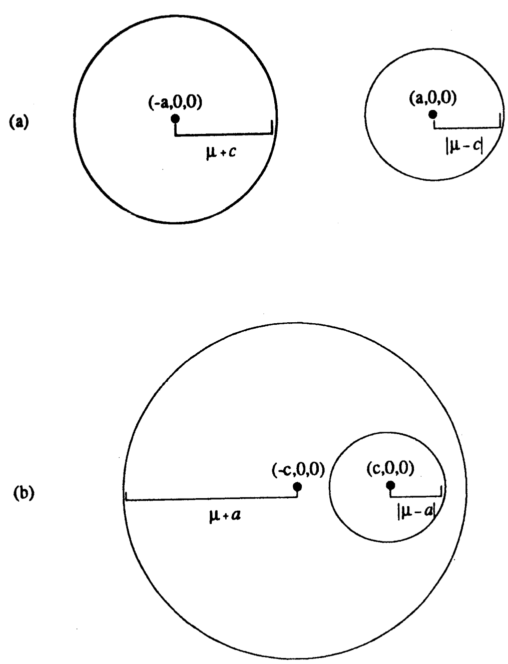


Fig. 2. (a) $y=0$, the exterior plane of symmetry. (b) $z=0$, the interior plane of symmetry.

The two circles in the plane $z=0$ are

$$(x-c)^2+y^2=(\mu-a)^2 \quad \text{and} \quad (x+c)^2+y^2=(\mu+a)^2.$$

Proof. The pair of circles in $y=0$ are found by factoring $(x^2+z^2-\mu^2+b^2)^2=4(ax-c\mu)^2$ into

$$\left[(x^2+z^2-\mu^2+b^2-2(ax-c\mu)\right]\left[(x^2+z^2-\mu^2+b^2+2(ax-c\mu)\right]=0,$$

which is equivalent, using $a^2=b^2+c^2$, to

$$\left[(x-a)^2+z^2-(\mu-c)^2\right]\left[(x+a)^2+z^2-(\mu+c)^2\right],$$

The same technique is used for the pair of circles in $z=0$, except the equation (1) for the cyclide is replaced by the following equivalent equation [Forsyth '12]:

$$(x^2+y^2+z^2-\mu^2-b^2)^2=4(cx-a\mu)^2-4b^2z^2. \quad \square \qquad (2)$$

Remark 4.4. The smaller circle in the $y=0$ plane never lies inside the larger one (i.e., $2a+|\mu-c|>\mu+c$). Conversely, the smaller circle in the $z=0$ plane never lies outside the larger one (i.e., $2c<|\mu-a|+\mu+a$).

Remark 4.4 motivates the following terminology.

Definition 4.5. The *exterior* (resp., *interior*) *plane of symmetry* of a cyclide is the plane of symmetry whose smaller circle never lies inside (resp., outside) the larger circle. The *exterior* (resp., *interior*) *circles* are the two circles that lie in the exterior (resp., interior) plane of symmetry.

Remark 4.6. $y=0$ is the exterior plane of symmetry and $z=0$ is the interior plane of symmetry for the normal form (1).

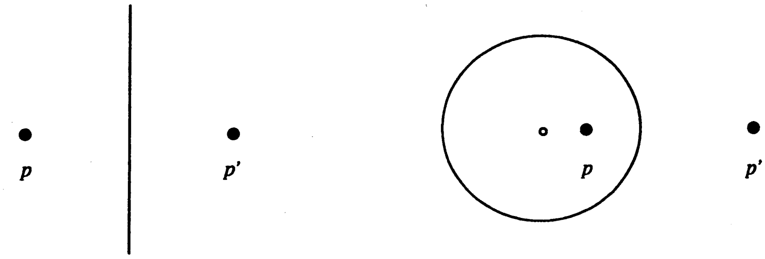
We can identify three configurations for the exterior and interior circles of a cyclide:

1. the smaller exterior circle lies completely outside the larger exterior circle, and the smaller interior circle lies completely inside the larger interior circle;
2. the smaller exterior circle intersects the larger exterior circle, and the smaller interior circle lies completely inside the larger interior circle;
3. the smaller exterior circle lies completely outside the larger exterior circle, and the smaller interior circle intersects the larger interior circle.

Associated with these three configurations are the three classes of cyclides: *ring* cyclides, *self-intersecting* cyclides, and *horned* cyclides, respectively (see Fig. 1). The horned cyclide splits further into *singly-horned* (when the smaller interior circle is tangent to the larger interior circle) and *doubly-horned* (otherwise). The *horns* of a horned cyclide are the intersections of its interior circles.

Remark 4.7. Using the normal form (1) and Fig. 2, we note that the exterior circles intersect if and only if $\mu \geqslant a$, while the interior circles intersect if and only if $\mu \leqslant c$. (For example, $2c+(a-\mu)\geqslant \mu+a \Rightarrow 2c \geqslant 2\mu$.) Thus, a cyclide is horned if $\mu \leqslant c$, ring if $c<\mu<a$, and self-intersecting if $\mu \geqslant a$.

The above development gives us a very useful way of defining a cyclide. The following result can be easily proved using Fig. 2.



(a) (b)

Fig. 3. (a) Reflection in a line. (b) Inversion in a circle.

Notation 4.8. Let $C(c, r, o)$ be the circle with center c and radius r that lies in the plane with normal o.

Lemma 4.9. *A cyclide is fully determined by its exterior circles and its type, or its interior circles and its type. In particular, the cyclide with exterior circles* $C_1(c_1, r_1, o)$ *and* $C_2(c_2, r_2, o)$, $r_1 \geqslant r_2$, *is the cyclide with normal form* (1) *where*

- $a = \| c_2 - c_1 \| /2$,
- $\mu = (r_1 - r_2)/2$ *if the cyclide is horned*; $\mu = (r_1 + r_2)/2$ *otherwise,*
- $c = r_1 - \mu$, *and*
- $b = \sqrt{a^2 - c^2}$,

under the rigid transformation that rotates the y-axis to the vector o, *and translates the origin to* $(c_1 + c_2)/2$. [3]

The cyclide with interior circles $C_1(c_1, r_1, o)$ *and* $C_2(c_2, r_2, o)$, $r_1 \geqslant r_2$, *is the cyclide with normal form* (1) *where*

- $c = \| c_2 - c_1 \| /2$,
- $\mu = (r_1 + r_2)/2$ *if the cyclide is self-intersecting*; $\mu = (r_1 - r_2)/2$ *otherwise,*
- $a = r_1 - \mu$, *and*
- $b = \sqrt{a^2 - c^2}$,

under the rigid transformation that rotates the z-axis to the vector o, *and translates the origin to* $(c_1 + c_2)/2$.

Remark 4.10. In this paper, we are consistently able to represent a cyclide by two circles in a plane of symmetry. All of the algorithms that deal with a cyclide use this representation (Theorems 7.2 and 8.8).

Cayley also used two circles in a plane of symmetry to define the cyclide. However, his method of deriving the cyclide from these two circles is quite different (see [Chandru et al. '89, p.285]).

Chandru, Dutta, and Hoffmann [Chandru et al. '89] observe that the input of a cyclide by two (exterior) circles in its plane of symmetry is natural for blending, where the circles can encode two (extremal) positions of a rolling ball blend.

5. Inversion

In this section, we formally introduce the inversion map. As suggested by Fig. 3, inversion is the generalization of reflection. While a point is reflected in a line or plane, it is inverted with respect to a circle or sphere.

[3] Note that the normal form (1) assumes that the exterior circles lie in the plane $y = 0$, centered at the origin.

