On the Lower Degree Intersections of Two Natural Quadrics I: Algorithms*†

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In general, two quadric surfaces intersect in a space quartic curve. However, the intersection frequently degenerates to a collection of plane curves. In this paper, we investigate this problem for natural quadrics. Three algorithms are presented: to detect and compute 1) conic intersections, 2) linear intersections and 3) tangency and disjointness. These methods reveal the relationship between the planes of the degenerate intersections and the quadrics. Using the theory developed in the paper, we also present a new and simplified proof of a necessary and sufficient condition for conic intersection. Since only elementary geometric routines such as line intersection are used, the algorithms are intuitive and easily implementable.

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1 Introduction

Natural quadrics consist of the spheres, the circular cylinders and the right cones. Many papers have been dedicated to this class because, in most CAGD systems, they are still the most useful and natural objects to model mechanical parts. In this paper, we will investigate the degenerate intersection of natural quadrics. In general, two quadric surfaces intersect

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Table 1: Possible Intersection Types of Two Natural Quadrics

Number of Conics	*Number of Lines*	*Number of Points*
2	0	0
1	1	0
0	1, 2, 3, 4	0
0	0	1,2

in a space quartic curve. However, the intersection may degenerate to a collection of plane curves. Using a simple geometric argument, it can be shown that there are four types of degenerate intersection for natural quadrics, as outlined in Table 1. These degenerate cases are frequent and important in CAGD, because degeneracies are often required by design. We shall develop a method of detecting and computing degenerate intersections of natural quadrics. Detection is important, because degenerate intersections can be computed more easily, and also allow simpler treatment of important problems. For example, if two natural quadrics have conic intersection, the two surfaces can be blended using a family of cyclides (Boehm [4], Pratt [29], and Sabin [32]); and if two quadrics have planar intersection, they can be blended using a quadric or cubic surface, rather than a quartic (Warren [40]).

The paper is divided into nine sections. Section 2 reviews related work. Section 3 introduces the notation used in the paper. In Section 4, our algorithm for detecting and calculating conic intersections is presented in its most general form without concern for degenerate cases. By observing that natural quadrics with planar intersection must have coplanar axes, the computation of the intersection is reduced to this plane of the axes. Then planar intersections are determined by testing the height from a point on this plane to both surfaces. These computations only involve line intersections[1] and simple arithmetic operations. Section 5 discusses how to handle the degenerate cases so that the presentation of our algorithm is complete. Sections 6 and 7 present simple algorithms for detecting and computing linear intersections, tangency and disjointness. In Section 8, we present a necessary and sufficient condition for conic intersection. Section 9 gives our conclusion.

2 Previous Work

The intersection of two surfaces is a fundamental and challenging problem in CAGD. Many powerful algorithms have been developed for computing the intersection curve of two algebraic surfaces: for example, Abhyankar–Bajaj [1], Garrity–Warren [9], Hoffmann [15], and Manocha–Canny [21]. For two quadric surfaces, the computation is simpler and has theoretically been solved in the last century (see, for example, Bromwich [5]). In the context of CAGD, Sabin [31] is one of the earliest rigorous solutions of this problem. During the past

[1]This should not be surprising since the ancient Greek geometers already recognized that the natural quadrics are "linear" objects with circular shape.

decade, starting with Levin [18, 19], algebraic methods based on the pencil of two quadrics have been successfully used to attack quadric intersection. Other examples are Farouki–Neff–O'Connor [8], Miller [22], Ocken–Schwartz–Sharir [26], O'Connor [27], and Sarraga [34].

Recent attention has been drawn to the conditions that cause degenerate intersection curves, since a general algorithm may be insufficient for degenerate cases. Farouki–Neff–O'Connor [8] is a complete study for general quadric surfaces, using the pencil technique.

Samuel–Requicha–Elkind [33] and Hakala–Hillyard–Nourse–Malraison [14] are two of the first studies of natural quadrics by the CAGD community. Piegl [28] seems to be the first paper concerning the conditions that cause the intersection of two natural quadrics to degenerate to a conic. He distinguishes eight cases for cylinder–cone intersection, in which two of them give conic intersection. However, from this enumeration, it is not clear how to devise an extension to the cone–cone case, which is more general and more subtle than the cylinder–cone case.

Concurrently with the development of our work on degenerate intersection of natural quadrics [36], Goldman and Miller [10, 11, 23, 24] developed conditions for degenerate intersection of two natural quadrics. They use the pencil method, in which the singular members of the one-parameter family $\mathcal{Q}_1 + \lambda\mathcal{Q}_2 = 0$ are computed, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two quadric surfaces. If $\mathcal{Q}_1 \cap \mathcal{Q}_2$ has planar intersection, we can find some $\bar{\lambda}$ such that $\mathcal{Q}_1 + \bar{\lambda}\mathcal{Q}_2$ is of rank two or less and thus can be factored into the product of two planes. If the two quadrics have planar intersections, then the two planar members in the pencil are computed (Miller–Goldman [23]). Finally, these planar members are intersected with the natural quadric to compute the intersection curve (Miller–Goldman [24]).

In our method, we depart from the pencil method and use geometric rather than algebraic arguments. These help to reveal the structure of the problem, simplify the development, and allow detection and computation of the degenerate intersections to be done at the same time. We also consider the question of disjointness: that is, whether the surfaces intersect at all.

3 Notation

We need to develop some terminology and notation.

Definition 3.1 *A* **natural** *quadric is a sphere, a circular cylinder, or a right cone. An* **axial** *natural quadric is a natural quadric with an axis, namely a cylinder or a cone.*

Definition 3.2 *A* **conic** *intersection is a composite intersection curve such that the highest degree among all component curves is two (Cases 1 and 2 in Table 1). A* **linear** *intersection is a composite intersection curve such that the highest degree among all component curves is one (Case 3 of Table 1). A* **point** *intersection is a composite intersection such that the components are all isolated points (Case 4 of Table 1).* **Tangency** *means a point intersection. Hence a common tangent line is a linear intersection.*

Definition 3.3 *Two surfaces are* **congruent** *if and only if there exists a rigid motion transforming one into the other.*

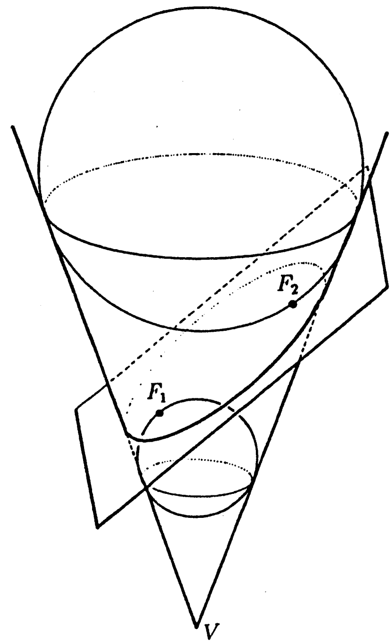

Figure 1: The Dandelin Spheres

Notation: $\mathcal{C}(V, \ell, \alpha)$ is the cone with vertex V, axis ℓ and half angle α. $\mathcal{Z}(\ell, r)$ is the cylinder with axis ℓ and radius r. $\mathcal{S}(O, r)$ is the sphere with center O and radius r. $\overleftrightarrow{AB}, \overrightarrow{AB}, \overline{AB}$ and $|\overline{AB}|$ are the line, the ray, the segment, and the length of the segment determined by two points A and B.

4 Conic Intersections

In this section, we present our conic intersection algorithm in a generic form, ignoring degenerate cases. The idea is to reduce the surface intersection problem to a simple planar line intersection problem. Section 4.1 presents some basic results which play a fundamental role in our study. Section 4.2 introduces the concept of the axial plane and proves our first structural result for conic intersection. Section 4.3 discusses the concept of height at some point on the axial plane and its computation.

4.1 Some Fundamental Results

In this section, we establish that a necessary condition for conic intersection is coplanar axes. The following classical result is useful.

Theorem 4.1 (Dandelin Sphere) *Let the plane P intersect the axial natural quadric Q in a conic. There are one or two spheres inscribed in Q and tangent to P. The tangent points of these spheres on P are the foci of the intersection conic (Figure 1).*

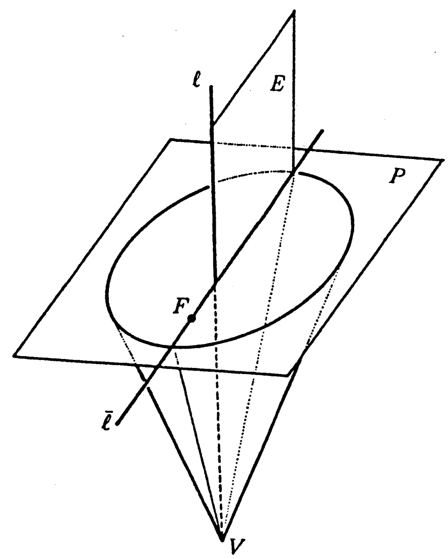

Figure 2: Relation of Plane P and Plane E

These spheres are called *focal spheres* or *Dandelin spheres*, after the Belgian mathematician Germinal P. Dandelin [6] who discovered the relationship of the inscribed sphere and the foci of the conic section in 1822.[2] The Dandelin sphere has a direct application in the computation of the intersection conic of a plane and an axial natural quadric (Johnstone–Shene [16] and Miller–Goldman [24]).

