

Finding generators and directrix curves on algebraic ruled surfaces*

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Abstract

Ruled surfaces are a very useful class of surfaces for solid modeling. They share properties of both curves and surfaces, which not only makes them easy to work with but also useful in the difficult transition between curves and surfaces in solid modeling. They are a naturally occurring surface in man-made objects, and simple ruled surfaces are already widely used in most solid models.

In this paper, the two key algorithms that must be developed in order to use algebraic ruled surfaces, and to realize their $2\frac{1}{2}$ -dimensional nature, are presented: (1) how to find the generator through a given point, and (2) how to find a plane directrix curve. We then immediately show how these two algorithms can be used to solve two operations on algebraic ruled surfaces: parameterization and intersection. We also show how to identify that an algebraic surface is ruled from its equation. The theory developed in this paper serves to clarify some of the classical definitions of ruled surfaces.

1 Introduction

Surfaces are more complicated objects than curves. Consequently, a number of problems in geometric modeling have been solved for curves but not for surfaces. Ruled surfaces (which

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are surfaces that can be generated by sweeping a line through space, or more specifically, sweeping a line along a directrix curve) are a useful class of surfaces, because they bridge the gap in complexity between curves and surfaces. Algorithms for ruled surfaces tend to be easier to develop than algorithms for arbitrary surfaces, since a ruled surface can be defined by a curve (the directrix curve), or alternatively by a collection of lines.

Ruled surfaces are not only of interest because they are easy to work with. Many of the surfaces that solid modelers use as primitives from which to build models are ruled surfaces. Three of the four most common surfaces (the plane, cylinder, cone, but not the sphere) are ruled, as are four of the six quadric surfaces (adding the hyperboloid of one sheet and the hyperbolic paraboloid). Ruled surfaces also arise naturally in man-made objects, since the sweeping of a line through space is a fundamental operation in machining. Gordon says “warped cylinders with three directrices are widely used in engineering (in designing rowing screws, propellers, automobile bodies, etc.)” [8, p. 206]. They are prevalent in the design of ship hulls and aircraft bodies [3], [7, p. 228]. Finally, ruled surfaces bear a strong relationship to the generalized cylinder method of shape representation that is widely used by the vision community [16, 21].

In this paper, we show how to perform the crucial reduction step from ruled surfaces to curves: finding a directrix curve. Intuitively, the directrix curve is the curve that the line sweeps along in generating the ruled surface. We prove that a plane directrix curve can always be found. Next, we present the complementary reduction step from ruled surfaces to lines: finding the direction of the line at a given point of the directrix curve. This can be viewed as a decomposition of the ruled surface into lines. The solutions to these two problems allow one to retrieve the generating sweep of a ruled surface.

Once the generators and directrix curve of a ruled surface are known, operations with the ruled surface are much simplified. To exhibit this, we show how to parameterize a ruled surface and how to intersect a ruled surface with another surface.

The paper is structured as follows. Section 2 presents key definitions, some of which are purposely changed from the typical ones in the literature in order to satisfy the more constructive requirements of this paper. Section 3 establishes some basic facts about singularities that are important for the rest of the paper. The core of the paper is in Sections 4

and 5, where a method for finding the generator through a point, for identifying a ruled surface, and for finding a directrix curve are presented. This work also clarifies the notion of ruled surface, by proving the equivalence of two definitions of ruled surface. Section 6 discusses the parameterization of ruled surfaces and Section 7 develops the advantages of a line decomposition of a ruled surface, including intersection. The paper ends with some conclusions and directions for future work.

The results of this paper extend [11]. For other work on ruled surfaces, the reader is referred to discussions of lofted surfaces (e.g., Farin [6], Rogers and Adams [19]) and Ravani's work on patching with ruled surfaces [17, 18].

All curves and surfaces in this paper, including ruled surfaces, are assumed to be non-linear, irreducible, and algebraic. Recall that a surface is algebraic if it can be defined by a polynomial $f(x, y, z) = 0$. It is reducible if it can be expressed as the union of two algebraic surfaces.

2 Definitions

In this section, we introduce definitions for the ruled surface and its key components. There are several choices for the definition of a ruled surface. The weakest definition is that a ruled surface is (1) a surface that is equal to the union of a set of lines. (Equivalently, through every point of a ruled surface R , there exists a line that is completely contained in R .) A stronger definition that is the usual definition in the mathematical literature [5, 22] and the one that we shall use is (2) a surface that can be generated by smoothly sweeping a line through space (see Figures 1 and 2). (More formally, a ruled surface is the image of a continuous map $S : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$, where $S(t)$ is a line for all $t \in I$. We also insist that the map is one-to-one almost everywhere: we do not want sweeps that return to sweep over a portion of the surface a second time.) An even stronger definition is (3) a surface that can be generated by sweeping a line along a plane algebraic curve. This definition is motivated by the observation that all of the quadric ruled surfaces can be generated by sweeping a line along a line or ellipse. We shall prove that this strongest definition is actually an equivalent definition when the surface is algebraic (Corollary 5.3).

A straight line that lies in a ruled surface is called a *generator* of the surface. Usually, there is only one generator through a point of a ruled surface. If there are two or more generators through almost every point of a ruled surface, then the ruled surface is *doubly ruled*, otherwise it is *singly ruled*. The only doubly ruled algebraic surfaces are the plane, the hyperbolic paraboloid, and the hyperboloid of one sheet [9].

A simple definition of a directrix curve in the literature is a curve on the ruled surface that is intersected by every generator of the surface [22]. However, this definition does not lend itself well to use in geometric modeling algorithms. For example, even the vertex of a cone would be a directrix curve under this definition. In particular, it should be easy to generate the surface from the directrix curve. Thus, we use a more pragmatic definition: a *directrix curve* of the ruled surface R is a curve C on R such that the generators that intersect C , choosing only one generator for every point of C , are sufficient to generate R (*i.e.*, these generators densely cover R , leaving at most a set of zero measure¹ uncovered). A directrix curve C is (almost) *strong* if (almost) every point of C is intersected by exactly one generator.