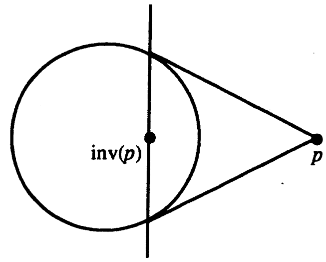


Fig. 4. The relation between inversion and polars.

In the following discussion, we shall present definitions and results for the sphere; however, most results also apply to inversion in a circle, by simply replacing sphere by circle and plane by line.

Definition 5.1. $\text{inv}_S(p)$, the *inverse of a point* $p \in \mathbb{R}^3$ in the sphere S of radius R centered at c, is the point $p' \in \overrightarrow{cp}$ such that $\text{dist}(c, p) * \text{dist}(c, p') = R^2$. S, c and R are called the *sphere of inversion*, the *center of inversion*, and the *radius of inversion*, respectively. As a special case, the inverse of a point in a plane is the reflection of that point in the plane.

Note that, under inversion, the interior of the sphere of inversion is mapped to its exterior and vice versa, while the sphere is mapped to itself.

Lemma 5.2. *Let S be the sphere of inversion with center c and radius R.*

$$\text{inv}_S(p) = \frac{R^2}{\| p - c \|^2}(p - c) + c.$$

Proof. The inverse of p is $p' = c + k(p - c)$, where $k \| p - c \| * \| p - c \| = R^2$. □

The most important property of inversion is that, in general, it maps circles to circles. This property will be examined in detail in Section 10. [Pedoe '57] is an excellent source of proofs for Properties (1)–(5).

(1) The inverse of a circle is a circle or a line. It is a line exactly when the circle passes through the center of inversion. [4]

(2) The inverse of a sphere is a sphere or a plane. It is a plane exactly when the sphere passes through the center of inversion.

Several other properties of inversion are indirectly important.

(3) $\text{inv}_S^{-1} = \text{inv}_S$, i.e., $\text{inv}_S(\text{inv}_S(p)) = p$.

Combining properties (1) and (3), we have

(4) The inverse of a line L is a circle through the center of inversion or a line. It is a line exactly when L passes through the center of inversion, in which case the inverse of L is L.

[4] This is the key to Peaucellier's linkage for drawing a straight line.

Property (4) is also true for planes.

(5) Inversion is conformal. That is, the angle between two intersecting curves is preserved under inversion.

Definition 5.3. If the point p lies outside the sphere S, the *polar* of p with respect to S is the plane that intersects S in the points of tangency from p (Fig. 4).

(6) If p lies outside S, $\text{inv}_S(p)$ lies on the polar of p with respect to S [Coolidge '16]. Thus, $\text{inv}_S(p)$ is the intersection of p's polar and the line to the center of S (Fig. 4).

(7) Lines of curvature are preserved under inversion [Fischer '86].

A direct consequence of (7), and (1), is that cyclides are mapped to cyclides.

(8) The inverse of a Dupin cyclide is a Dupin cyclide, if the center of inversion does not lie on the cyclide. [5]

In special cases, the inverse of a cyclide can be simpler than a cyclide.

(9a) Any Dupin cyclide can be inverted to a torus.
(9b) Any singly-horned Dupin cyclide can be inverted to a cylinder.
(9c) Any double-horned Dupin cyclide can be inverted to a cone.
(9d) Any self-intersecting Dupin cyclide can be inverted to a cone.

These important cases will be proved and discussed in Section 8.

6. A new method for intersection

In this section, we present our method for the exact intersection of any ringed surface with a quadric or cyclide. We begin with intersection with a quadric surface, then intersection with a horned or self-intersecting cyclide, and finally intersection with a ring cyclide. Our goal is to reduce everything to a degree-4 computation. This is necessary for exact computation, since equations of higher degree do not have closed form solutions [Hernstein '75] and must be solved numerically.

Consider the intersection of a ringed surface with a quadric surface, which illustrates the importance of circle decomposition in intersection. Using a circle decomposition of the ringed surface (that is, a decomposition of the surface into circles, $\bigcup_{t \in I} \text{circle}(t)$), the ringed/quadric intersection can be reduced to a circle(t)/quadric intersection. This can be interpreted as an infinite number of circle/quadric intersections. Treating the parameter t as a constant, we can solve symbolically for the intersection of the circle with the quadric (since it is a degree-4 computation). The resulting parameterization in t is the intersection curve.

Circle decomposition also immediately simplifies the intersection of a ringed surface with a cyclide, by reducing it to circle(t)/cyclide intersection, which is a symbolic degree-8 computation. The secret to reducing the intersection to a degree-4 computation is to reduce the circles to lines. In particular, exact ringed/cyclide intersection requires inversion as well as circle decomposition. Recall that inversion can reduce the degree, as in the mapping of a circle to a line, or a horned cyclide to a cylinder.

In some cases, the intersection of a ringed surface and a cyclide quickly reduces to the intersection of a ringed surface and a quadric. Recall that a horned cyclide can be inverted to

[5] Under the more general definition of a Dupin cyclide as a surface whose lines of curvature are circles or lines, the inverse of a Dupin cyclide is always a Dupin cyclide.

ringed ∩ cyclide

↓

circle(t) ∩ cyclide

↓

circle(t) ∩ torus

↓

line(t) ∩ cyclide(t)

Fig. 5. Ringed/cyclide intersection (last two arrows are inversion maps).

a cylinder or cone, and a self-intersecting cyclide can be inverted to a cone (Properties (9b)–(9d) of inversion). Suppose that the cyclide is horned or self-intersecting, and the inversion map f maps it to a cylinder or cone. By circle decomposition, the ringed/cyclide intersection can be reduced to a circle(t)/cyclide intersection. Since circles are inverted to circles, this can be further reduced to a circle(t)/cylinder or circle(t)/cone intersection by the inversion f: $\bigcup_{t \in I}[f(\text{circle}(t))] \cap f(\text{cyclide})$. This is the intersection of a ringed surface and a quadric, which has been solved above. The desired result is the inversion $f(C)$ of this curve C.

We can now restrict our attention to the intersection of a ringed surface with a ring cyclide, which is the most challenging problem. The ringed/cyclide intersection is first reduced to $\text{circle}_1(t)$/cyclide intersection. The degree of the computation is then halved from 8 to 4 through inversion (Fig. 5). We reduce the (ring) cyclide to a (ring) torus T via inversion (Property (9a) of inversion). This inversion maps the circles of the ringed surface to circles. That is, the $\text{circle}_1(t)$/cyclide intersection is reduced via inversion to a $\text{circle}_2(t)$/torus intersection. This is an (infinite) collection of circle/torus intersections, still a degree-8 computation.

Remark 6.1. The reason that we first reduce the cyclide to a torus is that it is easier to compute the inverse of a torus with respect to an arbitrary point (Section 7), a facility that is needed in the next step.

We reduce further to an (infinite) collection of line intersections, by inverting each circle to a line. The reduction of each circle of $\text{circle}_2(t)$ to a line requires a different center of inversion for each circle, since the center of inversion must lie on the circle in order to invert it to a line. To solve this problem, we use a symbolic center of inversion: $\text{circle}_2(t)$ will have the center of inversion $(\text{circle}_2(t))(k)$, where k is a constant but t is a free parameter. Under this symbolic inversion, the torus is inverted to a symbolic cyclide. Thus, the collection of circle/torus intersections, $\bigcup_{t \in I}[\text{circle}_2(t) \cap \text{torus } T]$, is reduced to a collection of line/cyclide intersections: $\bigcup_{t \in I}[\text{line}(t) \cap \text{cyclide}(t)]$, where

$$\text{line}(i) = \text{inv}_{(\text{circle}_2(i))(k)}\text{circle}_2(i) \quad \text{and} \quad \text{cyclide}(i) = \text{inv}_{(\text{circle}_2(i))(k)}T.$$

In order to solve for the intersection line (t) and cyclide(t), we substitute the parameterization of line(t) into the implicit equation of cyclide(t), yielding a degree-4 equation in t which can be solved symbolically, yielding the points of intersection of line(t)/cyclide(t). The result of line(t)/cyclide(t) is the parameterization (in t) of the intersection curve of the ringed surface and the original cyclide, after a series of two inversions (Fig. 5). Thus, we need to invert this result, first via the symbolic center of inversion $(\text{circle}_2(t))(k)$ and then via the center of inversion used for the original cyclide-to-torus inversion. The final result is the desired intersection curve.