Using the Dandelin sphere, we can prove the following important technical lemma.

Lemma 4.1 *Let P be a plane cutting an axial natural quadric in a conic C. Then the plane E determined by the major axis of C and the axis of the axial natural quadric is perpendicular to P.*

Proof: Note that P cannot contain the axis ℓ of the axial natural quadric, since otherwise the intersection is two lines, not a conic. If P is perpendicular to ℓ, the intersection is a circle, and the lemma is obviously true. If P is not perpendicular to ℓ, consider the Dandelin sphere D yielding the focus F (Figure 2). The plane E containing ℓ and perpendicular to P cuts the surface, the plane P and the Dandelin sphere D into two symmetric parts. Therefore $\bar{\ell} = E \cap P$ must be an axis of the intersection conic. Note that the tangent point F lies on $\bar{\ell} = E \cap P$ because of symmetry. Hence $\bar{\ell}$ must be the major axis. ∎

We can immediately use this lemma to develop a necessary condition for conic intersection.

[2]The complete foci–directrix relation was determined by Pierce Morton [25] a few years later. Dandelin's beautiful idea is covered in many texts on conic section theory. See, for example, Berger [3], Drew [7] or Macaulay [20], for geometric development, and Taylor [39] for historical material along this line. Moreover, many descriptive geometry texts (Bereis [2], Kramer [17], Rehbock [30] and Schmid [35]) have more detailed discussions about the Dandelin sphere and its applications.

Lemma 4.2 *If two axial natural quadrics have a conic intersection, their axes are coplanar.*

Proof: Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two axial natural quadrics. If $\mathcal{Q}_1 \cap \mathcal{Q}_2$ contains a circle, then the axes must coincide. Otherwise, let P be the plane containing one of the intersection conics. By Lemma 4.1, the plane $\mathcal{H}$ determined by the major axis of the intersection conic and the axis of $\mathcal{Q}_1$ is perpendicular to P. The same is true for $\mathcal{Q}_2$. Hence both axes of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ lie in plane $\mathcal{H}$. ∎

4.2 The Axial Plane

In this section, we will discuss the axial plane and identify the planes that must contain the conics of a conic intersection.

Definition 4.1 *If the axes of two axial natural quadrics are distinct and coplanar, they determine a plane called the* **axial plane.**[3] *This plane cuts each surface in a pair of straight lines, called the* **skeletal pair**. *The lines in a skeletal pair are called* **skeletal lines**.

Consider the non-degenerate case in which neither vertex of the axial natural quadrics lies on a skeletal line. That is, the four skeletal lines form a *complete quadrilateral* with six intersection points. (We assume that we are working in the Euclidean plane with the line at infinity so that any two lines intersect in a point.) These six points generate three lines not in any skeletal pair, called *diagonals*. However, since we are not interested in the diagonal containing vertices, we restrict to diagonals that do not contain a vertex.[4] Figure 3 shows some examples of skeletal pairs and their diagonals. Parts of the diagonals that lie in the interior of both surfaces are indicated by thick line segments. In Figure 3(c) and (d), lines in one skeletal pair are parallel to their counterpart from the other pair. Hence two intersection points are at infinity and the diagonal containing them is the line at infinity. In Figure 3(e), notice that one diagonal includes an intersection point at infinity.

The following lemma characterizes the non-degenerate case mentioned above.

Lemma 4.3 *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two axial natural quadrics with coplanar and distinct axes. Let the axial plane cut $\mathcal{Q}_i$ in the skeletal pair S_i. Then S_1 and S_2 form a complete quadrilateral (i.e., non-degenerate) if and only if both $V_1 \notin S_2$ and $V_2 \notin S_1$ hold, where V_i is the vertex of $\mathcal{Q}_i$.*

Proof: ($\Rightarrow$) A complete quadrilateral has four sides, three diagonals, and six intersection points. It is clear that $V_1 \notin S_2$ and $V_2 \notin S_1$ hold.

($\Leftarrow$) Suppose $V_1 \notin S_2$ and $V_2 \notin S_1$. We have four distinct lines, no three collinear. Hence they form a complete quadrilateral. ∎

[3] If the axes are identical, then the two quadrics are either identical, disjoint or have an intersection of two circles that is easy to compute.

[4] Since the axial plane cuts a cylinder in two parallel lines and parallel lines meet at the point of infinity along the lines' direction, a cylinder has its vertex at the point of infinity along the direction of any line on its surface.

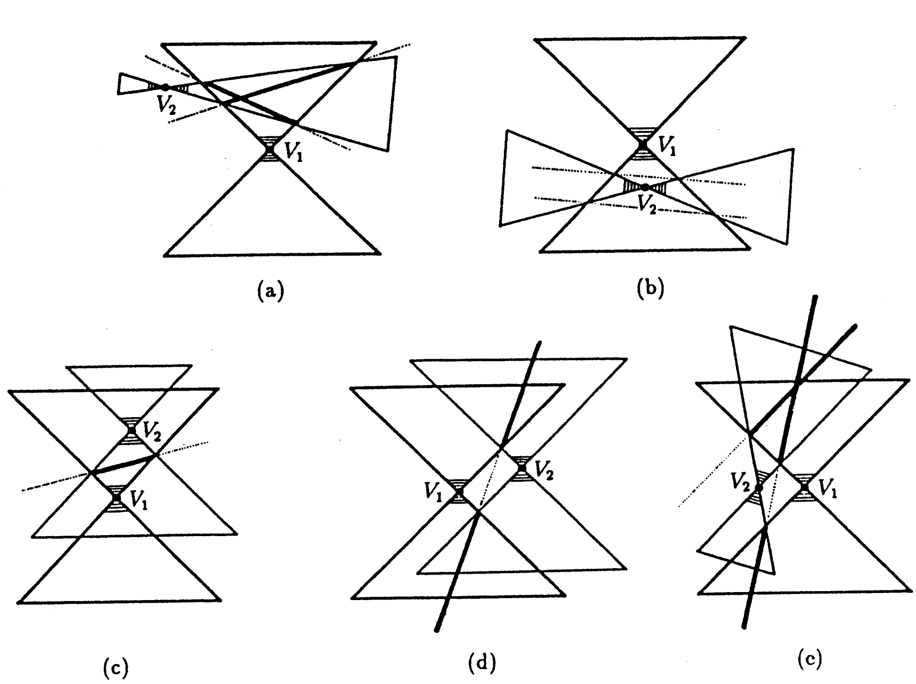

Figure 3: Some Examples of Diagonals

Now suppose two axial natural quadrics with distinct axes have a conic intersection. Let one of the intersection conics be C. Since the axial plane $\mathcal{H}$ cuts both surfaces into two symmetric parts, C is symmetric about $\mathcal{H}$. In particular, if P is the plane containing C, $P \cap \mathcal{H}$ is the major axis of C (Figure 4). By Lemma 4.1, P is perpendicular to the axial plane $\mathcal{H}$. Since C belongs to both surfaces, the points of $C \cap \mathcal{H}$ are intersection points of the two skeletal pairs. Also $P \cap \mathcal{H}$ must be a diagonal.

Lemma 4.4 *If two axial natural quadrics with coplanar and distinct axes have conic intersection, then the conics lie in the planes through a diagonal and perpendicular to the axial plane.*

Thus in order to test if the given surfaces intersect in conics, we can test if the intersections of P with each surface coincide.