Example 2.1 *An elliptical cross-section of a hyperboloid of one sheet is a directrix curve. An elliptical cross-section of a circular cylinder or cone is a strong directrix curve.*

As it is presently defined, there is no assurance that the ruled surface can be generated by smoothly sweeping a line along a directrix curve. In particular, the lines that intersect a directrix curve are enough to generate the surface, but it is not clear that it is possible to find a smooth sweep through them. We now establish this fact for almost-strong directrix curves.

Lemma 2.1 *Let R be a ruled surface and let S be an almost-strong directrix curve of R . Then R can be generated by sweeping a line along S .*

Proof: By definition, R can be generated by sweeping a line through space. We shall show that the line must actually sweep along S . Suppose that the sweep leaves S at x . It

¹The set A is *dense* in the set B if for each x in B and each open set J containing x , J meets A [2]. The set C is of *zero measure* in the set B if $B \setminus C$ is dense in B . For example, a finite set of points and a line are both of zero measure in a cylinder.

must return to x again and this time stay on the curve S , otherwise points of S in some neighbourhood of x will not be visited by the sweep. It suffices to show that the sweep is always in the same orientation when it returns to x to resume the sweep along S . This is sufficient since it shows that it is possible to sweep smoothly along S without leaving: if the sweep ever leaves S at x , then one can simply follow the alternate direction of the sweep from x that stays on S rather than leaving. Since this sweep passes through all of the points of S , which is an almost-strong directrix curve, it densely covers the surface (*i.e.*, generates the surface).

Suppose that the sweep leaves S at x . If there is only one generator through x , then the sweep must be in the same orientation when it returns to x . If there is more than one generator through x , then the sweep can return to x in any of the orientations of these generators but when it returns to stay on the curve S it must be in the same orientation as when it left the curve at x originally, for the following reason. Of two adjacent points on an almost-strong directrix curve, only one can be intersected by two or more generators (by definition) and only one of these two or more generators can smoothly change into the single generator of the adjacent point. Therefore, if $x_{-\epsilon}$ and $x_{+\epsilon}$ are the two points of S infinitesimally close to x on either side, there is a unique generator through x , say L_x , that smoothly changes into the generators through $x_{-\epsilon}$ and $x_{+\epsilon}$. When the sweep leaves S at x it must be in the orientation L_x , and when it returns to sweep through x and continue along S to the previously unvisited neighbourhood of x , it must also be in the orientation L_x . ■

Lemma 2.1 shows that the term ‘directrix curve’ is well chosen: a directrix curve does indeed direct how the line should be swept through space. In Section 5, we will show that every singly ruled surface has an almost-strong directrix curve.

3 Facts about singularities of surfaces

We begin with some theory about the singularities of a surface, which play a key role in the development of our algorithms (as they do in most algorithms that deal with curves and surfaces). Geometrically, a *singularity of a surface* is a point at which the tangent plane is undefined [9]. Algebraically, a singularity of the surface $f(x, y, z) = 0$ is a point P such

that $(\text{grad } f)(P) = 0$ and $f(P) = 0$ [15], where $(\text{grad } f)(P)$ is the gradient of f , the vector of its partial derivatives. To see the equivalence of these definitions, recall that the tangent plane of a nonsingular point P is the plane perpendicular to the surface normal at P and $(\text{grad } f)(P)$ is the surface normal of a nonsingular point.

Before we can continue, we must present a very important theorem from algebraic geometry.

Theorem 3.1 (Bezout's Theorem [10, 15])

- (a) *An algebraic curve of degree m and an algebraic curve of degree n have at most mn intersections, unless one of the curves is contained in the other curve.*
- (b) *An algebraic curve of degree m and an algebraic surface of degree n have at most mn intersections, unless the curve is contained in the surface.*
- (c) *An algebraic surface of degree m and an algebraic surface of degree n intersect in a collection of algebraic curves, unless one of the surfaces is contained in the other surface. The sum of the degrees of these curves of intersection is at most mn .*

Lemma 3.1 *The set of singularities of an irreducible algebraic surface is a finite set of algebraic curves and a finite set of points.*

Proof: The proof is an application of the surface-surface and curve-surface versions of Bezout's Theorem to the intersection of four surfaces. By Theorem 3.1c, the solution set of $f_x = 0, f = 0$ is a finite set of algebraic curves, since f is irreducible and f is not a component of f_x (being of higher degree). Thus, $(f_x = 0, f = 0), f_y = 0$ is a finite set of points and algebraic curves (Theorem 3.1b). Similarly, $(f_x = 0, f = 0, f_y = 0), f_z = 0$ is a finite set of points and algebraic curves. ■

We now move on to the actual algorithms for ruled surfaces.

4 Computing the generator through a given point

In sweeping a line through space, it is not enough to know the curve along which to sweep. One must also know the direction of the line at every point of the curve. Thus, a crucial algorithm for ruled surfaces is computing the generator through a point. It is only necessary to find generators through nonsingular points, since we shall find a directrix curve that only contains a finite number of singular points. The formula for the generator through a nonsingular point P of the ruled surface $f(x, y, z) = 0$ should depend upon f and P .