The computation of the implicit equation of cyclide(t), which is needed in the intersection of line(t) and cyclide(t), requires some elaboration. Consider the inversion from the torus to cyclide(t). (Section 7 has the full discussion of this inversion.) The torus T will be defined by a special pair of exterior circles (Theorem 7.2). If the center of inversion (circle$_2$(t))(k) is outside the torus for a specific value of t, it turns out that these exterior circles are mapped to exterior circles of the cyclide (Theorem 7.2), and the implicit equation of the cyclide can therefore be determined by applying the first half of Lemma 4.9, which translates from exterior circles to an implicit equation. However, if the center of inversion is inside the torus for a specific value of t, it turns out that the exterior circles of the torus are mapped to interior circles of the cyclide (Theorem 7.2), and the implicit equation of the cyclide must instead be determined by applying the second half of Lemma 4.9. Thus, we shall find the implicit equations of *two* cyclides that together define cyclide(t): $(c_1(x, y, z))(t)$ and $(c_2(x, y, z))(t)$, where

$$\text{cyclide}(t) = \begin{cases} c_1(t) & \text{if } (\text{circle}_2(t))(k) \text{ lies outside the torus } T, \\ c_2(t) & \text{if } (\text{circle}_2(t))(k) \text{ lies inside the torus } T. \end{cases}$$

We can identify the intervals where cyclide(t) $= c_1(t)$ and the remaining intervals where cyclide(t) $= c_2(t)$ by computing the intersections of $\{(\text{circle}_2(t)(k) \mid t \in I\}$ with the torus. We then compute both line(t) $\cap\, c_1(t)$ and line(t) $\cap\, c_2(t)$, and choose the appropriate one for each t.

Remark 6.2. When the center of inversion lies on the torus, the inverse of the torus is a (cubic) parabolic cyclide rather than a quartic cyclide. Thus, cyclide(t) is $c_1(t)$ on some intervals $t \in (t_i, t_{i+1})$, $c_2(t)$ on the remaining intervals $t \in (t_j, t_{j+1})$, and $c_3(t)$ for $t = t_i$, $i = 1, \ldots, N$, where $c_3(t)$ is a parabolic cyclide. To avoid the complication of computing $c_3(t)$, we ignore the values $t = t_i$ when computing the line(t)/cyclide(t) intersections, thus ignoring a finite number of points from the intersection. In fact, since the intersection curve is continuous, these 'missing' points are limits of their neighbours, and are automatically filled in.

To review, here is our algorithm for ringed/cyclide intersection, including a guide to the sections of the paper that fill in the details of each step.

The Circle Method

Step 1. Express ringed $\cap$ cyclide as $\bigcup_{t \in I}$[circle(t) $\cap$ cyclide].

Step 2. If the cyclide is horned or self-intersecting, then:

(a) Reduce $\bigcup_{t \in I}$[circle(t) $\cap$ cyclide] to $\bigcup_{t \in I}$[circle(t) $\cap$ quadric], by inverting the cyclide to a cylinder or cone (Section 9) and mapping each circle under this inversion (Section 10). Let the inversion map be f.

(b) Solve this ringed/quadric intersection as described above (a symbolic intersection of a circle and quadric). If the intersection is $\alpha(t)$, then the ringed/cyclide intersection curve is $f(\alpha(t))$.

(c) Exit.

Step 3. (We now know that we are working with a ring cyclide.) Reduce $\bigcup_{t \in I}$[circle(t) $\cap$ cyclide] to $\bigcup_{t \in I}$[circle(t) $\cap$ torus], by inverting the cyclide to a torus (Section 8) and mapping each circle under this inversion (Section 10). Let the center of inversion be c.

Step 4. Let $t_1, t_2, \ldots, t_N$ be the parameter values of the intersections of the curve $[(\text{circle}(t))(k) \mid t \in I]$ with the torus.[6] Identify the intervals (t_{i_1}, t_{i_1+1}), $(t_{i_2}, t_{i_2+1}), \ldots$, (t_{i_m}, t_{i_m+1}) where $(\text{circle}(t))(k)$ lies outside the torus.

Step 5. By inverting the circle (Section 10) and torus (Section 7) with respect to the center of inversion $(\text{circle}(t))(k)$, reduce $\bigcup_{t \in I}[\text{circle}(t) \cap \text{torus}]$ to $\bigcup_{t \in I}[\text{line}(t) \cap \text{cyclide}(t)]$.

Step 6. Find two implicit equations that define cyclide(t), as described above:

$$\text{cyclide}(t) = \begin{cases} c_1(t) & \text{if } t \in \bigcup_{j \in 1,\ldots,m}\{(t_{i_j}, t_{i_j+1})\}, \\ c_2(t) & \text{otherwise}. \end{cases}$$

Step 7. Viewing t as a symbolic constant, compute $\text{line}(t) \cap c_1(t)$ for its four intersections $p_{1,1}(t)$, $p_{2,1}(t)$, $p_{3,1}(t)$, $p_{4,1}(t)$.[7] Similarly, compute $\text{line}(t) \cap c_2(t)$ for its four intersections $p_{1,2}(t)$, $p_{2,2}(t)$, $p_{3,2}(t)$, $p_{4,2}(t)$. Let

$$p_i(t) = \begin{cases} p_{i,1}(t) & \text{if } t \in \bigcup_{j \in 1,\ldots,m}\{(t_{i_j}, t_{i_j+1})\}, \\ p_{i,2}(t) & \text{otherwise}. \end{cases}$$

Step 8. Invert $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ to $q_1(t)$, $q_2(t)$, $q_3(t)$, $q_4(t)$, using the symbolic center of inversion $(\text{circle}(t))(k)$ (and the same radius of inversion as in Step 5).

Step 9. Invert $q_1(t)$, $q_2(t)$, $q_3(t)$, and $q_4(t)$ to $r_1(t)$, $r_2(t)$, $r_3(t)$, $r_4(t)$, using the center of inversion c (and the same radius of inversion as in Step 3).

Step 10. The intersection of the original ringed surface and cyclide is $\bigcup_{t \in I}[r_1(t), r_2(t), r_3(t), r_4(t)]$.

Remark 6.3. If numeric computation is preferred, then the above method of intersection can be interpreted in another way. The ringed/cyclide intersection can be approximated by a finite collection of circle/cyclide intersections, and the above method can be used to reduce the circle/cyclide intersections to line/cyclide intersections, which can be solved (numerically) more efficiently.