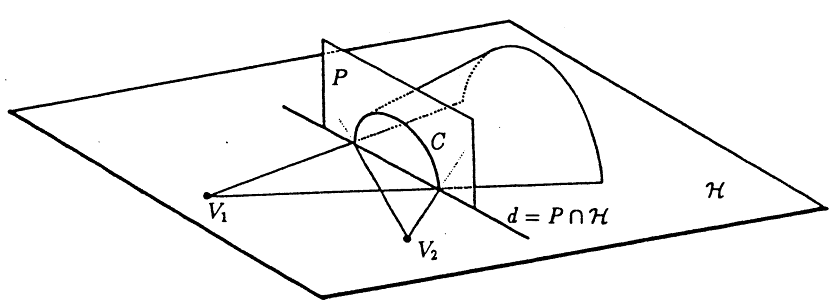

Figure 4: The Axial Plane and the Intersection Conic

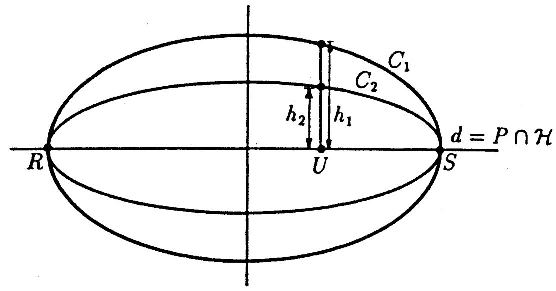

Figure 5: Equal Height Implies Equal Conics

4.3 The Height and Its Computation

Lemma 4.4 tells us that the planes through a diagonal and perpendicular to the axial plane are important. Let P be such a plane, through the diagonal $d = P \cap \mathcal{H}$, and suppose that it intersects the two given surfaces in C_1 and C_2 (Figure 5). We want to develop a method to test if $C_1 = C_2$. C_1 and C_2 must have the same major axis, $d = P \cap \mathcal{H}$, by symmetry with $\mathcal{H}$. In particular, if R and S are the intersections of the diagonal d and any skeletal pair, then C_1 and C_2 have major axis $d = \overleftrightarrow{RS}$. Moreover, if both R and S are finite (hence C_1 and C_2 are central conics), then the direction of their minor axes are the same: through the midpoint of $\overline{RS}$ and perpendicular to $\mathcal{H}$. Thus the question reduces to testing if the length of the minor axes of C_1 and C_2 are the same. We shall use a height computation to check this. In the following, assume that both R and S are finite. The case of infinite R or S will be addressed at the end of this section.

To compare C_1 and C_2, pick any point U on d, construct a line through U and perpendicular to $\mathcal{H}$, and compute the distance from U to the surface along this line. This distance is called the *height* at U. Because the surface is symmetric about the axial plane $\mathcal{H}$, the height from U to either side of the surface are equal. Hence the concept of height is well-defined. Since there are two surfaces, we have two heights at U, h_1 and h_2. If $h_1 = h_2$, $C_1 = C_2$. To see this more formally, let $a = \frac{1}{2}|\overline{RS}|$ be the semi-major axis length. If the midpoint of $\overline{RS}$ is taken to be the coordinate origin, C_i's equation is $\frac{1}{a^2}x^2 + B_i y^2 = 1$, where B_i is unknown. Then, by plugging $x = u$, the x-coordinate of U, and $y = h_i$ into the equation, we have $B_i = \frac{a^2-u^2}{a^2h_i^2}$. Hence, if $h_1 = h_2$, $B_1 = B_2$ and thus $C_1 = C_2$.

The height computation is not difficult. From U construct a plane perpendicular to the surface axis ℓ meeting it at K. This plane intersects the surface in a circle. Let $d_1 = |\overline{UK}|$. If the surface is a cone with vertex V, let $d_2 = |\overline{VK}|$ (Figure 6(a)). Hence the circle's radius is $d_2 \tan\alpha$, where α is the cone angle. Obviously the squared height at U is $(d_2 \tan\alpha)^2 - d_1^2$ (Figure 6(c)). If the surface is a cylinder, the circle always has radius r, the radius of the cylinder, and the squared height is $r^2 - d_1^2$ (Figure 6(b)). Note that, if U lies outside of the circle, the squared height will be negative. This is equivalent to saying that we have an imaginary y-coordinate at U.

We have shown how to detect and calculate the first intersection conic, using two height computations. It would appear that another height computation is needed to calculate the second intersection conic, using the other diagonal. However, by calculating the height from

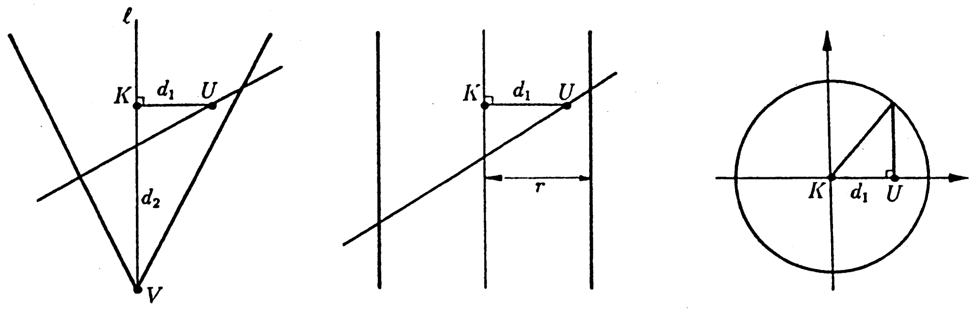

Figure 6: Height Computation

the intersection point X of the two diagonals, we deal with both conics at the same time and it is possible to use only two squared height computations in total. If the diagonals are parallel, X can be any point on either diagonal. Since, in this case, the intersection conics lie on parallel planes and parallel planar intersections are similar, the second conic can be determined from the first conic (Johnstone–Shene [16]).

The above discussion assume that R and S are finite. If one of R and S, say S, is at infinity, the intersection conic must be a parabola and the computation is very similar, still involving two height computations. R is the vertex of the parabola, while the line $d_\perp$ through R and perpendicular to $\mathcal{H}$ gives the direction of the directrix. If $d, d_\perp$ and R are chosen to be the x-axis, the y-axis and the origin, the parabola has equation $y^2 = 4fx$, where $|f|$ is the unknown focal length. Using the height computation, with u and its squared height h_u^2, we have $f = h_u^2/(4u)$ and hence $y^2 = \left(\frac{h_u^2}{u}\right)x$.

We are now ready to present the conic intersection algorithm for $\mathcal{Q}_1 \cap \mathcal{Q}_2$, where $\mathcal{Q}_i$ is an axial natural quadric with vertex V_i and axis ℓ_i.

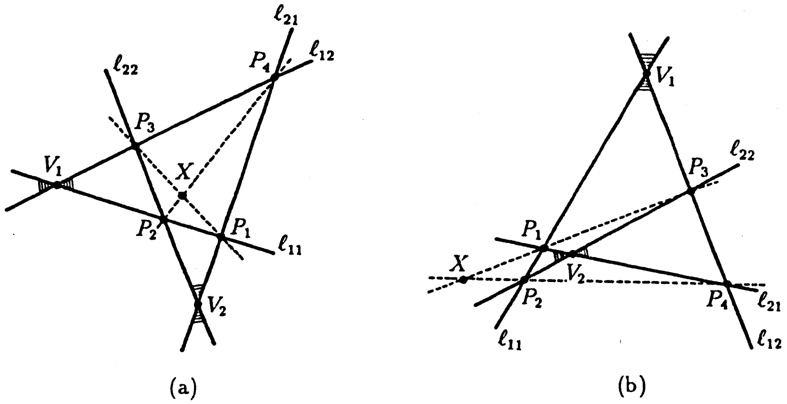

Figure 7: Points and Lines Used in Determining the Intersection

Algorithm: Conic-Intersection (Non-degenerate Case)

1. If the axes are not coplanar, then $\mathcal{Q}_1$ and $\mathcal{Q}_2$ do not have conic intersection; else let $\mathcal{H}$ be the axial plane.

2. If $\ell_1 = \ell_2$ and $V_1 \neq V_2$, the intersection consists of two circles; if $\ell_1 = \ell_2$ and $V_1 = V_2$, the intersection is the common vertex or the entire surface.

3. On the axial plane, let the skeletal pair $\mathcal{Q}_1 \cap \mathcal{H}$ (resp., $\mathcal{Q}_2 \cap \mathcal{H}$) be ℓ_{11} and ℓ_{12} (resp., ℓ_{21} and ℓ_{22}).

4. If $V_1 \in \ell_{21} \cup \ell_{22}$ or $V_2 \in \ell_{11} \cup \ell_{12}$, we have a degenerate case (Section 5).

5. Let $P_1 = \ell_{11} \cap \ell_{21}, P_2 = \ell_{11} \cap \ell_{22}, P_3 = \ell_{22} \cap \ell_{12}$ and $P_4 = \ell_{21} \cap \ell_{12}$ (Figure 7).

6. Let $X = \overleftrightarrow{P_1P_3} \cap \overleftrightarrow{P_2P_4}$, the intersection of the diagonals. If the diagonals are parallel, go to step 10.

7. Compute the squared heights h_1^2 and h_2^2 to both surfaces at X.

8. If $h_1^2 = h_2^2$, we have conic intersection.

9. If P_1 and P_3 are both finite, the intersection on diagonal $\overleftrightarrow{P_1P_3}$ is a central conic. If P_3 is at infinity, the intersection on diagonal $\overleftrightarrow{P_1P_3}$ is a parabola. The intersection conic on $\overleftrightarrow{P_2P_4}$ can be computed similarly.