Clearly, the line $P + tV$ is a generator through P if and only if $f(P + tV) = 0$ for all t . $f(P + tV) = 0$ is an equation of degree n in the four variables $V = (v_1, v_2, v_3)$ and t , where n is the degree of the surface f . P will be treated as a symbolic constant in this equation, since the formula for P 's generator should depend upon P . We use the following two observations to remove two of the variables from the equation:

- (1) A generator through a point P of a ruled surface is contained in P 's tangent plane [22]. This implies that $V \cdot (\text{grad } f)(P) = 0$ if $P + tV$ is a generator through P . This is a linear equation in $v_1, v_2,$ and v_3 , which can be used to solve quickly for one, say v_i , in terms of the other two. (If P is a singularity, then $(\text{grad } f)(P) = 0$ and the above equation is trivial rather than linear. Thus, the formula that we develop will not be valid for singular points of the surface.)
- (2) Only V 's direction is important, not its length. Therefore, one of V 's coordinates (say v_j) can be set to 1. One can always choose $j \neq i$, because if v_j is the only nonzero coordinate of V , it cannot be a linear combination of $v_{j \oplus 1}$ and $v_{j \oplus 2}$. Both possibilities, $v_{i \oplus 1} = 1$ and $v_{i \oplus 2} = 1$ (where \oplus is addition mod 3), should be considered independently because the choice of $v_{i \oplus 1} = 1$ is invalid if $v_{i \oplus 1}$ is actually zero in the solution.

Two variables are eliminated by (1) and (2). Thus, (1) and (2) reduce $f(P + tV) = 0$ to an equation E in t and (without loss of generality) v_1 . All that remains is to solve for v_1 in E . Viewed as a polynomial in t , E has an infinite number of roots. Thus, each of its coefficients must be zero, where the coefficients are polynomials in v_1 . This creates a system

of at most n equations in v_1 . The system of equations will have one or two solutions for v_1 , depending on whether the surface is singly or doubly ruled. (See Section 4.1.)

The system of univariate equations in v_1 can be solved by applying resolvents [4] from elimination theory [23] (Du and Goldman [4] solves precisely this problem). Another method of solving for v_1 is outlined in Remark 4.2. If all solutions are invalid (e.g., complex), then we must have set the wrong v_j to 1. Therefore, we return to step (2) and set the other variable to 1. Finally, each v_1 solution is grown into a $V = (v_1, v_2, v_3)$ vector.

Example 4.1 $f(x, y, z) = y^2 - x^2 - z = 0$ is the equation of a hyperbolic paraboloid. $\text{grad } f(x, y, z) = (-2x, 2y, -1)$ and $V \cdot \text{grad } f(P) = -2p_1v_1 + 2p_2v_2 - v_3$, which can be used to set $v_3 = 2p_2v_2 - 2p_1v_1$. By setting $v_2 = 1$, $f(P + tV)$ becomes $t^2(1 - v_1^2)$. This polynomial is identically zero (when viewed as a polynomial in t), leading us to conclude that all of its coefficients are zero. That is, $1 - v_1^2 = 0$ or $v_1 = \pm 1$. (This is a rather trivial system of equations.) Therefore, $V = (1, 1, 2p_2 - p_1), (-1, 1, 2p_2 + p_1)$. See the red generators in Figure 1.

In this example, notice that there is no linear or constant term in $f(P + tV) = t^2(1 - v_1^2)$. This is not a coincidence. After the elimination of two variables using (1) and (2), $f(P + tV)$ will never have a linear or constant term, and thus the system of equations in v_1 actually has at most $n - 2$ equations. Since $f(P) = 0$ (P lies on the surface), the constant term of $f(P + tV)$ is zero. Similarly, since $V \cdot \text{grad } f(P) = 0$ (by our choice of v_i in (1)), the constant term of $V \cdot \text{grad } f(P + tV)$ is zero; since $V \cdot \text{grad } f(P + tV)$ is the derivative of $f(P + tV)$ with respect to t , this implies that the linear term of $f(P + tV)$ is zero.

Example 4.2 Consider the ruled cubic $f(x, y, z) = x^2 - y^3 + xyz = 0$, called Cayley's ruled cubic [22]. We have

$$\begin{aligned} f(P + tV) &= t^3(v_1v_2v_3 - v_2^3) \\ &+ t^2(v_1^2 + v_1v_2p_3 + v_1v_3p_2 - 3v_2^2p_2 + p_1v_2v_3) \\ &+ t(2p_1v_1 + p_1p_2v_3 + p_1p_3v_2 + p_2p_3v_1 - 3p_2^2v_2) \end{aligned}$$

which yields the system of equations (ignoring the linear term which will disappear):

