The Bézier tangential surface system: a robust dual representation of tangent space

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Abstract

This paper develops a robust dual representation for the tangent space of a rational surface. This dual representation of tangent space is a very useful tool for visibility analysis. Visibility constructs that are directly derivable from the dual representation of this paper include silhouettes, bitangent developables and kernels.

It is known that the tangent space of a surface can be represented by a surface in dual space, which we call a tangential surface. Unfortunately, a tangential surface is usually infinite. Therefore, for robust computation, the points at infinity must be clipped from a tangential surface. This clipping requires two complementary refinements, the first to allow clipping and the second to do the clipping. First, three cooperating tangential surfaces are used to model the entire tangent space robustly, each defined within a box. Second, the points at infinity on each tangential surface are clipped away while preserving everything that lies within the box. This clipping only involves subdivision along isoparametric curves, a considerably simpler process than exact trimming to the box. The isoparametric values for this clipping are computed as local extrema from an analysis using Sederberg's piecewise algebraic curves.

A construction of the tangential surface of a parametric surface is outlined, and it is shown how the tangential surface of a Bézier surface can be expressed as a rational Bézier surface.

Keywords: tangent space, dual space, tangential surface, Bézier surface, piecewise algebraic curve.

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1 Introduction

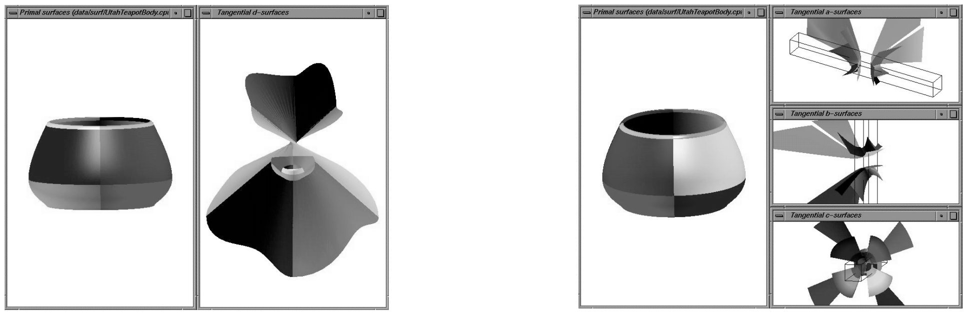
This paper addresses the representation of a parametric surface's tangent space. Rather than computing individual tangent planes, which is well understood, we want to capture the entire tangent space, which will facilitate an analysis of the interaction of the tangent space with a scene. This interaction plays an important role in the visibility analysis of a scene of smooth objects, including occlusion culling, lighting, and motion planning. Consider three examples. The silhouette of a surface is the locus of points whose tangent plane passes through the viewpoint. It defines the boundaries of visibility from the viewpoint. The kernel of a surface is the locus of points not touched by any tangent plane of the surface. It is the region that sees the entire surface. A bitangent developable is the envelope of a one-parameter family of bitangent planes between two surfaces. It defines a boundary of visibility between two surfaces and a visual event in a scene of surfaces. These examples show the importance of a surface's tangent space in visibility analysis.

The tangent space of a parametric surface is a two-parameter family of planes. Using the geometric duality between planes and points in 3-space, this tangent space may be reinterpreted as a two-parameter family of points in dual space, or a surface (Figure 1a). We will call this dual representation of tangent space a tangential surface.

Definition 1 *Let $S(s,t)$ be a parametric surface in 3-space. The* **tangential surface** *of S is the surface S^* where $S^*(s,t)$ is the dual of the tangent plane at $S(s,t)$.*

A dual representation of tangent space is very useful in visibility analysis, as many visibility computations reduce to surface intersections in dual space. Consider three examples. The computation of a surface's silhouette reduces to the intersection of a plane (the dual of the viewpoint) with the tangential surface. The kernel of a surface can be computed from the convex hull of its tangential surface. A bitangent developable of two surfaces reduces to a component of the intersection of their tangential surfaces. In general, the tangential surface is an ideal tool for the computation of visual events that occur as the viewpoint interacts with the tangent spaces of a scene of surfaces.