10. Use any point on the first diagonal $\overleftrightarrow{P_1P_3}$ to detect and calculate the conic intersection. If the intersection is a conic, use any point on the second diagonal $\overleftrightarrow{P_2P_4}$ to compute the second conic.

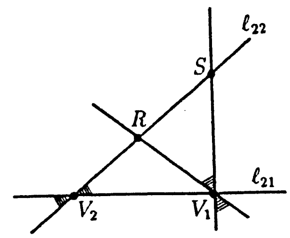

Figure 8: $V_1 \in S_2$ and $V_2 \notin S_1$

5 Degenerate Cases

In the previous section, we ignored the degenerate case when the vertex of one surface lies on a skeletal line. There are four cases: (1) $V_1 \in S_2$ and $V_2 \notin S_1$, (2) $V_1 \notin S_2$ and $V_2 \in S_1$, (3) $V_1 \in S_2$ and $V_2 \in S_1$, and (4) $V_1 = V_2$. This section will focus on the first three cases only. We will show that the first two cases cannot have conic intersection, while the third always delivers a conic intersection. The fourth case will be discussed in Section 6.

5.1 Exactly One Vertex Lies on the Other Surface

In this section, we will show if $V_1 \in S_2$ and $V_2 \notin S_1$ then Q_1 and Q_2 cannot have conic intersection. The argument also works for $V_1 \notin S_2$ and $V_2 \in S_1$.

Suppose V_1 lies on the skeletal line ℓ_{21} from S_2. The lines of S_1 intersect the other skeletal line ℓ_{22} of S_2 in two points, say R and S (Figure 8). If Q_1 and Q_2 have conic intersection, the plane P containing one of the intersection conics C must contain one of R and S, but not both, which can be seen as follows. If both R and S lie on P, then $P \cap Q_1$ is a conic, while $P \cap Q_2$ is a line because R, S and V_2 are collinear. Thus we have a contradiction. Also P does not contain V_1, otherwise $P \cap Q_1$ is linear while $P \cap Q_2$ is a conic. Without loss of generality, let $R \in P$. Then P must cut the other skeletal line from S_2 somewhere, but not at V_1 or V_2. This is impossible since $S_1 \cap S_2 = \{R, S, V_1\}$. Hence $Q_1 \cap Q_2$ has no conic intersection.

Indeed, $Q_1 \cap Q_2$ cannot have linear intersection. If it contains a line, this line must contain both V_1 and V_2 and thus the intersection line is $\ell_{21} = \overleftrightarrow{V_1V_2}$. Then the axial plane cuts Q_1 in three lines, two in S_1 and $\ell_{21} = \overleftrightarrow{V_1V_2}$, which is impossible. In summary we have the following lemma.

Lemma 5.1 *Let Q_1 and Q_2 be two axial natural quadrics with coplanar and distinct axes. If only one vertex lies on the other surface, then $Q_1 \cap Q_2$ cannot have conic or linear intersection.*

In fact, the intersection curve is a space quartic with one isolated point. However we will not elaborate this issue here.

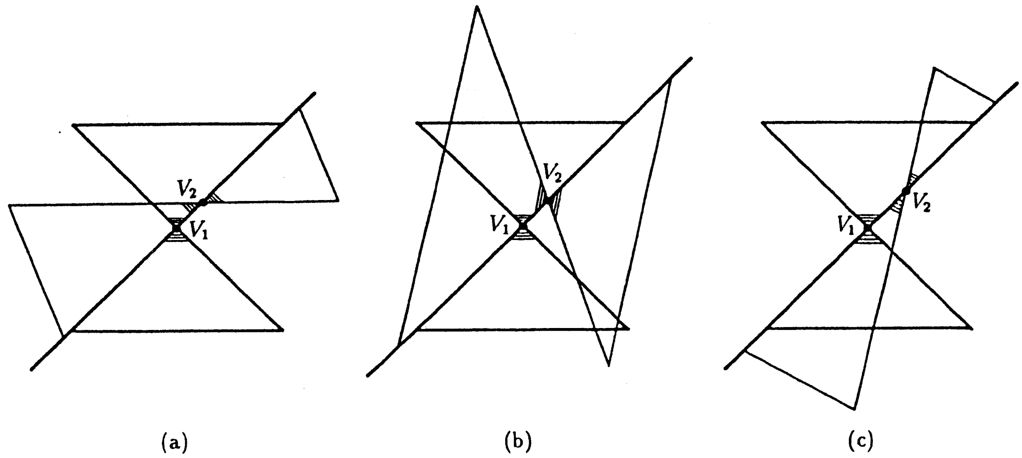

Figure 9: The Common Line $\overleftrightarrow{V_1V_2}$ on the Axial Plane

5.2 A Double Line and Conic

If $V_1 \in S_2, V_2 \in S_1$ and $V_1 \neq V_2$, on the axial plane we have a common line $\overleftrightarrow{V_1V_2}$ (Figure 9). Since the plane containing this line and perpendicular to the axial plane is tangent to both surfaces, this is a double line. By Bezout's theorem, the residue curve of the complete intersection is a conic. In this section we will find the residue conic. We will concentrate on the cone–cone case only. The same argument works for the cylinder–cone case. Since the intersection of two cylinders is strictly linear, it will not be discussed.

There are two possibilities to consider: $\ell_1 // \ell_2$ and ℓ_1 not parallel to ℓ_2, where ℓ_1 and ℓ_2 are the axes of the cones. For the former case, the two given cones have equal cone angles. Their intersection consists of the double line $\overleftrightarrow{V_1V_2}$ and a circle at infinity on which any point is the intersection of two corresponding parallel rulings, one from each cone. For a formal proof, see Theorem 8.2.

In general, ℓ_1 is not parallel to ℓ_2. The common inscribed sphere is crucial to the computation of the conic intersection in this case.

Lemma 5.2 *Let $O = \ell_1 \cap \ell_2$ and d be the distance from O to $\overleftrightarrow{V_1V_2}$. The sphere S with center O and radius d is a common inscribed sphere of both cones (Figure 10).*

Proof: Simple. ∎

Note that $d = d_1 \sin\alpha_1 = d_2 \sin\alpha_2$, where d_i is the distance from O to V_i.

Theorem 5.1 *Suppose two cones $\mathcal{Q}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{Q}_2(V_2, \ell_2, \alpha_2)$, ℓ_1 not parallel to ℓ_2, have a common line $\overleftrightarrow{V_1V_2}$ on the axial plane $\mathcal{H}$. Let P be the intersection point of the other two skeletal lines and Q the tangent point on $\overleftrightarrow{V_1V_2}$ of the common inscribed sphere. Then the intersection conic of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ lies in the plane through $\overleftrightarrow{PQ}$ and perpendicular to the axial plane $\mathcal{H}$.*

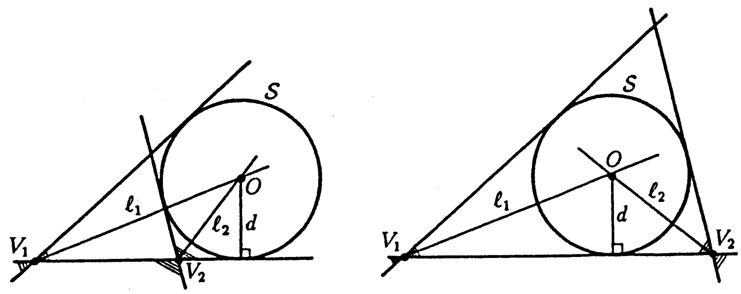

Figure 10: The Inscribed Sphere with center O and radius d

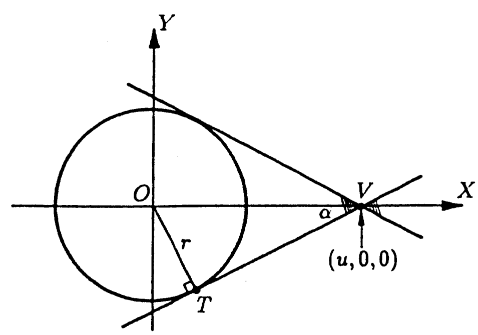

Figure 11: The Relation of c, r and u

Proof: We know that the conic must lie in a plane perpendicular to $\mathcal{H}$ and must pass through P. Thus we must only show that the plane containing the conic passes through Q.

Suppose the axial plane $\mathcal{H}$ is chosen to be the xy-plane with $O = \ell_1 \cap \ell_2$ the origin. Let $\mathcal{S}(O,r)$ be a sphere. Any cone with axis on the xy-plane and inscribed sphere $\mathcal{S}$ can be obtained by rotating a standard cone with equation $-c^2(x-u)^2 + y^2 + z^2 = 0$, where $(u,0,0), u > 0$, is the cone vertex and α the cone angle, $c = \tan\alpha$.