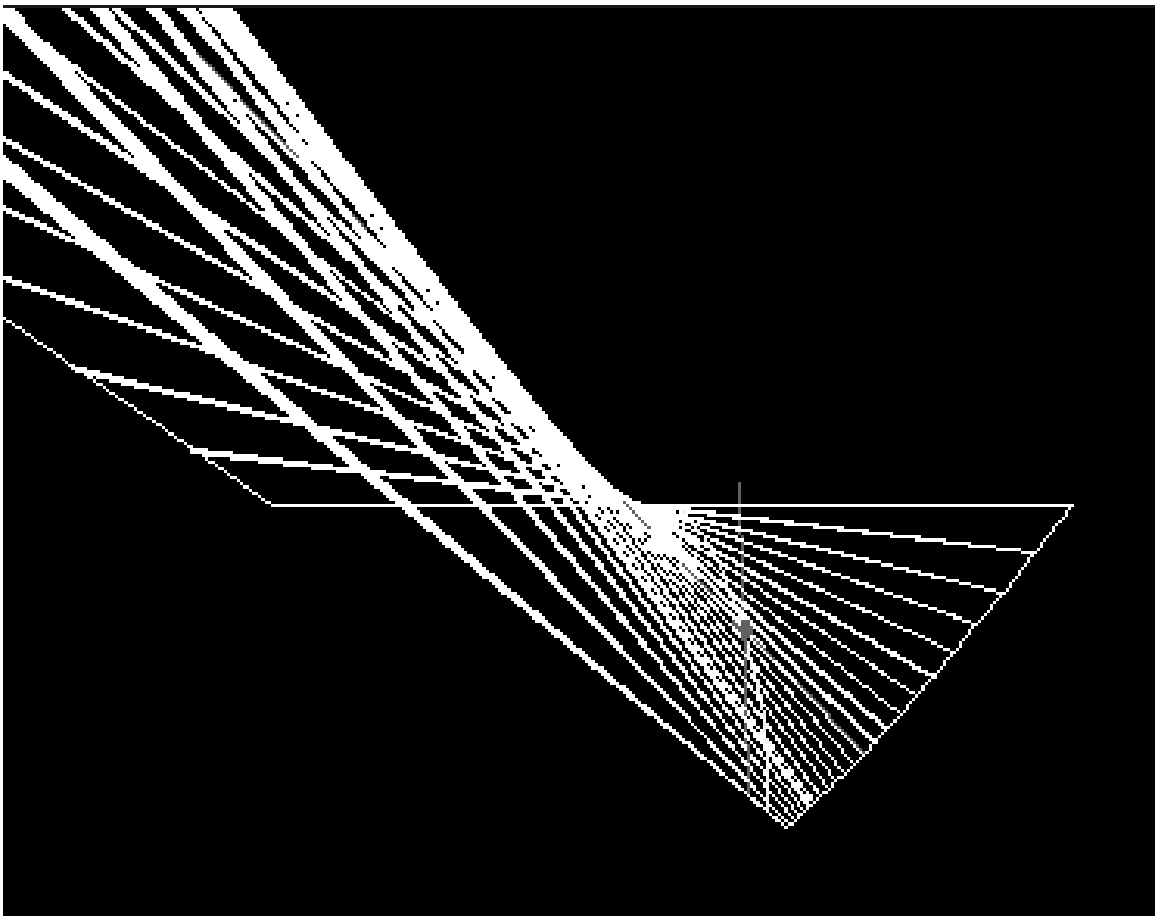


Figure 1: A hyperbolic paraboloid

$$v_1 v_2 v_3 - v_2^3 = 0$$

$$v_1^2 + v_1 v_2 p_3 + v_1 v_3 p_2 - 3v_2^2 p_2 + p_1 v_2 v_3 = 0$$

Using $V \cdot (\text{grad } f)(P) = V \cdot (2p_1 + p_2 p_3, -3p_2^2 + p_1 p_3, p_1 p_2) = 0$, one can solve for

$$v_3 = \frac{-v_1(2p_1 + p_2 p_3) - v_2(p_1 p_3 - 3p_2^2)}{p_1 p_2} \quad (1)$$

We then set $v_2 = 1$.² A solution of the above system yields $v_1 = \frac{p_2^2}{p_1 + p_2 p_3}$. Thus, the unique generator through a point $P = (p_1, p_2, p_3)$ of Cayley's ruled cubic is $P + tV$ where

$$V = (v_1, v_2, v_3) = \left(\frac{p_2^2}{p_1 + p_2 p_3}, 1, \frac{p_2}{p_1 + p_2 p_3} + \frac{p_3}{p_2} \right) \quad (2)$$

For example, the generator through $(1, 1, 0)$ is $(1, 1, 0) + t(1, 1, 1)$. See the red generator in Figure 2.

²Note that if $v_2 = 0$, the coefficient of the quadratic term of $f(P + tV)$ is a quadratic equation in v_1 whose only solution is $v_1 = 0$ (unless $p_1 + p_2 p_3 = 0$, which implies P is a singularity as shown below in Remark 4.1). This leads to the invalid solution $V = (0, 0, 0)$.