Remark 6.4. Since the intersection curve, $\text{circle}(t) \cap \text{cyclide}$, is computed in terms of the intersection $\text{line}(t) \cap \text{cyclide}(t)$ (at the distance of two inversion maps), we should verify that the inversions do not lose some of the intersections, and that the final line/cyclide intersection captures the same intersection curve as the circle/cyclide intersection. The only danger is that some of the points of intersection are mapped to infinity under the inversion, which can only happen if the point of intersection is the center of inversion. This is clearly not a problem with the first inversion (Step 3): the center of inversion does not even lie on the cyclide. We have to be more careful with the second inversion (Step 5, from circle(t) to line(t)). Fortunately, although the symbolic center of inversion always lies on circle(t), it does not lie on the torus, except at a finite number of points. Thus, at most a finite number of points of the intersection curve are lost (and moreover, as discussed in Remark 6.2, these points are filled in as limits of their neighbours). Thus, the line/cyclide intersection does indeed capture the same intersection curve as the circle/cyclide intersection.

[6] The parametric curve $(\text{circle}(t))(k)$ is of the same degree as the directrix curve of the ringed surface. Its intersections with the torus are found by substituting its parametric equation into the implicit equation of the torus and solving numerically for t.

[7] For each of the intersections $p_{i,1}(t)$, there will be intervals of t for which $p_{i,1}(t)$ is real and other intervals for which $p_{i,1}(t)$ is complex. The latter intervals can be found, if necessary, through an analysis of the radicals in the intersections [Turnbull '39].

7. The inverse of a torus

In this section, we show how to compute the inverse of a torus. This is needed in Step 5 of the intersection algorithm. The inverse of a torus is an easy computation for the following reason. We know that the inverse of a cyclide is a cyclide (Property (8) of inversion). The inverse of a cyclide is easy to compute if the center of inversion lies on a plane of symmetry. This plane remains a plane of symmetry and the inverse cyclide is determined by the effect of the inversion on the two circles in this plane of symmetry (Lemma 4.9). For this reason, the inverse of a torus is always easy to compute, since a torus has an infinite number of planes of symmetry which cover space. That is, the center of inversion always lies in a plane of symmetry. The only difficulty is deciding whether the plane of symmetry becomes an exterior or interior plane of symmetry. The following lemma shows how to decide.

Note that we don't need to consider the inverse of an arbitrary torus. Like cyclides, we can distinguish between ring, horned, and self-intersecting tori, and we only need to consider the inverse of ring tori (i.e., tori whose exterior circles are disjoint). This is enough because the torus in Step 5 of the algorithm is a ring torus, being the inverse of a ring cyclide.

Lemma 7.1 *Consider the inverse of a ring torus. Let P be the exterior plane of symmetry of the torus that contains the center of inversion. The exterior circles in P map to interior circles if and only if the center of inversion lies inside one of the exterior circles.*

Proof. Let C_1 and C_2 be the two exterior circles in the plane P. We will prove that inv(C_2) contains inv(C_1) if and only if C_2 contains the center of inversion. Suppose C_2 contains the center of inversion c. In particular, in every direction from c, C_2 is encountered before C_1. After inversion, in every direction from c, inv(C_1) will be encountered before inv(C_2), because the closer a point is to the center of inversion, the further away it is mapped from the center of inversion. Since inv(C_2) is a circle that contains the center of inversion, this implies that inv(C_2) contains inv(C_1).

Suppose C_2 does not contain the center of inversion. Circles are stretched radially out from the center of inversion, or shrunk radially in. Thus, in order for C_1 to map inside C_2, it must lie in the sector defined by C_2 as well as the annulus defined by C_2 (Fig. 6). It is impossible to satisfy both of these criteria, since C_1 and C_2 are disjoint. □

This leads immediately to the desired result. Note that we can assume that the center of inversion does not lie on the torus, since this will be the case in Step 5 of the intersection algorithm (Remark 6.2).

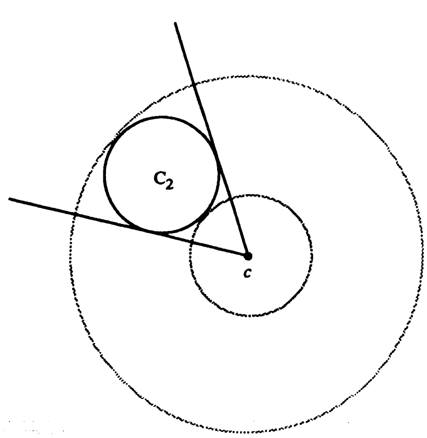


Fig. 6. C_1 cannot map inside C_2.

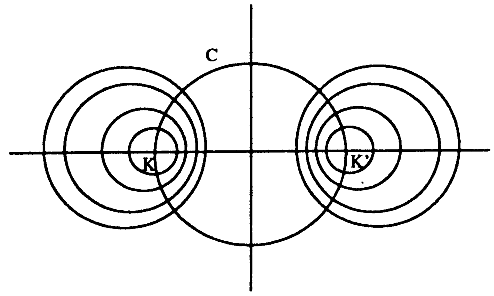


Fig. 7. A coaxal system with radical axis A, all orthogonal to the circle C.

Theorem 7.2. *Let T be a ring torus and let S be a sphere of inversion with center $c \notin T$. The inverse of T is the ring cyclide with*

interior circles $\text{inv}_S(E_1)$, $\text{inv}_S(E_2)$ *if the center of inversion lies inside T,*
exterior circles $\text{inv}_S(E_1)$, $\text{inv}_S(E_2)$ *otherwise*

where E_1 and E_2 are the exterior circles of T that lie in the same plane as the center of inversion.

Section 10 discusses how to compute the inverse of a circle (for $\text{inv}_S(E_1)$ and $\text{inv}_S(E_2)$), and Lemma 4.9 shows how to recover a cyclide from its interior or exterior circles. In the following section, we consider the map from a cyclide to a torus, which is needed in Step 3 of the intersection algorithm and in Section 11 on circle decomposition.

8. The mapping of a cyclide to a torus

The inverse of a torus is always a cyclide, but the inverse of a cyclide is rarely a torus. Nevertheless, it is always possible to find a center of inversion that maps a cyclide to a torus. It is clear from Fig. 2 that a cyclide is a torus if and only if $c = 0$, if and only if the interior circles are concentric. Thus, a cyclide can be mapped to a torus by inverting its interior circles (or its exterior circles) to concentric interior circles. We attack this problem in the following section.

8.1. Mapping to concentric circles

There is a classical method for inverting two non-intersecting circles into concentric circles.

Definition 8.1. The *power* of a point P with respect to a circle C, $\text{power}_C(P)$, is the product of the (signed) distances from P to the closest point and the furthest point of the circle.[8] Thus,

$$\text{power}_C(P) = (\|P - c\| - r)(\|P - c\| + r) = \|P - c\|^2 - r^2,$$

where c and r are the center and radius of C.

Remark 8.2. The power of a point can be thought of as a distance measure of the point from a circle. Both the closest and furthest points of the circle are used, rather than just the closest

[8] The classical definition of $\text{power}_C(P)$ is the product of signed distances from P to any pair of points U and V on C, such that U, V, and P are collinear. This product is well defined. Our definition chooses the closest and furthest points of the circle for U and V.

point. This encodes more information: if only the closest point is used, the inside of the circle cannot be distinguished from the outside. The power of a point is positive when the point lies outside the circle and negative when the point lies inside.

Definition 8.3. The *radical axis* of two coplanar circles C_1 and C_2 is the locus of points P such that $\text{power}_{C_1}(P) = \text{power}_{C_2}(P)$ (Fig. 7). A *coaxal system* of circles is a collection of circles, each pair of which have the same radical axis.

Remark 8.4. The radical axis of two circles is a line perpendicular to the line between their centers [Johnson '29]. There are three types of coaxal systems of circles:

(1) a collection of concentric circles;

(2) a collection of circles through one or two fixed points;

(3) a collection of non-intersecting circles, all of which are orthogonal to a fixed circle, and all of whose centers are collinear.