Let $\mathcal{S}(O,r)$ be the common inscribed sphere of both cones (Lemma 5.2). By rotation we can assume without loss of generality that these two cones have a common skeletal line $y + r = 0$. It suffices to show that the planar members of the pencil formed by these two transformed cones both pass through $Q = (0,-r,0)$, the tangent point of the common line and the common inscribed sphere.

If the cone $-c^2(x-u)^2 + y^2 + z^2 = 0$ has an inscribed sphere $x^2 + y^2 + z^2 = r^2$, then $c^2u^2 = r^2(1+c^2)$. To see this, note that the cone angle α must be an acute angle. Let one of the skeletal lines be tangent to the inscribed sphere at T (Figure 11). We have $c = \tan\alpha = \frac{|\overline{OT}|}{|\overline{VT}|}$. Since $r = |\overline{OT}|$ and $|\overline{TV}| = \sqrt{u^2 - r^2}$, we have $c^2 = \frac{r^2}{u^2-r^2}$ and hence $c^2u^2 = r^2(1+c^2)$. Note that this relation is an invariant under rotation transformation, since r and c are constants, and u is the distance from the cone's vertex to O, which does not change under rotation.

After rotation, the first cone is a standard cone rotated by α, or

$$(1-c^2)y^2 + 2cxy + 2rcx - 2c^2ry - r^2(1+c^2) + z^2 = 0, \tag{1}$$

while the second cone, which is obtained by rotating another standard cone $-d^2(x-v)^2 + y^2 + z^2 = 0$, is

$$(1-d^2)y^2 + 2dxy + 2rdx - 2d^2ry - r^2(1+d^2) + z^2 = 0, \tag{2}$$

where $d = \tan\beta$ or $d = \tan(\pi + \beta)$. Subtracting (2) from (1) and applying $c^2u^2 = r^2(1+c^2)$ and $d^2v^2 = r^2(1+d^2)$, we have

$$y^2 - \frac{2}{c+d}xy - \frac{2r}{c+d}x + 2ry + r^2 = 0.$$

This equation can be factored into two linear factors:

$$(y+r)\left(y - \frac{2}{c+d}x + r\right) = 0.$$

$y+r=0$ and $y - \frac{2}{c+d}x + r = 0$ are clearly the two planar members. Note that both planes are perpendicular to the xy-plane as we expected and both planes pass through $Q = (0, -r, 0)$ as desired. ∎

Based on the above theorem, the following algorithm computes the intersection conic.

Algorithm: Common Line and Conic Intersection

1. If $\ell_1 // \ell_2$, the intersection consists of only one finite component $\overleftrightarrow{V_1V_2}$, the common line on the axial plane.
2. If ℓ_1 and ℓ_2 are not parallel, let $O = \ell_1 \cap \ell_2$. Let P be the intersection point of the other two skeletal lines. From O drop a perpendicular to $\overleftrightarrow{V_1V_2}$ meeting it at Q.
   - P is finite : Let X be the midpoint of $\overline{PQ}$ and compute the squared height h_X^2 to C_1 (or C_2). The intersection is a central conic with major axis $\overleftrightarrow{PQ}$, center X and the length of its semi-major axis is $\frac{1}{2}|\overline{PQ}|$. The minor axis is the line through X and perpendicular to $\mathcal{H}$ and the length of the semi-minor axis is $\sqrt{|h_X^2|}$. Note that if $h_X^2 > 0$, the conic is an ellipse, otherwise it is a hyperbola.
   - P is at infinity : The intersection is a parabola with major axis $\overleftrightarrow{PQ}$ and vertex Q. The focal length of this parabola can be determined using the technique of Section 4.3.

6 Linear Intersections

This section handles the last degenerate case: $V_1 = V_2$. We will show that $V_1 = V_2$ implies linear intersection. (Recall that under our definition, a linear intersection is an intersection

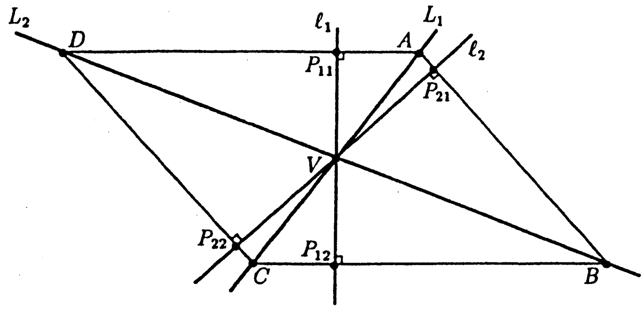

Figure 12: Lines L_1 and L_2

whose components are all linear.) For the cylinder–cylinder case, $V_1 = V_2$ means all rulings on both surfaces are parallel, since the vertex of a cylinder is the point at infinity along the direction given by the axes. In this case, the intersection is either empty, a common tangent, or two parallel lines. $V_1 = V_2$ is obviously impossible for the cylinder–cone case. Thus, we need only consider cone–cone intersection.

Let $\mathcal{C}_1(V, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V, \ell_2, \alpha_2)$ be two cones with the common vertex V and distinct axes ℓ_1 and ℓ_2. Let P_{11} and P_{12} be distinct points on ℓ_1 such that $|\overline{P_{11}V}| = |\overline{P_{12}V}| = \cos\alpha_1$, while P_{21} and P_{22} are distinct points on ℓ_2 such that $|\overline{P_{21}V}| = |\overline{P_{22}V}| = \cos\alpha_2$. The lines on the axial plane through P_{11} and P_{12} and perpendicular to ℓ_1, and through P_{21} and P_{22} and perpendicular to ℓ_2 form a parallelogram $ABCD$ (Figure 12). Denote the two diagonals $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ of this parallelogram by L_1 and L_2 respectively. The following lemma characterizes L_1 and L_2.

Lemma 6.1 *A point P, $P \neq V$, on $\mathcal{H}$ lies on $L_1 \cup L_2$ if and only if the ratio of distances from V to the perpendicular feet of P on ℓ_1 and ℓ_2 is $\frac{\cos\alpha_1}{\cos\alpha_2}$.*

Proof: See Figure 12. ∎

L_1 and L_2 in turn characterize the intersection lines of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Theorem 6.1 *Two cones $\mathcal{C}_1(V, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V, \ell_2, \alpha_2)$, $\ell_1 \neq \ell_2$, always intersect in lines. The lines are the intersection of either cone with the plane through $L_i, i = 1, 2$, and perpendicular to the axial plane.*

Proof: First, we will show that the plane through L_i and perpendicular to the axial plane $\mathcal{H}$ contains one pair of intersection lines of $\mathcal{C}_1 \cap \mathcal{C}_2$ by computing the squared heights to both cones at an arbitrary point $T \in L_i, T \neq V$. (This is obviously true for $T = V$.) From T, drop a perpendicular to ℓ_i meeting it at $T_i, i = 1, 2$ (Figure 13(a)). By the above lemma, we can assume $|\overline{VT_i}| = t\cos\alpha_i, i = 1, 2$, for some $t > 0$. Let $t_i = |\overline{TT_i}|$ and $p = |\overline{VT}|$. Then the squared height from T to $\mathcal{C}_i$ is $h_i^2 = t^2\sin^2\alpha_i - t_i^2$ (Figure 13(b)). From $\triangle TT_iV$, since $\angle TT_iV = \frac{\pi}{2}$, $p^2 = t^2\cos^2\alpha_i + t_i^2 = t^2 - t^2\sin^2\alpha_i + t_i^2$ and hence $t^2\sin^2\alpha_i - t_i^2 = t^2 - p^2$. Therefore, we have $h_1^2 = t^2 - p^2 = h_2^2$. These equal heights imply that the plane through

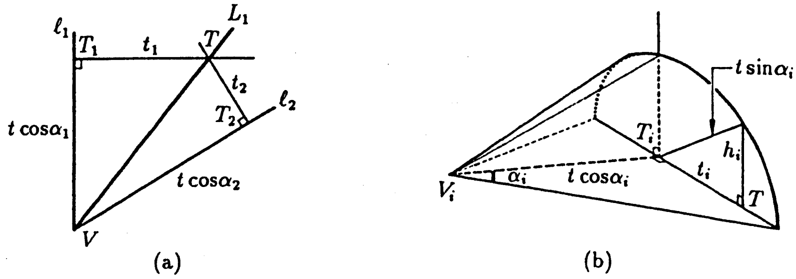

Figure 13: Quadrangle TT_1VT_2 and the Height at T

L_i and perpendicular to $\mathcal{H}$ cuts the cones in a common intersection curve. Since this plane passes though the common vertex V, the common intersection curve must be a pair of lines. According as $h_1^2 = h_2^2$ is positive, zero, or negative, the intersection lines are distinct real, double real, or conjugate imaginary. With the above argument, we have already found four lines of intersection. By Bezout's theorem, these are the only intersections of $\mathcal{C}_1 \cap \mathcal{C}_2$. ∎

The computation procedure is simple and consists of the following steps:

Algorithm: Linear Intersection ($V_1 = V_2$)

1. Determine the axial plane $\mathcal{H}$.
2. Determine the parallelogram $ABCD$ (Figure 12).
3. For points A and D, compute the squared heights h_A^2 and h_D^2. One can use the method in the proof of Theorem 6.1.
4. Based on the sign of h_A^2 we have three cases:
   (a) $h_A^2 > 0$: On the line through A and perpendicular to the axial plane, choose the two points at distance h_A from A. Connecting these two points to V gives two intersection lines.
   (b) $h_A^2 = 0$: The intersection line is L_1, the line through V and A.
   (c) $h_A^2 < 0$: The intersection lines are imaginary. Thus, for our purpose, the intersection consists of only one point, the vertex V.
5. Do the same thing using D and h_D^2.