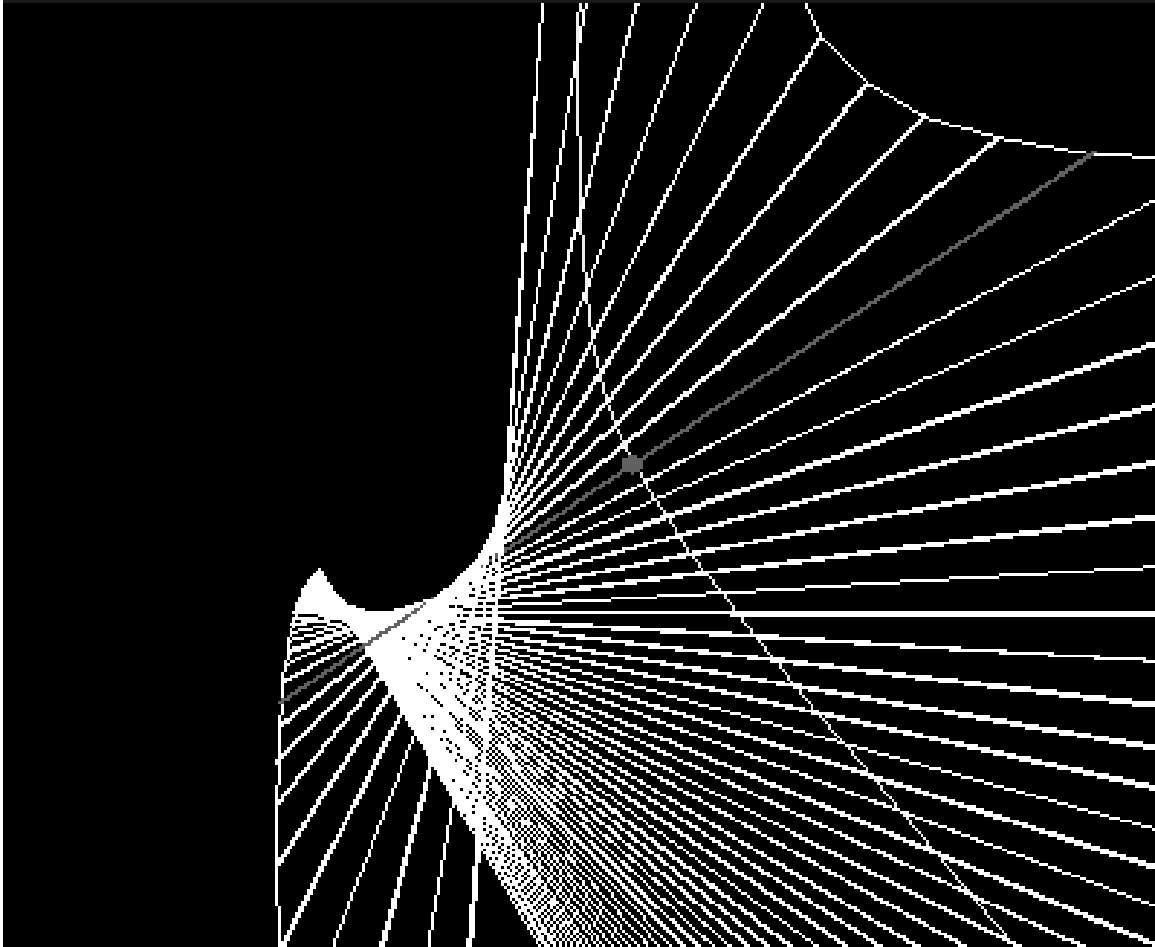


Figure 2: Cayley's ruled cubic

Remark 4.1 Notice that the denominators of the formula (2) only vanish if P is a singularity. To see this, note that if either p_1 or p_2 is zero, then the other must also be zero (since $f(P) = 0$), and P will lie on the z -axis. If $p_1 = -p_2p_3$, then $p_2 = 0$ (since $f(P) = 0$), and P will again lie on the z -axis. However, the entire z -axis of Cayley's ruled cubic is singular (i.e., $\text{grad } f = 0$). Therefore, the denominators of (1) and (2) only vanish at singularities. However, the formula says nothing about generators through singularities.

Remark 4.2 There is another way of solving for the final variable v_1 . All of the coefficients of the Taylor series of $f(P + tV) = 0$ must be zero. The first coefficient is $V \cdot (\text{grad } f)(P)$ (which we have already used in step (1)) while the second coefficient is $V^t H(P) V$, where $H(P)$ is the Hessian of f at P . This is a quadratic equation that can be solved symbolically for v_1 , as long as $H(P) \neq 0$. If $H(P) = 0$, we can use the third coefficient (or the first coefficient that is not identically zero) to solve for v_1 . The disadvantage of this method is that we must compute many higher derivatives of $f(X)$ and evaluate them at P . For example, the Hessian involves all six double derivatives of f .

4.1 The number of generators through a point

It is useful to know how many generators we should expect to find through each point. We now show that there is one generator through almost every point of a ruled surface. We then show that the special points struck by more than one generator are singularities (unless the surface is doubly ruled). Finally, we characterize the points struck by an infinite number of generators, which will become important in the section on directrix curves.

Lemma 4.1 *There is one generator through almost every point of a singly ruled surface.*

Proof : Consider our method for finding the generator through a point. We derive a system of equations in v_1 and the symbolic constant P , where P represents an arbitrary nonsingular point. Each solution for v_1 (which may depend upon P) generates a vector V such that $P + tV$ is a generator through P . Suppose that the system has α simultaneous solutions for v_1 : $v_{1,1}, v_{1,2}, \dots, v_{1,\alpha}$. We claim that almost all of the points of the surface will be struck by α generators. The only points that might not be struck by α generators are singularities and

points such that two of the solutions for v_1 become identical. That is, because the above method, in particular step (1), assumes that P is nonsingular, it cannot be used to make any conclusions about singularities; and if, for example, the solutions are $v_{1,1} = 1 - p_1$ and $v_{1,2} = 1 + p_1$, then most points will be struck by two generators but a point such as $P = (0, y)$ will be struck by only one generator. The singularities of $f(x, y, z)$ form a set of zero measure as a subset of $f(x, y, z)$ (Lemma 3.1). The points P_0 such that $v_{1,i}(P_0) = v_{1,j}(P_0)$, $i \neq j$ also form a set of zero measure (as a subset of the surface $f(x, y, z)$). To see this, view a solution for v_1 as a polynomial in P and view this polynomial as a surface. Two distinct surfaces intersect in a set of zero measure with respect to any surface (Bezout's Theorem 3.1c). Therefore, the points of the surface that are not struck by α generators form a set of zero measure. Moreover $\alpha = 1$, otherwise the surface is doubly ruled. ■

Corollary 4.1 *A point of a singly ruled surface that is intersected by two or more generators is a singularity.*

Proof: The only points that may have an unusual number of generators are the singularities and the points such that two of the solutions for v_1 become identical (Lemma 4.1). The latter set is empty, because there is only one solution to start with. ■

For the two nonplanar doubly ruled surfaces, it is easy to establish that every point is intersected by exactly two generators. In light of Corollary 4.1, it is interesting to note that doubly ruled surfaces are nonsingular.

If a point struck by more than one generator is a singularity (Corollary 4.1), then a point struck by an infinite number of generators must be the worst kind of singularity. This special type of singularity deserves more attention, because of its importance later in the paper.

Definition 1 *A pathological singularity of a singly ruled surface is a point that is struck by an infinite number of generators.*

In the following lemma, we show that a surface can have at most one pathological singularity. This illustrates a general property of algebraic curves and surfaces: unusual things tend to occur only a finite number of times. (Bezout's theorem also makes this point.)

Definition 2 A **general cone** (*resp.*, *cylinder*) is a ruled surface of lines from a curve to a finite (*resp.*, *infinite*) vertex.

Lemma 4.2 A ruled surface has a pathological singularity if and only if it is a general cone or cylinder. A cone or cylinder has only one pathological singularity, its vertex.

Proof : Ruled surfaces are split into two classes: developable and not developable. There are several ways to distinguish a developable surface.