We are interested in the latter type.

Definition 8.5. Consider a coaxal system of type (3), all of whose circles are orthogonal to the circle C and all of whose centers lie on the line L. The *limiting points* of this coaxal system are the two points of intersection of C and L.

The points K and K' are the limiting points of the coaxal system in Fig. 7.

Lemma 8.6 [Johnson '29]. Two non-intersecting circles are inverted to concentric circles if a limiting point of their coaxal system is chosen as the center of inversion.

The following lemma shows how to find this limiting point.

Lemma 8.7 [Johnson '29]. *Let C_1 and C_2 be two non-intersecting circles, with centers c_1 and c_2. Let P be the intersection of their radical axis and $\overleftrightarrow{c_1c_2}$. The limiting points of the coaxal system of C_1 and C_2 lie on $\overleftrightarrow{c_1c_2}$, at distance $\sqrt{\text{power}_{C_1}(P)} = \sqrt{\text{power}_{C_2}(P)}$ from P.*

It is simple to compute P, the unique point on the line between the centers such that $\text{power}_{C_1}(P) = \text{power}_{C_2}(P)$, and thus it is simple to compute the limiting point.

8.2. Reduction of cyclide to torus

We can now invert a cyclide to a torus using inversion with respect to a limiting point of the interior or exterior circles.

Theorem 8.8. *Let C be a cyclide. Let $C_1(c_1, r_1, o)$ and $C_2(c_2, r_2, o)$, $r_1 < r_2$, be*

the interior circles of C if C is a ring or self-intersecting cyclide,
the exterior circles of C if C is a horned cyclide.

The cyclide C is mapped to a torus under an inversion whose center is the limiting point of C_1 and C_2. Moreover, the interior circles of the torus are the inverses of C_1 and C_2.

Proof. We wish to map two non-intersecting circles in a plane of symmetry of C to two concentric circles in a plane of symmetry. This will successfully map C to a torus, since the only cyclide with concentric interior circles is the torus. By choosing the center of inversion to lie in a plane of symmetry of C, we guarantee that this plane remains a plane of symmetry. The non-intersecting condition enforces the use of interior circles for the self-intersecting cyclide, and exterior circles for the horned cyclide (Remark 4.7). Note that we could have used either interior or exterior circles for the ring cyclide. □

Remark 8.9 Note that the radius of inversion is arbitrary; and that the torus is not necessarily a ring torus (e.g., it may be self-intersecting).

Remark 8.10. The inversion of a singly-horned cyclide to a torus deserves additional comment, since it is a degenerate case. Let the exterior circles of the singly-horned cyclide be C_1 and C_2. One of these exterior circles is a point, a degenerate circle (say C_2). We can still talk of the coaxal system of C_1 and C_2, but C_2 is now a degenerate circle of the coaxal system. In particular, C_2 is a limiting point of the coaxal system. (This is to be expected, since given a circle of a coaxal system, the circle always contains one of the limiting points; thus, when the circle degenerates to a point, it degenerates to the limiting point.) This means that we have to take care with the choice of the center of inversion: if the two limiting points of the coaxal system of C_1 and C_2 are C_2 and P, we should choose P as the center of inversion. Otherwise, the process of inverting the cyclide to a torus is identical. The singly-horned cyclide will map to a 'horned' torus, for which one interior circle is a point.

Since a torus is fully defined by its interior circles, and Section 10 will show how to compute the inverse of a circle, Theorem 8.8 fully defines the mapping of a cyclide to a torus.

9. The mapping of a cyclide to a quadric

The previous section showed that every cyclide can be simplified to a torus, through inversion. In this section, we show that certain cyclides (the horned cyclide and the self-intersecting cyclide) can also be mapped to a cylinder or a cone by inversion. This is a very useful mapping, since the degree of the surface drops.

9.1. Reduction of horned cyclides to cylinders and cones

Lemma 9.1. *A singly-horned cyclide is mapped to a circular cylinder if its horn is used as the center of inversion. A doubly-horned cyclide is mapped to a circular cone if one of its horns is used as the center of inversion.*

Proof. Although this result is well-known, we provide the proof since it is an important result. All circles of a horned cyclide pass through the horns. Thus, if one of the horns is used as the center of inversion, the circles are all mapped to lines and the inverse of the horned cyclide is a ruled surface. This inverse must also be a general Dupin cyclide (i.e., a surface whose lines of curvature are circles *or lines*). [9] But the only general Dupin cyclides that are ruled are the cylinder and cone.

All of the circles of a doubly-horned cyclide pass through the second horn, so this horn maps to a cone vertex and the cyclide maps to a cone. The circles of a singly-horned cyclide only share the one horn, so this cyclide maps to a cylinder. □

Remark 9.2. The interior circles are fundamental in the reduction of a horned cyclide to a cone/cylinder. The center of inversion is a horn, which is an intersection of the interior circles. Moreover, since the center of inversion lies in an (interior) plane of symmetry, the

[9] See the footnote to Property (8) of inversion.

interior circles map to lines in a plane of symmetry; and these lines clearly define the cone/cylinder. For the doubly-horned cyclide, their intersection defines the cone vertex and their angle defines the cone angle; for the singly-horned cyclide, they are parallel and their bisector is the cylinder's axis.

9.2. Reduction of self-intersecting cyclides to cones

By a similar argument, the self-intersecting cyclide can also be mapped to a cone. In Lemma 9.1, we mapped a cyclide to a cone/cylinder by inverting with respect to an intersection of the interior circles. For the self-intersecting cyclide, we use the exterior circles.

Lemma 9.3. *A self-intersecting cyclide is mapped to a cone by an inversion centered at one of the intersections of its exterior circles.*

Proof. Let C be the self-intersecting cyclide, and let i and j be the two intersections of its exterior circles. Let i be the center of inversion. It is not true that all the circles (lines of curvature) on the cyclide pass through i, but it is still true that the circles (line of curvature) of the cyclide that pass through i cover the cyclide. [10] This is enough to guarantee that $\text{inv}_S(C)$ is a ruled surface. Moreover, by the same argument, the lines on $\text{inv}_S(C)$ all pass through $\text{inv}_S(j)$. We conclude that $\text{inv}_S(C)$, which must be a general Dupin cyclide, is a circular cone. □

Remark 9.4. Just as with the doubly-horned cyclide (Remark 9.2), the vertex of the cone and the cone angle are defined by the image of the two exterior circles.

10. The inverse of any circle

The inverse of a circle is usually a circle, and this inverse circle is well understood in two dimensions: i.e., when the plane of the circle contains the center of inversion and the inversion is thus restricted to a plane. However, the literature is silent on the general case of the inverse of a circle in three dimensions. In this section, we consider this inverse of a *space* circle in detail.

Definition 10.1. A *plane circle* is a circle whose plane contains the center of inversion. A *space circle* is a circle whose plane does not contain the center of inversion.

For the mapping of a cyclide to a torus, only the inverse of a plane circle is needed, since one can work completely in a plane of symmetry of the cyclide (Section 8). However, the inverse of a space circle is needed in many other places of the intersection algorithm, such as the reduction of circle(t)/cyclide intersection to circle(t)/torus intersection (Step 3 of the algorithm in Section 6) and our circle decomposition of a cyclide (Section 11).

A circle is fully defined by its center, its radius, and its orientation in space. In the following sections, we examine how inversion affects these three components.

[10] To see this, one can invert C to a self-intersecting torus, for which the statement is clear because of the symmetry of the torus. Inversion preserves the property.