7 Tangency and Disjointness

In this section, we will study when two axial natural quadrics are tangent to each other in a finite number of points. The following lemma provides a necessary condition for tangency.

Lemma 7.1 *If two axial natural quadrics with distinct vertices are tangent at an isolated point, their axes must be skew and each vertex must lie in the exterior of the other surface.*

Proof: Suppose two axial natural quadrics are tangent at isolated points. Let P be one of these tangent points. Since axial natural quadrics are ruled, the tangent plane at P is determined by the two lines $\overleftrightarrow{V_1P}$ and $\overleftrightarrow{V_2P}$, where V_1 and V_2 are the vertices. Since P is an isolated point of tangency, the two surfaces lie on different sides of the common tangent plane; otherwise, P becomes the singular point of the intersection curve. Therefore their axes are skew.

If V_1 lies in the interior of the other surface $\mathcal{Q}_2$, the rulings of $\mathcal{Q}_1$ can only intersect $\mathcal{Q}_2$ transversally, because $\mathcal{Q}_2$ is convex. If $V_1 \in \mathcal{Q}_2$ and both surfaces have an isolated tangent point P, then $\overleftrightarrow{V_1P}$ is a common tangent. This contradicts the isolated point assumption. ∎

We have three cases to consider: cylinder–cylinder, cylinder–cone and cone–cone. Two cylinders are tangent at an isolated point if and only if their axes are skew and the length of the common perpendicular of the cylinders' axes is equal to the sum of the two cylinders' radii. Cone–cone tangency is presented in Theorem 7.1 and cylinder–cone tangency in Corollary 7.1.

Consider two cones. We shall find two planes that bound one cone tightly and test the other cone against the two bounding planes for tangency.

Definition 7.1 *Let $\mathcal{C}(V,\ell,\alpha)$ be a cone, and let v be a line through V and outside $\mathcal{C}$. The two planes through v and tangent to $\mathcal{C}$ are called the* **bounding planes** *of $\mathcal{C}$ with respect to v. The bounding planes divide the space into four quadrants. The two quadrants containing $\mathcal{C}$ are called the* **bounding regions** *of $\mathcal{C}$ with respect to v, denoted by $B(\mathcal{C},v)$.*

The bounding regions yield another necessary condition for tangency.

Lemma 7.2 *Let $\mathcal{C}_1(V_1,\ell_1,\alpha_1)$ and $\mathcal{C}_2(V_2,\ell_2,\alpha_2)$ be two cones, V_1 outside $\mathcal{C}_2$ and V_2 outside $\mathcal{C}_1$. If $\ell_2 \cap B(\mathcal{C}_1,\overleftrightarrow{V_1V_2}) \neq \emptyset$, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect transversally.*

Proof: Suppose that $\ell_2 \cap B(\mathcal{C}_1,\overleftrightarrow{V_1V_2}) \neq \emptyset$. Consider the plane containing V_1 and ℓ_2. This plane intersects $B(\mathcal{C}_1,\overleftrightarrow{V_1V_2})$, since ℓ_2 does. Since this plane also passes through both vertices, it cuts the two cones in two pairs of intersecting lines. If none of the four lines are parallel, the two cones will intersect transversally. If one line from each pair are parallel, the two remaining lines intersect both parallel lines. Thus the two cones again intersect transversally. ∎

The following theorem characterizes the disjointness and tangency of two cones.

Theorem 7.1 (Tangency and Disjointness) *Let $\mathcal{C}_1(V_1,\ell_1,\alpha_1)$ and $\mathcal{C}_2(V_2,\ell_2,\alpha_2)$ be two cones, V_1 outside $\mathcal{C}_2$, V_2 outside $\mathcal{C}_1$ and ℓ_2 outside $B(\mathcal{C}_1,\overleftrightarrow{V_1V_2})$. Let S be a point on ℓ_2, $S \neq V_2$ and let $d = |\overline{SV_2}|$. Let the bounding planes of $\mathcal{C}_1$ with respect to $\overleftrightarrow{V_1V_2}$ be T_1 and T_2*

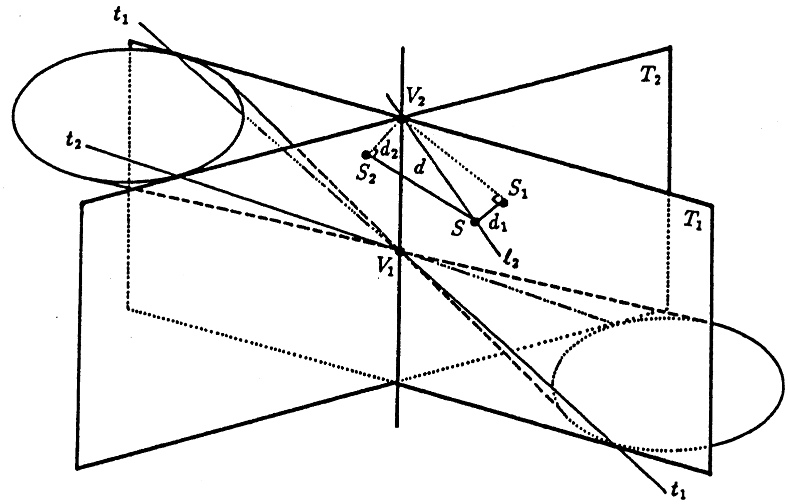

Figure 14: The Construction for Testing Tangency

tangent to C_1 in the lines t_1 and t_2 respectively (Figure 14). From S drop a perpendicular to T_i meeting it at $S_i, i = 1, 2$. Let $d_i = |\overline{SS_i}|, i = 1, 2$. We have the following characterizations for tangency and disjointness.

1. *$d_1, d_2 > d \sin \alpha_2$: The two surfaces are disjoint.*

2. *$d_1 > d_2 = d \sin \alpha_2$: The two surfaces are tangent at $t_2 \cap \overleftrightarrow{V_2 S_2}$. If this point does not exist, they are disjoint.*

3. *$d_2 > d_1 = d \sin \alpha_2$: The two surfaces are tangent at $t_1 \cap \overleftrightarrow{V_2 S_1}$. If this point does not exist, they are disjoint.*

4. *$d_1 = d_2 = d \sin \alpha_2$: The two surfaces are tangent at $t_1 \cap \overleftrightarrow{V_2 S_1}$ and $t_2 \cap \overleftrightarrow{V_2 S_2}$. One or both of these may not exist. If both do not exist, they are disjoint.*

5. *$d_1 < d \sin \alpha_2$ or $d_2 < d \sin \alpha_2$: The two surfaces intersect transversally.*

Proof: The key element of this proof is the sphere centered at S with radius $d \sin \alpha_2$. This sphere is inscribed in the cone C_2. We know that ℓ_2 does not lie in $B(C_1, \overleftrightarrow{V_1 V_2})$. If the distances from S to the two planes T_1 and T_2 are both greater than the radius of the sphere, then not only ℓ_2 but the entire cone lies outside $B(C_1, \overleftrightarrow{V_1 V_2})$. Therefore proposition 1 holds.

Note that the cone C_2 is tangent to T_2 if and only if $d_2 = d \sin \alpha_2$. $d_1 > d_2 = d \sin \alpha_2$ implies that C_2 does not intersect T_1, except at V_2. Therefore, C_2 intersects T_2 in one line. If this line, $\overleftrightarrow{V_2 S_2}$, intersects t_2, then C_1 and C_2 are tangent there since this is the only point at which they intersect. If these two lines do not intersect at all, the two cones do not intersect since they lie in different regions in space. Therefore proposition 2 holds. Similar arguments can prove proposition 3 and proposition 4.

If $d_2 < d \sin \alpha_2$, the plane T_2 intersects the cone C_2 in two lines. They cannot both be parallel to t_2, which is the common tangent line of C_1 and T_2. That is, one of these lines must intersect t_2 and therefore the two cones cannot be tangent to each other. The same argument applies for $d_1 < d \sin \alpha_2$. ∎

For the case of cylinder–cone, the cylinder's vertex goes to infinity and hence the line $\overleftrightarrow{V_1V_2}$ is parallel to the axis of the cylinder. That is, $\overleftrightarrow{V_1V_2}$ is replaced by the line through the cone's vertex and parallel to the cylinder's axis. A similar argument to Theorem 7.1 gives the following result for cylinder–cone tangency and disjointness.