- (1) Developable surfaces can be bent, without tearing, into a plane.
- (2) Developable surfaces have zero curvature at every point.
- (3) A ruled surface is developable if along each generator the tangent planes are all identical [14, p. 140].
- (4) A ruled surface is developable if and only if neighbouring generators intersect [9, p. 207]. The points of intersection of consecutive generators define a curve, called the *line of striction* (or edge of regression). For the cone and cylinder, the line of striction is a point rather than a curve.
- (5) A ruled surface is developable if it is a cone, a cylinder, or the tangential developable of its line of striction [14, p. 141], [9]. The *tangential developable* of a curve is the surface swept out by the tangents of this curve.

Definitions (1) and (3) give a good intuition, but we are interested in definitions (4) and (5). By definition (4), a ruled surface with a pathological singularity is a developable surface. Moreover, its line of striction must contain the pathological singularity. The tangents at the pathological singularity of the line of striction certainly cannot represent the infinite number of generators through this point. Therefore, the ruled surface is not the tangential developable of its line of striction. We conclude that a surface with a pathological singularity is a cone or a cylinder. ■

Remark 4.3 Lemma 4.2 shows that it is easy to compute a pathological singularity. If a ruled surface has a pathological singularity, then it is the intersection of two arbitrary generators of the surface. These generators can be found with the method of Section 4.

4.2 Is this surface ruled?

The methods that we have described for finding the generator(s) $P + tV$ through a point P can also be used to check if a surface is ruled. Given the surface $f(x, y, z) = 0$, one solves for the generator $P + tV$ through the symbolic point P . If no V can be found or it is invalid (e.g., complex), then the surface is not ruled.

Example 4.3 $f(x, y, z) = x^2 + y^2 - z = 0$ is the equation of an elliptic paraboloid.

$\text{grad } f(x, y, z) = (2x, 2y, -1)$ and $V \cdot \text{grad } f(P) = 2p_1v_1 + 2p_2v_2 - v_3$, which can be used to set $v_3 = 2p_1v_1 + 2p_2v_2$. Then $f(P + tV) = t^2(v_1^2 + v_2^2)$, whose coefficients must be zero. Thus, $v_1^2 + v_2^2 = 0$. Whether v_1 or v_2 is set to 1, this does not have a real solution. We conclude that the paraboloid is not ruled.

Example 4.4 Consider the cubic surface $f(x, y, z) = x^2 + xz^2 + y^3 = 0$.

$$f(P + tV) = t^3(v_2^3 + v_1v_3^2) + t^2(v_1^2 + 3p_2v_2^2 + 2p_3v_1v_3 + p_1v_3^2) + t(2p_1v_1 + p_3^2v_1 + 3p_2^2v_2 + 2p_1p_3v_3)$$

Using $V \cdot (\text{grad } f)(P) = 0$, we solve for v_3 and substitute into $f(P + tV)$. Consider the resulting quadratic term of $f(P + tV)$, which as a polynomial in v_1 is:

$$\frac{v_1^2(4p_1^2 - 3p_3^4) + v_1(12p_1p_2^2v_2 - 6p_2^2p_3^2v_2) + (9p_2^4v_2^2 + 12p_1p_2p_3^2v_2^2)}{4p_1p_3^2} = 0 \quad (3)$$

Its discriminant ($b^2 - 4ac$ of the quadratic formula) is

$$\frac{3v_2^2(-p_1p_2)}{p_3^2}. \quad (4)$$

The solution for v_1 (and thus the generator through P) will be complex if and only if this discriminant is negative. To show that the surface is not ruled, it suffices to show that the discriminant (4) is negative for a positive measure of points P on the surface.

If $v_2 = 0$, then (3) reduces to $\frac{v_1^2(4p_1^2 - 3p_3^4)}{4p_1p_3^2} = 0$, implying $v_1 = 0$ almost everywhere and hence $V = 0$, an illegal value for V . Thus, $v_2 \neq 0$ almost everywhere. The discriminant is negative whenever $v_2 \neq 0$ and $p_1p_2 > 0$. We find that, for all z and for all $x < -z^2$, $(x, -\sqrt[3]{x^2 + xz^2}, z)$ is a point on the surface with $x < 0$ and $y < 0$ (since $x^2 + xz^2 > 0$). Therefore, there is no real generator through these points, which form a set of positive measure on the surface. We conclude that the surface $x^2 + xz^2 + y^3 = 0$ is not ruled.

5 Finding a directrix curve

In the theory of ruled surfaces, directrix curves are a very important structure that will often be used in algorithms. Moreover, the directrix curve will be the main means of translating ruled surface problems to plane curve problems. Therefore, the development of a method for finding a directrix curve on a ruled surface is crucial. The following example suggests how this might be done.

Example 5.1 *We would like to show that a planar cross-section of the ruled surface can be used as a directrix curve. However, a plane parallel to a cylinder's axis will not intersect the cylinder in a directrix curve, although all other planes will create an elliptical directrix curve. This suggests that one should not choose a plane that is parallel to a generator of the cylinder. As another example, a planar cross-section of a cone that passes through the cone's vertex will not generate a directrix curve, although all other cross-sections shall. This suggests that points of the ruled surface that have several generators passing through them (the singularities) can cause problems.*

We use these observations to find a directrix curve.

Theorem 5.1 *Let R be a ruled surface and P be a plane. $R \cap P$ is a directrix curve if P satisfies the following restrictions:*

- (a) P intersects R
- (b) P is not parallel to G , for almost all generators G through nonsingular points of R

and

- (c) P does not contain any of the following:
 - (i) a pathological singularity
 - (ii) an entire irreducible singularity curve of R .