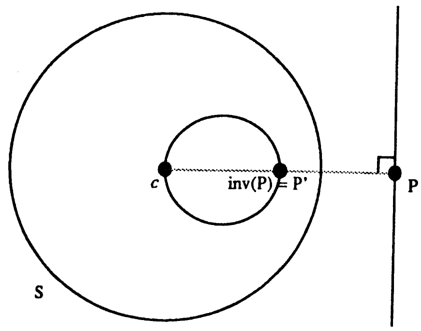


Fig. 8. The inverse of a line.

10.1 Radius

Theorem 10.2. *Let S be the sphere of inversion with center c and radius R.*

1. *Let D be a plane circle or a sphere, of radius r, $c \notin D$. The radius of* $\mathrm{inv}_S(D)$ *is*

$$\frac{R^2}{|\mathrm{power}_D(c)|}r \tag{3}$$

2. *Let D be a line or plane, $c \notin D$. The radius of* $\mathrm{inv}_S(D)$ *is*

$$\frac{R^2}{2\,\mathrm{dist}(c, D)} \tag{4}$$

3. *Let D be a space circle of radius r and center d, lying in the plane p. The radius of* $\mathrm{inv}_S(D)$ *is*

$$\begin{cases} \textit{radius of } \mathrm{inv}_S(p) & \text{if } c \in E, \\ \dfrac{\textit{radius of } \mathrm{inv}_S(E) \cdot \textit{radius of } \mathrm{inv}_S(p)}{\textit{dist}(\textit{center of } \mathrm{inv}_S(E), \textit{ center of } \mathrm{inv}_S(p))} & \textit{otherwise} \end{cases} \tag{5}$$

where E is the sphere with the same center and radius as D.

Proof. 1. The result for plane circles is classical [Coolidge '16, Johnson '29].

2. Suppose D is a line (or plane), $c \notin D$. Let P be the closest point of D to c. Since the inverse of D is a circle (or sphere) through c and $\overleftrightarrow{cP}$ is an axis of symmetry (Fig. 8), the diameter of $\mathrm{inv}_S(D)$ is dist(c, $\mathrm{inv}_S(P)$) $= R^2/\mathrm{dist}(c, P)$.

3. Let D be a space circle, the intersection of the sphere E and the plane p. $\mathrm{inv}_S(D) = \mathrm{inv}_S(E) \cap \mathrm{inv}_S(p)$. $\mathrm{inv}_S(E)$ and $\mathrm{inv}_S(p)$ are orthogonal, since E and p are (Property (5) of Section 5). $\mathrm{inv}_S(p)$ is a sphere (i.e., $c \notin p$, because D is a space circle). There are two cases based on whether $\mathrm{inv}_S(E)$ is a sphere or a plane. Suppose $c \in E$. Then $\mathrm{inv}_S(E)$ is a plane, which must pass through the center of $\mathrm{inv}_S(p)$ since $\mathrm{inv}_S(E)$ and $\mathrm{inv}_S(p)$ are orthogonal. Thus, the radius of $\mathrm{inv}_S(D)$ is identical to the radius of $\mathrm{inv}_S(p)$.

Suppose $c \notin E$. Then $\mathrm{inv}_S(E)$ is a sphere. Let the center of $\mathrm{inv}_S(E)$ be C_1, and the center of $\mathrm{inv}_S(p)$ be C_2. Any plane containing the centers of the two spheres intersects the two spheres in two circles (Fig. 9), which intersect in X and Y. The center of $\mathrm{inv}_S(D)$ is the midpoint of $\overline{XY}$, i.e., $\overline{XY} \cap \overline{C_1C_2}$. (The two spheres are generated by rotating the two circles about the axis $\overline{C_1C_2}$; thus, $\mathrm{inv}_S(D)$ is generated by rotating X and Y about $\overline{C_1C_2}$.) Let the center of $\mathrm{inv}_S(D)$ be C_3, so that the desired radius of $\mathrm{inv}_S(D)$ is dist(C_3, X). Since $\mathrm{inv}_S(E)$ and $\mathrm{inv}_S(p)$ are orthogonal, ΔXC_1C_2 is a right triangle. Its area is $\frac{1}{2}C_1X * C_2X$. Since

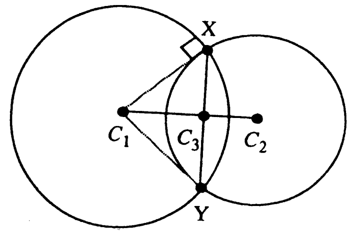


Fig. 9. C_3 is the center of the space circle.

$\angle C_1C_3X$ is also a right angle, C_3X is the altitude of ΔXC_1C_2, whose area is therefore $\frac{1}{2}C_1C_2 * C_3X$. Equating these two expressions for the area, we have $C_3X = (C_1X * C_2X)/C_1C_2$. □

Remark 10.3. The formulas (3) and (4) both involve the factor R^2/d, where d is the distance of the object from the center of inversion (Remark 8.2). This factor is also important in the mapping of a point under inversion (Lemma 5.2). It encodes the amount of stretching of a point or plane circle under inversion.

10.2. Center

The amount of stretching that a point undergoes during inversion is nonlinear. The relative stretching (or shrinking) of a point far away from the circle of inversion is much more than that of a point close to the circle of inversion. For this reason, the center of a circle does not map to the center of the new circle under inversion. However, it turns out that the new center can be found with two or three applications of the inversion map.

Theorem 10.4. *Let S be the sphere of inversion with center c and radius R.*

1. *Let D be a plane circle or a sphere, of center d, $c \notin D$. The center of* $\text{inv}_S(D)$ *is*

$$\text{inv}_S(\text{inv}_D(c)). \tag{6}$$

2. *Let D be a line or plane, with equation $f(x, y, z) = 0$ and normal N, $c \notin D$. The center of* $\text{inv}_S(D)$ *is*

$$\text{inv}_S(\text{reflection}_D(c)) = c - \frac{R^2}{2f(c)}N. \tag{7}$$

3. *Let D be a space circle with center d, lying in the plane p. The center of* $\text{inv}_S(D)$ *is*

$$\begin{array}{ll} \textit{center of } \text{inv}_S(p) & \textit{if } c \in E, \\ \text{inv}_{\text{inv}_{S(p)}}(\textit{center of } \text{inv}_S(E)) & \textit{otherwise} \end{array} \tag{8}$$

where E is the sphere with the same center and radius as D.

Proof. 1. This is a classical result [Coolidge '16, Davis '49, Johnson '89].

2. It is classical that the center of $\text{inv}_S(D)$ is $\text{inv}_S(\text{reflection}_D(c))$ [Coolidge '16]. To get the formula, note that the reflection of c in the plane D is

$$c - 2\frac{f(c)}{\|N\|}\frac{N}{\|N\|},$$

since c lies at distance $|f(c)|/\|N\|$ from D in the direction $(\text{sign}(f(c)))N$ [Snyder & Sisam '14]. An application of Lemma 5.2 yields the result.