Corollary 7.1 *Let $\mathcal{C}(V, \ell_1, \alpha)$ and $\mathcal{Z}(\ell_2, r)$ be a cone and a cylinder, V outside $\mathcal{Z}$, and ℓ_2 outside $B(\mathcal{C}, v)$, where v is the line through V and parallel to ℓ_2. Let S be a point on ℓ_2. Let the bounding planes of $\mathcal{C}$ with respect to v be T_1 and T_2 tangent to $\mathcal{C}$ in the lines t_1 and t_2 respectively. From S drop a perpendicular to T_i meeting it at $S_i, i = 1, 2$. Let $d_i = |\overline{SS_i}|, i = 1, 2$. We have the following characterizations for tangency and disjointness.*

1. *$d_1, d_2 > r$: The two surfaces are disjoint.*
2. *$d_1 > d_2 = r$: The two surfaces are tangent at exactly one point, the intersection point of t_2 and the tangent line of $\mathcal{Z}$ and T_2.*
3. *$d_2 > d_1 = r$: The two surfaces are tangent at exactly one point, the intersection point of t_1 and the tangent line of $\mathcal{Z}$ and T_1.*
4. *$d_1 = d_2 = r$: The two surfaces are tangent at the two points given by case 2 and case 3.*
5. *$d_1 < r$ or $d_1 < r$: The two surfaces intersect transversally.*

8 A Necessary and Sufficient Condition for Conic Intersection: The Common Inscribed Sphere

It is well known by descriptive geometers that two axial natural quadrics with intersecting axes have conic intersection if they have a common inscribed sphere. For example, Bereis [2] and Rehbock [30] use this property implicitly, while Gordon and Sementsov-Ogievskii [13, p. 283] state it explicitly and indicate that it is also true for revolute quadrics. However, in this literature, the existence of a common inscribed sphere is only stated as a sufficient condition for conic intersection.[5]

Recently, Goldman and Miller [10, 11] and Shene and Johnstone [36] have shown that the common inscribed sphere is actually a necessary and sufficient condition. The method and proof of Goldman and Miller is a case-by-case algebraic analysis, while the method and proof of Shene and Johnstone is geometric.[6] In this section, using Lemma 4.1 and Lemma 4.2 as

[5] In [12] Goldman and Warren point out that this condition is provably not necessary for revolute quadrics.

[6] For a complete proof, the interested reader should consult Shene–Johnstone [37]. In the second part of this paper [38], we will present yet another proof, which offers deeper insight into this problem and provides a new technique for designing other algorithms.

our starting point, we will provide a new and very short algebraic proof for the common inscribed sphere criterion. The purpose of this proof is to illustrate how the use of the axial plane can simplify algebraic proofs.

The following lemma provides all of the necessary theory for our proof.

Lemma 8.1 *Two axial natural quadrics with coplanar and distinct axes have conic or linear intersection if and only if the parallel projection of their intersection curve to the axial plane factors. The projection curve is of second degree.*

Proof: Suppose the axial plane is taken to be the xy-plane. Because of symmetry, there will be no z term in the equations of the given axial natural quadrics, $Q_1(x,y) + z^2 = 0$ and $Q_2(x,y) + z^2 = 0$, where $Q_1(x,y)$ and $Q_2(x,y)$ are second degree polynomials of x and y. Note that eliminating the z terms from both equations is equivalent to performing a parallel projection of the intersection curve to the xy-plane. Thus the projection curve has equation $Q_1(x,y) - Q_2(x,y) = 0$, which is obviously a second degree curve.

($\Rightarrow$) Recall that the intersection conics lie on two planes both perpendicular to the axial plane. Thus the projection generates two straight lines. Since the projection curve is of second degree and it has two lines as components, it must factor.

($\Leftarrow$) If the projection curve factors into two lines, the intersection curve is contained in two planes through the lines and perpendicular to the axial plane. Each such plane cuts the surfaces in a conic or two lines. ∎

By Lemma 4.2, we can assume without loss of generality that the two given axial natural quadrics have coplanar axes. We have two theorems, based on whether the axes are parallel or intersect.

Theorem 8.1 (Common Inscribed Sphere Criterion) *Two axial natural quadrics with intersecting and distinct axes have conic intersection if and only if they have a common inscribed sphere.*

Proof: We will prove this theorem using two cones. The cases of cone–cylinder and cylinder–cylinder are similar. We can simplify the calculation by making the axial plane the xy-plane. We can also assume without loss of generality that the intersection point of the two axes is the coordinate origin. By an appropriate rotation bringing one cone's axis to the x-axis, the corresponding equation becomes $-c^2(x-u)^2 + y^2 + z^2 = 0$, where $(u,0,0)$ is the cone's vertex. The second cone, which is the rotation of a cone $-d^2(x-v)^2 + y^2 + z^2 = 0$ by angle $\theta \neq 0, \pi$, has the equation

$$-d^2(x\cos\theta - y\sin\theta - v)^2 + (x\sin\theta + y\cos\theta)^2 + z^2 = 0.$$

Eliminating the z^2 term gives the following projection curve on the xy-plane:

$$[c^2 - d^2 + (1+d^2)\sin^2\theta]x^2 - (1+d^2)\sin^2\theta y^2 + 2\sin\theta\cos\theta(1+d^2)xy$$

$$+2(d^2 v\cos\theta - c^2 u)x - 2d^2 v\sin\theta y + (c^2u^2 - d^2v^2) = 0.$$

The discriminant of this conic (after some reduction) is

$$\begin{aligned}\Delta &= \begin{vmatrix} c^2 - d^2 + (1+d^2)\sin^2\theta & \sin\theta\cos\theta(1+d^2) & d^2v\cos\theta - c^2u \\ \sin\theta\cos\theta(1+d^2) & -(1+d^2)\sin^2\theta & -d^2v\sin\theta \\ d^2v\cos\theta - c^2u & -d^2v\sin\theta & c^2u^2 - d^2v^2 \end{vmatrix} \\ &= \sin^2\theta(1+c^2)(1+d^2)\left[\frac{d^2v^2}{1+d^2} - \frac{c^2u^2}{1+c^2}\right].\end{aligned}$$

Therefore the projection curve factors if and only if $\Delta = 0$, if and only if $\frac{d^2v^2}{1+d^2} = \frac{c^2u^2}{1+c^2}$. However, from the proof of Theorem 5.1, we have $r_1^2 = \frac{c^2u^2}{1+c^2}$, where r_1 is the radius of the inscribed sphere of the first cone with center the origin. Similarly, for the second cone, we have $r_2^2 = \frac{d^2v^2}{1+d^2}$, where r_2 is the radius of the inscribed sphere of the second cone with center the origin. Now it is obvious that $\Delta = 0$ if and only if the radii of the inscribed spheres are equal. That is, if and only if the two given cones have a common inscribed sphere. ∎

The following corollary, also reported in Goldman–Miller [10, 11], is a simple consequence of Theorem 8.1.

Corollary 8.1 *Consider two axial natural quadrics with ℓ_1 and ℓ_2, $\ell_1 \neq \ell_2, O = \ell_1 \cap \ell_2$, as their axes.*

- *Two cones $C_1(V_1, \ell_1, \alpha_1)$ and $C_2(V_2, \ell_2, \alpha_2)$ have a conic intersection if and only if $d_1 \sin\alpha_1 = d_2 \sin\alpha_2$, where d_i is the distance from O to V_i.*
- *A cone $\mathcal{C}(V, \ell_1, \alpha)$ and a cylinder $\mathcal{Z}(\ell_2, r)$ have a conic intersection if and only if $d_1 \sin\alpha_1 = r$, where d_1 is the distance from O to V_1.*
- *Two cylinders $\mathcal{Z}_1(\ell_1, r_1)$ and $\mathcal{Z}_2(\ell_2, r_2)$ have conic intersection if and only if $r_1 = r_2$.*

Proof: The radius of the inscribed sphere of C_i with center O is $d_i \sin\alpha_i$. ∎

Theorem 8.2 (Parallel Axes) *Two axial natural quadrics with parallel and distinct axes have conic intersection if and only if they have equal cone angles.*

Proof: Again assume that the axial plane is the xy-plane. The cone with axis the x-axis and vertex the origin has equation $-c^2x^2 + y^2 + z^2 = 0$, while the other cone with axis parallel to the x-axis and vertex $(u, v, 0)$ has equation $-d^2(x-u)^2 + (y-v)^2 + z^2 = 0$. Note that since the axes are distinct, $v \neq 0$. Eliminating the z^2 term gives the following projection curve on the xy-plane:

$$(c^2 - d^2)x^2 + 2d^2ux - 2vy + (v^2 - d^2u^2) = 0.$$

This conic's discriminant is the following:

$$\Delta = \begin{vmatrix} c^2 - d^2 & 0 & d^2u \\ 0 & 0 & -v \\ d^2u & -v & v^2 - d^2u^2 \end{vmatrix} = -v^2(c^2 - d^2).$$

Thus, the projection curve factors if and only if $\Delta = 0$, if and only if $c^2 = d^2$. ∎

Remark 8.1 *Theorem 8.2 also satisfies the common inscribed sphere criterion, since the two cones have a common inscribed sphere with center at the point at infinity along the two parallel axes.*

9 Conclusions

This paper has successfully developed a fully geometric technique for detecting and computing the planar intersections of two natural quadric surfaces, as well as their tangency and disjointness. The fundamental method involves a comparison of two heights from a point in the axial plane to the surfaces. In developing the method, the structure of planar intersection has been revealed, such as that the axes are coplanar and the planes containing the intersection curves are perpendicular to this plane.