Proof: The curve $R \cap P$ satisfies the first requirement of a directrix curve: the generators that intersect $R \cap P$ do indeed densely cover the surface, since almost all generators through nonsingular points intersect P and thus $R \cap P$ (condition b). (The generators through nonsingular points suffice to cover the surface by Lemma 3.1.) We must next show that the generators that intersect $R \cap P$ still densely cover the surface if we restrict to one generator through each point of $R \cap P$. The proof for doubly ruled surfaces is straightforward and we only consider singly ruled surfaces. By Corollary 4.1, it suffices to show that $R \cap P$ contains only a finite number of surface singularities, each of which is intersected by only a finite number of generators. By Lemma 3.1, the singularities of R consist of a finite set of algebraic curves and a finite set of points. Because of restriction c(ii), the plane P can intersect every singularity curve in at most a finite number of points (Theorem 3.1(a)). Therefore, P contains only a finite number of singularities. Restriction c(i) guarantees that each of these is intersected by only a finite number of generators. ■

Example 5.2 Consider the hyperbolic paraboloid $f(x, y, z) = y^2 - x^2 - z = 0$. In Example 4.1, we showed that the two generators through a point P of this surface are $P + t(\pm 1, 1, 2p_2 \mp p_1)$. None of these generators are parallel to the $x = 0$ plane, and there are no singularities to avoid (since $f_z = -1 = 0$ is impossible). Therefore, we can find a directrix curve from the cross-section of the surface by the plane $x = 0$, yielding the parabola $\{x = 0, y^2 - z = 0\}$. See the yellow directrix curve in Figure 1.

Example 5.3 Consider Cayley's ruled cubic $x^2 - y^3 + xyz = 0$. We will show that the cross-section of the surface by the plane $y = 1$ is a valid directrix curve. (A plane $y = k$ is chosen so that the resulting directrix curve is conic rather than cubic.) From Example 4.2, we know that the generator through a nonsingular point P of Cayley's ruled cubic is $P + t(\frac{p_2^2}{p_1 + p_2 p_3}, 1, \frac{p_2}{p_1 + p_2 p_3} + \frac{p_3}{p_2})$. None of these generators are parallel to the plane $y = 1$, since the y -component of their defining vectors is nonzero. Moreover, $y = 1$ does not contain any singularities of the surface: if a point $P = (p_1, p_2, p_3)$ lies on $y = 1$, then $(\text{grad } f)(P) = (2p_1 + p_3, -3 + p_1 p_3, p_1)$ which cannot equal $(0, 0, 0)$ (if the third component is 0, then the second is not). Since $y = 1$ satisfies all of the restrictions of Theorem 5.1, it defines a valid directrix curve $\{y = 1, x^2 - 1 + xz = 0\}$. See the yellow directrix curve in Figure 2.

Remark 5.1 *An imprecise suggestion of Theorem 5.1 is given in Sommerville [22, p. 382]: “in general any plane section [of a ruled surface], not containing any generator, is a directrix curve” There is no elaboration of what ‘in general’ means, and there is no proof of the statement. A proof would have been of no use to us anyway, since Sommerville’s definition of directrix curve is different from ours. This is a good example of the necessity for more constructive and exact results for solid modeling.*

Corollary 5.1 *A randomly chosen planar cross-section of a ruled surface is, with probability one, a directrix curve.*

Proof: There are a finite number of singularity curves (Lemma 3.1). There is at most one pathological singularity (Lemma 4.2). Finally, almost all planes satisfy condition (b) of Theorem 5.1. To see this, suppose a plane is parallel to an infinite number of generators of a fixed ruled surface (and does not contain a pathological singularity). These generators intersect the plane at its line at infinity (in projective space) in infinitely many different points. Therefore, this line at infinity must be contained in the ruled surface (by Bezout’s Theorem (b)). Since the ruled surface can contain only a finite number of lines at infinity, there are at most a finite number of planes that are parallel to an infinite number of generators (*i.e.*, that do not satisfy condition (b)). We conclude that the set of planes that satisfy the restrictions of Theorem 5.1 are dense in the set of planes. ■

Corollary 5.2 *A ruled surface always has a plane algebraic directrix curve.*

A more deterministic method of finding a directrix curve is to guarantee that the conditions of Theorem 5.1 are fulfilled, leading to the following algorithm (see Examples 5.2 and 5.3 above).

Algorithm *Directrix Curve*

- (1) Find the symbolic formula for the generator through a nonsingular point of the ruled surface R (Section 4).
- (2) Compute the singular curves and isolated singular points of R .

- (3) Compute the pathological singularity of R , if it exists (Remark 4.3).
- (4) Choose a plane P that does not completely contain any of these (finite number of) singular curves, isolated singularities, or pathological singularities.
- (5) If P intersects almost all generators (using the formula in (1)), then $R \cap P$ is a directrix curve, otherwise go to (4).

For a singly ruled surface, the directrix curve of Theorem 5.1 will be almost-strong, since it will contain a finite number of singularities and singularities are the only points of a singly ruled surface that can be intersected by more than one generator (Corollary 4.1). This establishes two very interesting corollaries.

Corollary 5.3 *An algebraic ruled surface can be generated by sweeping a line along a plane algebraic curve. That is, for algebraic surfaces, the last definition of ruled surface in Section 2 is actually equivalent to (not stronger than) the definition we are using.*

Proof: A singly ruled surface has a plane directrix curve that is almost-strong (Theorem 5.1). Apply Lemma 2.1. For doubly ruled surfaces, notice that they can be generated by sweeping a line along a line or ellipse. ■

Corollary 5.4 *There is a unique way of generating a singly ruled surface by sweeping a line through space.*

Proof: Let R be a singly ruled surface. It has an almost-strong directrix curve S . In the proof of Lemma 2.1, it was shown that every sweep of R contains a sweep along all of S . We claim that, for singly ruled surfaces, every sweep of R is exactly a sweep along S . A sweep along S already generates R , by definition of almost-strong directrix curve. Thus, every generator that does not intersect S will be singular (Corollary 4.1), since every point of this line is already contained in a generator through S . We conclude that there are only a finite number of generators that do not intersect S (Lemma 3.1). Thus, every sweep of R is a sweep along S . However, there is a unique way of sweeping a line along S : the line must sweep smoothly along S (with no changes of direction, because of the injective nature

of a sweep), there is no flexibility in the choice of generator through a nonsingular point, and there is no choice in the direction to take at a singularity (because the sweep must be continuous). ■