3. Let D be a space circle. $\text{inv}_S(D) = \text{inv}_S(E) \cap \text{inv}_S(p)$. Recall from the proof of Theorem 10.2 that, if $c \in E$, $\text{inv}_S(E)$ is a plane that passes through the center of the sphere $\text{inv}_S(p)$, so the center of $\text{inv}_S(D)$ is identical to the center of $\text{inv}_S(p)$. Suppose $c \notin E$. $\text{inv}_S(D)$ is the intersection of the two spheres $\text{inv}_S(E)$ and $\text{inv}_S(p)$, with centers C_1 and C_2. Any plane containing the centers of the two spheres intersects the two spheres in two circles (Fig. 9), which intersect in X and Y. As we saw in Theorem 10.2, the center of $\text{inv}_S(D)$ is $\overline{XY} \cap \overline{C_1C_2}$. We can find this point using inversion, as follows. $\text{inv}_S(E)$ and $\text{inv}_S(p)$ are orthogonal, since E and p are. That is, C_1's and C_2's tangents are normal at X and Y. Thus, C_2's tangents at X and Y pass through C_1. That is, $\overleftrightarrow{XY}$ is the polar of C_1 with respect to the circle centered at C_2. By Property (6) of Section 5, $\overleftrightarrow{XY} \cap \overline{C_1C_2}$ is the inverse of C_1 in the circle centered at C_2. That is, the center of $\text{inv}_S(D)$ is the inverse of C_1 in $\text{inv}_S(p)$. Note that by symmetry, the center of $\text{inv}_S(D)$ is also the inverse of C_2 in $\text{inv}_S(E)$. □

It is straightforward to reduce the space circle formulas (5) and (8) to algebraic formulas, by applying the inversion map of Lemma 5.2 and the results for spheres and planes.

10.3. Orientation

Definition 10.5. The *orientation* of a plane is its normal vector. The orientation of a circle is the orientation of the plane that contains it.

Theorem 10.6. *Let S be the sphere of inversion with center c and radius R.*
1. *The orientation of a plane circle does not change under inversion.*
2. *Let D be a sphere with center d, such that $c \in D$. The orientation of $\text{inv}_S(D)$ is $d - c$.*
3. *Let D be a space circle with center d, lying in the plane p. The orientation of $\text{inv}_S(D)$ is*

$$\begin{cases} \text{the orientation of } \text{inv}_S(E) & \text{if } c \in E, \\ \text{the vector from the center of } \text{inv}_S(p) \text{ to the center of } \text{inv}_S(E) & \text{otherwise} \end{cases}$$

where E is the sphere with the same center and radius as D.

Proof. 2. See Fig. 8.
3. If $c \in E$, $\text{inv}_S(E)$ is a plane that contains $\text{inv}_S(D)$. If $c \notin E$, recall that $\text{inv}_S(D)$ is generated by rotating X about the axis C_1C_2, in Fig. 9. Thus, C_1C_2 is a normal to the plane that contains $\text{inv}_S(D)$. □

10.4. Circle-to-line inversion

The only case that we have not dealt with is the inverse of a circle that contains the center of inversion. In this case, we know that the circle maps to a line. Circle-to-line inversion is very important in our algorithm (Step 5). A good proof of the following well-known fact can be found in [Pedoe '57].

Lemma 10.7. *Let D be a circle that contains the center of inversion c of the sphere of inversion S. Let $\overline{cP'}$ be the diameter of D that contains c (Fig. 8). Then the inverse of D is the line through $\text{inv}_S(P')$ and perpendicular to $\overline{cP'}$.*

The inverse of an arbitrary circle in 3-space is now fully understood.

11. Circle decomposition of a cyclide

In this section, we consider the circle decomposition of a special ringed surface, the cyclide. For other ringed surfaces, we assume that a circle decomposition is given directly, but this is not necessary for the cyclide. There are two ways to see that a cyclide is no more than a collection of circles. Its lines of curvature, which cover the surface, are all circles. Alternatively, it is the inverse of a torus, which is clearly a collection of circles, and circles are preserved under inversion. This observation leads immediately to a simple method for the circle decomposition of a cyclide. The circle decomposition of a torus is easy, and we know how a circle maps under inversion.

Definition 11.1. $C(t) = [(c(t), r(t), o(t)]$, $t \in I \subset \mathbb{R}$, is a *circle decomposition* of a surface S if

- $C(t)$ is the circle with center $c(t)$, radius $r(t)$, and orientation $o(t)$,
- $S = \bigcup_{t \in I} C(t)$, and
- $c(t)$, $r(t)$, and $o(t)$ all change smoothly with t.

Lemma 11.2. *A ring torus with interior circles $C_1(c, r_1, o)$ and $C_2(c, r_2, o)$, $r_1 > r_2$, has circle decomposition*

$$\text{center}(t) = \text{mid}(t), \quad \text{radius}(t) = \frac{r_1 - r_2}{2}, \quad \text{orientation}(t) = o \times (c - \text{mid}(t)),$$

where mid(t) *is a parameterization of the circle with center c, radius$(r_1 + r_2)/2$, and orientation o.*

A self-intersecting or horned torus with interior circles $C_1(c, r_1, o)$ and $C_2(c, r_2, o)$, $r_1 > r_2$, has circle decomposition

$$\text{center}(t) = \text{mid}(t), \quad \text{radius}(t) = \frac{r_1 + r_2}{2}, \quad \text{orientation}(t) = o \times (c - \text{mid}(t)),$$

where mid(t) *is a parameterization of the circle with center c, radius $(r_1 - r_2)/2$, and orientation o.*

Lemma 11.3. *Let C be a cyclide and let the torus $T = \text{inv}_S(C)$ be its inverse, where inversion is done with respect to the center of inversion in Theorem* 8.8. *If $c(t)$ is a circle decomposition of T, then*

$$\textit{center of } \text{inv}_S(c(t)), \qquad \textit{radius of } \text{inv}_S(c(t)), \qquad \textit{orientation of } \text{inv}_S(c(t))$$

is a circle decomposition of C. The center, radius, and orientation of the circles of the latter decomposition can be determined using the results of Section 10.

Remark 11.4. A self-intersecting or horned cyclide can also be mapped to a cylinder or cone, which has a simple circle decomposition.

Remark 11.5. The analogous problem for ruled surfaces is to find a line decomposition. In [Johnstone '93], the present author shows how to achieve this line decomposition, by finding generators and a directrix curve on the ruled surface.

Remark 11.6. The circle decomposition of a quadric surface is discussed in [Johnstone & Shene '90]. Recall that the centers lie on a line, and the radius function is quadratic.

The above method is simple and, since it uses inversion, it is a natural method for our intersection algorithm. However, there are other methods for decomposing a cyclide into circles. An excellent method is due to Cayley [Cayley 1896]: if C_1 and C_2 are the two interior circles of a cyclide and P is one of their centers of similitude, [11] then the circles of the cyclide rotate about the center of similitude. That is, a circle of the cyclide lies in a plane through P (and perpendicular to the plane of the interior circles), and its diameter is defined by the intersections of this plane with the two interior circles. This method is also reported in [Chandru et al. '89, p. 282] and [Coolidge '16, p. 270].

Forsyth [Forsyth '12] gives formulae for the radius and center of the circles on a cyclide. However, the formulae are for the normal form (1) and involve trigonometric functions.

12. Advantages of a circle decomposition

We have already seen that the representation of a surface by circles is a great advantage in intersection. However, there are other reasons that a circle decomposition is advantageous.

- (circle(s))(t) is a very good parameterization. It is easy to define geometrically meaningful subsets of the surface by a range on t, and it is always of degree 2 in s.
- For the cyclide, the circle decomposition has the advantage of revealing the lines of curvature. These lines of curvature are important, for example, in defining principal patches from the cyclide [de Pont '84, Martin '82, Sharrock '85].
- It is simple to render the surface, since the circle is a basic graphical element.
- Rotation and translation of the surface is simple: it reduces to rotation and translation of the directrix curve (Definition 1), which may be simpler than rotation and translation of the entire surface. The radius of each circle, and the orientation of each circle relative to the directrix curve, does not change. Moreover, there is no danger of introducing errors into these radii and orientations due to floating point error during the manipulation.
- The surface has the representation of a generalized cylinder, which is a preferred representation for computer vision applications.
- Offset computation of the cyclide is simple [Pratt '89a, Martin '82, Dutta '89].