The techniques of this paper can be generalized to cover more general quadric surfaces, and we are working on this problem. However, some lemmas presented here will not hold for more general surfaces; for example, planar intersection need not imply coplanar axes and tangency need not imply skew axes. Another interesting question is whether one can compute the higher degree intersection curves of natural quadrics using the techniques presented in this paper.

In part two of this paper (Shene–Johnstone [38]), techniques used here will be further developed to obtain more powerful characterization results, which can in turn help in designing other detection and computations algorithms and uncovering more geometric insights of the conic intersections of two axial natural quadrics.

References

[1] Abhyankar, S. S., and Bajaj, C. L. Automatic Parameterization of Rational Curves and Surfaces IV: Algebraic Space Curves, *ACM Transactions on Graphics*, Vol. 8 (1989), No. 4 (October), pp. 325–334.

[2] Bereis, R. *Darstellende Geometrie I*, Akademie–Verlag, Berlin, 1964.

[3] Berger, M. *Geometry II*, translated from the French by M. Cole and S. Levy, Springer–Verlag, New York, 1987.

[4] Boehm, W. On Cyclides in Geometric Modeling, *Computer Aided Geometric Design*, Vol. 7 (1990), pp. 243–255.

[5] Bromwich, T. J. I'A *Quadratic Forms and Their Classification by Means of Invariant-Factors*, Hafner Publishing Company, New York, 1971.

[6] Dandelin, G. P. Mémoire sur quelques propriétés remarquables de la Focal Parabolique, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles–lettres de Bruxelles*, Vol. 2 (1822), pp. 171–202.

[7] Drew, W. H. *A Geometrical Treatise on Conic Sections*, fifth edition, Macmillan and Co., 1875.

[8] Farouki, R. T., Neff, C. A., and O'Connor, M. A. Automatic Parsing of Degenerate Quadric-Surface Intersection, *ACM Transaction on Graphics*, Vol. 8 (1989), No. 3, pp. 174–203.

[9] Garrity, T., and Warren, J. D. On Computing the Intersection of a Pair of Algebraic Surfaces, *Computer Aided Geometric Design*, Vol. 9 (1989), No. 2 (May), pp. 137–153.

[10] Goldman, R. N., and Miller, J. R. Detecting and Calculating Conic Sections in the Intersection of Two Natural Quadric Surfaces, Part I: Detection, Draft, October 1990.

[11] Goldman, R. N., and Miller, J. R. Combining Algebraic Rigor with Geometric Robustness for the Detection and Calculation of Conic Sections in the Intersection of Two Natural Quadric Surfaces, in *ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications*, June 1991, pp. 221–231.

[12] Goldman, R. N., and Warren, J. D. A Sufficient Condition for the Intersection of Two Quadrics of Revolution to Degenerate into a Pair of Conic Sections, Technical Report, Rice University, December 1990.

[13] Gordon, V. O., and Sementsov-Ogievskii, M. A. *A Course in Descriptive Geometry*, translated from Russian by Leonid Levant, MIR Publishers, Moscow, 1980.

[14] Hakala, D. G., Hillyard, R. C., Nourse, B. E., and Malraison, P. J. Natural Quadrics in Mechanical Design, *Proceedings of Autofact West 1*, Anaheim, CA., Nov. 1980, pp. 363–378.

[15] Hoffmann, C. M. *Geometric and Solid Modeling: An Introduction*, Morgan Kaufmann, San Mateo, CA, 1989.

[16] Johnstone, J. K. and Shene, C.-K. Computing the Intersection of a Plane and a Natural Quadrics, Technical Report 91–04, Department of Computer Science, The Johns Hopkins University, April 1991 (revised June 1991). Submitted for publication.

[17] Kramer, W. *Darstellende Geometrie*, Veb Deutscher Verlag der Wissenschaften, Berlin, 1959.

[18] Levin, J. Z. A Parametric Algorithm for Drawing Pictures of Solid Objects Composed of Quadric Surfaces, *Communications of ACM*, Vol. 19 (1976), No. 10, pp. 555–563.

[19] Levin, J. Z. Mathematical Models for Determining the Intersections of Quadric Surfaces, *Computer Graphics and Image Processing*, Vol. 11 (1979), pp. 73–87.

[20] Macaulay, F. S. *Geometrical Conics*, Cambridge University Press, London, 1895.

[21] Manocha, D., and Canny, J. A New Approach for Surface Intersection, in *ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications*, June 1991, pp. 209–219.

[22] Miller, J. R. Geometric Approaches to Nonplanar Quadric Surface Intersection Curves, *ACM Transactions on Graphics*, Vol. 6 (1987), No. 4, pp. 274–307.

[23] Miller, J. R., and Goldman, R. N. Detecting and Calculating Conic Sections in The Intersection of Two Natural Quadric Surfaces, Part II: Calculation, Draft, February 1991.

[24] Miller, J. R., and Goldman, R. N. Using Tangent Balls to Find Plane Sections of Natural Quadric Surfaces, Draft, April 1991.

[25] Morton, P. On the Focus of a Conic Section, *Transactions of the Cambridge Philosophical Society*, Vol. 3 (1827–1830), pp. 185–191.

[26] Ocken, S., Schwartz, J. T., and Sharir, M. Precise Implementation of CAD Primitives Using Rational Parameterizations of Standard Surfaces, in *Planning, Geometry, and Complexity of Robot Motion*, edited by Jacob T. Schwartz, Micha Sharir and John Hopcroft, Ablex Publishing Co., Norwood, New Jersey, 1987, pp. 245–266.

[27] O'Connor, M. A. Natural Quadrics: Projections and Intersections, *IBM Journal of Research and Development*, Vol. 33 (1989), No. 4 (July), pp. 417–446.

[28] Piegl, L. Geometric Method of Intersecting Natural Quadrics Represented in Trimmed Surface Form, *Computer Aided Design*, Vol. 21 (1989), No. 4, pp. 201–212.

[29] Pratt, M. J. Cyclides in Computer Aided Geometric Design, *Computer Aided Geometric Design*, Vol. 7 (1990), pp. 221-242.

[30] Rehbock, F. *Darstellende Geometrie*, second edition, Springer–Verlag, New York, 1964.

[31] Sabin, M. A. A Method for Displaying the Intersection Curve of Two Quadric Surfaces, *The Computer Journal*, Vol. 19 (1976), No. 4, pp. 336–338.

[32] Sabin, M. A. as communicated to Pratt in [29].

[33] Samuel, N. M., Requicha, A. A. G., and Elkind, S. A. Methodology and Results of an Industrial Part Survey, Technical Memorandum No. 21, Production Automation Project, University of Rochester, Rochester, NY, 1976.

[34] Sarraga, R. F. Algebraic Methods for Intersections of Quadric Surfaces in GMSOLID, *Computer Vision, Graphics, and Image Processing*, Vol. 22 (1983), pp. 222–238.

[35] Schmid, T. *Darstellende Geometrie*, G. J. Göschen'sche Verlagshandlung, Berlin and Leipzig, 1912.

[36] Shene, C.-K., and Johnstone, J. K. On the Planar Intersection of Natural Quadrics, in *ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications*, June 1991, pp. 233–242.

[37] Shene, C.-K., and Johnstone, J. K. On the Converse of an Elementary Geometry Theorem, in preparation.

[38] Shene, C.-K., and Johnstone, J. K. On the Lower Degree Intersections of Two Natural Quadrics II: Characterization Theorems, in preparation.

[39] Taylor, C. *An Introduction to Ancient and Modern Geometry of Conics*, Cambridge University Press, London, 1881.

[40] Warren, J. D. Blending Quadric Surfaces with Quadric and Cubic Surfaces, *Proceedings of the Third Annual Symposium on Computational Geometry*, Waterloo, Ontario, Canada 1987, pp. 341–347.