6 Parameterization of ruled surfaces

The previous section showed that a ruled surface is the direct offspring of a plane curve. In this section, we give an example of how an algorithm for a ruled surface can be simplified to an algorithm for a plane curve, using the plane directrix curve that we have found. We use the example of parameterization, which straightforwardly applies the methods that we have developed. Parameterization is a problem that has been solved for rational plane curves [1] but not for surfaces (of degree higher than three). It is important because both the implicit and the parametric representation of a curve or surface are useful, so solid modelers wish to translate between them. (The complementary problem of implicitization, the translation from a rational parameterization of the surface to an implicit equation, has well-known solutions using elimination theory [10, 13, 20].)

In order to parameterize a ruled surface, we proceed as follows. Given a surface $f(x, y, z)$, if necessary first test if it is ruled (Section 4). Then find a plane directrix curve C (Section 5) and a formula V_P for the generator through P (Section 4). The parameterization of $f(x, y, z)$ is $c(s) + V_{c(s)}t$, where $c(s)$ is a parameterization of the plane curve C . C can be parameterized by the method of Abhyankar and Bajaj [1] if it is rational. A similar technique can be used for extending other plane curve algorithms to ruled surfaces, using the fact that a ruled surface is defined by a directrix curve.

Example 6.1 *From Example 5.2, we know that a directrix curve of the hyperbolic paraboloid $y^2 - x^2 - z = 0$ is the parabola $\{x = 0, y^2 - z = 0\}$, which has the parameterization $(x, y, z) = (0, t, t^2)$. From Example 4.1, the generators through the point P are $P + s(\pm 1, 1, 2p_2 \mp p_1)$. We conclude that a parameterization of the hyperbolic paraboloid is $(x, y, z) = (0, t, t^2) + s(1, 1, 2t) = (s, s + t, t^2 + 2st)$.*

Example 6.2 *From Example 5.3, a directrix curve of Cayley's ruled cubic is*

$\{y = 1, x^2 - 1 + xz = 0\}$, which has the parameterization $(x, y, z) = (t, 1, \frac{1-t^2}{t})$, $t \neq 0$. From Example 4.2, the generator of Cayley's ruled cubic through the nonsingular point P is $P + s(\frac{p_2^2}{p_1+p_2p_3}, 1, \frac{p_2}{p_1+p_2p_3} + \frac{p_3}{p_2})$. We conclude that a parameterization of Cayley's ruled cubic is $(x, y, z) = (t, 1, \frac{1-t^2}{t}) + s(t, 1, \frac{1}{t}) = (t + st, 1 + s, \frac{1+s-t^2}{t})$.

7 Line decomposition and intersection

The generator formula, along with the directrix curve, yields a line decomposition of the ruled surface: *i.e.*, a description as $\cup_{t \in I} \{\text{line}(t)\}$. This line decomposition provides a simple algorithm for intersection of a ruled surface with another surface, if the degree of the second surface is at most four. In particular, the intersection of a ruled surface R with an algebraic surface S of degree $d \leq 4$ can be computed as follows:

- Reduce $R \cap S$ to $\cup_{t \in I} \{\text{line}(t) \cap S\}$.
- Interpreting t as a symbolic constant, solve the equation $\text{line}(t) \cap S$, which is of degree $d \leq 4$.
- This yields a parameterization in t for the intersection.

The representation of a ruled surface by a line decomposition $\text{line}(t)$ has many advantages other than intersection.

- $(\text{line}(s))(t)$ is a good parameterization.³ It is always of degree 1 in s and it has geometric meaning (e.g., it is easy to define meaningful subsets of the surface using a range on t).
- It is simple to render the surface, since the line is a basic graphical element.
- The surface has the representation of a generalized cylinder, which is a preferred representation for computer vision applications.

³This is the parameterization where $(\text{line}(s))(k)$ is the k^{th} line.

8 Conclusions

In this paper, we have developed algorithms to make it easier to incorporate the rich and interesting class of algebraic ruled surfaces into solid modeling systems. The important steps of finding a directrix curve and finding the generator through every point of the directrix curve have been presented. Algorithms like parameterization and intersection immediately become easy. We have also shown how to check if a surface is ruled, and established that a ruled surface always has a plane directrix curve.

We believe that the entire class of ruled surfaces (not just cylinders, cones, and other simple ruled surfaces) should become a widely available resource for the solid modeler, and we hope that this paper, by developing the necessary algorithms and clarifying the mathematics, has established their simplicity and elegance.

9 Future Work

A problem for future study is to find not just any plane directrix curve, but the *simplest* directrix curve. For example, the hyperbolic paraboloid has a line directrix curve (indeed, any of its generators is a directrix curve [22]), as do cubic ruled surfaces [5, 22].

Since intersection with a ruled surface is simple, one approach to surface intersection is to reduce the problem to intersection with a ruled surface. This is the approach taken by Levin in his paper on the intersection of quadric surfaces [12]. In particular, the intersection of two surfaces $f(x, y, z)$ and $g(x, y, z)$ is equivalent to the intersection of $f(x, y, z)$ and a linear combination of $f(x, y, z)$ and $g(x, y, z)$. If one can find a linear combination of the two surfaces that is a ruled surface, one can solve the intersection. Therefore, a natural open problem arising from this work is to find a ruled surface in the pencil, or more generally the ideal, of two surfaces. Levin has shown that there is always a ruled surface in the pencil of two quadric surfaces.

A future direction is to consider an extension of ruled surfaces: surfaces generated by sweeping low degree algebraic curves (e.g., circles) through space, rather than lines. This would enrich the class of surfaces while maintaining the simplicity of ruled surfaces.

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