13. Conclusions

In this paper, we have developed a method for the exact intersection of any ringed surface with any surface in the present vocabulary (quadrics and cyclides). This is a first step towards expanding the vocabulary of a solid modeler to include all ringed surfaces. The intersection algorithm is not only directly applicable to modeling with these surfaces, but it reveals that exact intersection of complex surfaces is feasible, and that the decomposition of surfaces into simpler components (in this case, circles) can be powerful. We suspect that alternatives such as this to traditional methods (e.g., elimination methods or implicit/parametric combinations) will prove very useful in handling other complicated intersections.

The two key techniques of this paper are circle decomposition and inversion. The circle decomposition immediately simplifies an apparently degree-$4n$ computation (where n is the degree of the ringed surface), ringed $\cap$ cyclide, to a degree-8 computation of circle(t) $\cap$ cyclide. The inversion map induces another simplification, transforming a cyclide to a torus (which is an easier surface to work with) and the circle to a line (which is a curve of lower degree). This leads to a degree-4 computation of the intersection.

[11] A *center of similitude* of C_1 and C_2 is a point that divides the line between the centers of C_1 and C_2 in the ratio of their radii [Pedoe '70]. That is, V is a center of similitude if $V \in \overleftrightarrow{C_1C_2}$ and $\|C_1V\| / \|C_2V\| = r_1/r_2$. V is the inner (resp., outer) center of similitude if $V \in \overline{C_1C_2}$ (resp., $V \notin \overline{C_1C_2}$).

We end by proposing some directions for future work. First, the extension of other algorithms (e.g., blending, representation, volume, ray casting) to all ringed surfaces is needed, so that we can fully incorporate this class into the vocabulary. Second, since we have seen the benefit of the description of surfaces as circles or lines sweeping along simple curves (ringed and ruled surfaces), this approach might be extended further, looking at other simple curves sweeping along simple curves. The work on generalized cyclides (e.g., [Degen '90]) is an example of work in this direction. Finally, the intersection of ringed surface with ringed surface, and ruled surface with ruled surface is an important problem that we are presently working on.

Acknowledgements

Discussions with Chris Hoffmann were very helpful. I thank C.-K. Shene for an improved proof of Theorem 10.2 and for Figure 1, and for useful discussions.

References

Bajaj, C.L., Hoffmann, C.M., Hopcroft, J.E. and Lynch, R.E. (1988), Tracing surface intersections, Computer Aided Geometric Design 5, 285–307.

Boehm, W. (1990), On cyclides in geometric modeling, Computer Aided Geometric Design 7, 243–255.

Cayley, A. (1896), On the cyclide, in: *The Collected Mathematical Papers of Arthur Cayley*, Vol. IX, Cambridge University Press, Cambridge, 64–78.

Chandru, V., Dutta, D. and Hoffmann, C. (1989), On the geometry of Dupin cyclides, Visual Computer 5, 277–290.

Chandru, V., Dutta, D. and Hoffmann, C. (1990), Variable radius blending using Dupin cyclides, in: Wozny, M.J. et al., eds., *Geometric Modeling for Product Engineering*, IFIP Proceedings, North-Holland, Amsterdam.

Coolidge, J.L. (1916), *A Treatise on the Circle and the Sphere*. Chelsea, Bronx, NY.

Davis, D.R. (1949), *Modern College Geometry*, Addison-Wesley, Cambridge, MA.

de Pont, J.J. (1984), Essays on the Cyclide Patch. PhD thesis, Cambridge University.

Degen, W.L.F. (1990), Generalized cyclides for use in CAGD, Technical report, Mathematisches Institut B, Universität Stuttgart.

Dutta, D. (1989), Variable-radius Blends and Dupin Cyclides. PhD thesis, School of Industrial Engineering, Purdue University, West Lafayette, IN.

Farouki, R.T. (1986), The approximation of non-degenerate offset surfaces, Computer Aided Geometric Design 3, 15–43.

Fischer, G., ed. (1986), *Mathematical Models: Commentary*. Vieweg, Braunschweig, Germany.

Forsyth, A.R. (1912), *Lectures on the Differential Geometry of Curves and Surfaces*. Cambridge University Press, Cambridge.

Hernstein, I.N. (1975), *Topics in Algebra*. Wiley, New York.

Hilbert, D. and Cohn-Vossen, S. (1952), *Geometry and the Imagination*. Chelsea, New York.

Hoffmann, C. (1988), Variable radius blending using cyclides. *NSF-IFIP Workshop on Geometric Modeling*, Sept. 1988.

Hoffmann, C. (1989), *Geometric and Solid Modeling*. Morgan Kaufmann, San Mateo, CA.

Johnson, R.A. (1929), *Modern Geometry*, Houghton Mifflin, Riverside Press, Boston, MA.

Johnstone, J.K. (1989), Working with ruled surfaces in solid modeling, in: Kaltofen, E. and Watt, S., eds., *Proceedings of Computers and Mathematics '89*, Springer, New York, 260–268.

Johnstone, J.K. (1993), Finding generators and directrix curves on algebraic ruled surfaces, to appear in Computer Aided Design.

Johnstone, J.K. and Shene, C.-K. (1990), The representation of all hyperboloids of one sheet by two lines, Technical report JHU-90-08, Department of Computer Science, The Johns Hopkins University, Baltimore, MD, revised July 1991.

Johnstone, J.K. and Shene, C.-K. (1992), Dupin cyclides as blending surfaces for cones, in: *The Mathematics of Surfaces V*, Oxford University Press.

Levin, J. (1976), A parametric algorithm for drawing pictures of solid objects composed of quadratic surfaces, Comm. ACM 19 (10), 555–563.

Martin, R.R. (1982), Principal Patches for Computational Geometry, PhD thesis, Cambridge University.
Martin, R.R., de Pont, J. and Sharrock, T. (1986), Cyclide surfaces in CAD, in: Gregory, J., ed., *The Mathematics of Surfaces I*, Clarendon Press, Oxford, 253–267.
Maxwell, J.C. (1868), On the cyclide, Quart. J. Pure Appl. Math. 9.
McLean, D. (1985), A method of generating surfaces as a composite of cyclide patches, The Computer J. 28 (4), 433–438.
Neff, C.A. (1990), Finding the distance between two circles in three-dimensional space, IBM J. Res. Develop. 34 (5), 770–775.
Pedoe, D. (1957), *Circles*, Pergamon Press, New York.
Pedoe, D. (1970), *Geometry: A Comprehensive Course*, Dover, New York.
Pratt, M.J. (1989a), Applications of cyclide surfaces in geometric modelling, in: Handscomb, D.C., ed., *The Mathematics of Surfaces III*, Clarendon Press, Oxford, 405–428.
Pratt, M.J. (1989b), Cyclide blending in solid modelling, in: Strasser, W. and Seidel, H.P., eds., *Theory and Practice of Geometric Modeling*, Springer, Berlin, 235–245.
Pratt, M.J. (1990), Cyclides in computer aided geometric design, Computer Aided Geometric Design 7, 221–242.
Sabin, M. (1989), Circles and cyclides—applying the absolute, in: Handscomb, D.C., ed., *The Mathematics of Surfaces III*, Oxford University Press, 145–151.
Sharrock, T.J. (1985), Surface Design with Cyclide Patches, PhD thesis, Cambridge University.
Snyder, V. and Sisam, C. (1914), *Analytic Geometry of Space*, Henry Holt, New York.
Sommerville, D.M.Y. (1934), *Analytic Geometry of Three Dimensions*. Cambridge University Press, Cambridge.
Turnbull, H.W. (1939), *Theory of Equations*, Oliver and Boyd, Edinburgh.