The purpose of this paper is to develop a robust representation for the tangential surface. A robust representation is necessary because a tangential surface will typically contain points at infinity (Section 3). Two strategies



(a) (b)

Figure 1: (a) The body of the Utah teapot and its tangential a_4-surface. (b) The tangential surface system of the Utah teapot's body.

are available for coping with the points at infinity on a tangential surface: working directly in projective space (where points at infinity are not a problem since they are represented directly) or working in Cartesian space and clipping away the points at infinity from the tangential surface, which confuse algorithms such as intersection. The latter option is preferable, because an interpretation of the tangential surface in projective space is highly convoluted in a smooth environment, either requiring a line field or a nonrational 3d-surface restricted to the unit sphere in 4-space, which complicates intersection considerably and leads to nonrational representations. The projective space solution is analyzed in more detail in Section 2.

The removal of points at infinity from a tangential surface is a two-step process. If a single tangential surface is used, then any clipping will remove some of the tangent space. Thus, the representation of tangent space must first be shared across several tangential surfaces. In Section 4, it is shown how the tangent space can be spread across three tangential surfaces in three dual spaces, using only the part of each tangential surface inside a box centered at the origin. This allows the infinite part of each tangential surface to be clipped away without compromising the representation of tangent space. Section 5 shows how to remove the points at infinity from a tangential surface without removing any of the surface inside the relevant box. This surface clipping uses subdivision along isocurves, an efficient form of clipping. The isocurves that define the clipping are found using techniques from Sederberg's piecewise algebraic curves. A rational Bézier representation is developed for the tangential surface of a Bézier surface in Section 6. We conclude in Section 7. The paper starts with a comparison to related work (Section 2) and a review of duality (Section 3).

2 Related work

The tangential surface is very similar to the tangential variety of a smooth surface S [5], which is the algebraic variety consisting of all tangent planes of S.

While the tangential surface represents the exact tangent planes of a surface, the Gauss map [1, 3] represents only their unit normals. The exact tangent plane is important in distinguishing parallel planes from coincident planes, as for example in the computation of bitangent planes.

The dual Bézier surface [7] of a surface S is a description of S as the envelope of a family of planes, using control planes to represent the surface rather than control points. It is a representation of S in primal space using a dual control structure (a collection of planes rather than points), while the tangential surface is a representation of the tangent space of S in dual space using a conventional primal control structure (Section 6). Other dual representations in geometric modeling include work on dual subdivision schemes, such as Sabin [11] and Dyn et. al. [2], and the representation of developable surfaces as curves in dual space, such as Pottmann and Farin [10].

Hertzmann and Zorin develop a dual representation for the tangent space of a polyhedral mesh in order to compute silhouettes [6]. They are interested in polyhedral meshes rather than parametric surfaces, and silhouettes rather than general visibility analysis, which frees them to use a solution in projective space.

Definition 2 Projective 3-space P^3 *is the space* $\{(x_1, x_2, x_3, x_4) : x_i \in \Re,$ *not all zero*$\}$ *under the equivalence relation* $(x_1, x_2, x_3, x_4) = k(x_1, x_2, x_3, x_4)$, $k \neq 0 \in \Re$. *The point* (x_1, x_2, x_3, x_4) *in projective 3-space is equivalent to the point* $(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4})$ *in Cartesian 3-space. The last coordinate* x_4 *is special and called the* **projective coordinate**. *Its zeroes are associated with points at infinity.*

To compute the silhouette of a polyhedral mesh, they dualize the mesh and intersect it with the viewpoint's dual plane. To solve the problem of points at infinity, dual space is modeled as the boundary of a hypercube in 4-space, a direct model of projective 3-space. In this solution, the tangent plane $a_1x_1 + a_2x_2 + a_3x_3 + a_4 = 0$ at a vertex of the polyhedral mesh is dualized to a vertex (a_1, a_2, a_3, a_4) of the dual mesh and then projected to the hypercube. Projection to the hypercube is necessary to get a unique representation for a point in projective space, which is crucial for subsequent intersection. Projection to the hypercube (polyhedral hypersphere) is feasible for the finite number of vertices on a polyhedral mesh, but not for a parametric surface. In particular, projection to the hypersphere is a nonrational map, yielding nonrational parametric surfaces. Moreover, intersection of surfaces in Cartesian 3-space is simpler than the intersection of surfaces on S^3 in 4-space. In other words, we must work with surfaces in Cartesian 3-space, rather than projective space, to maintain rational representations and simplify subsequent intersection.

This paper extends our work in [8], in which a robust dual representation of the tangent space of a curve was developed.

3 Geometric duality

We now review some of the theory of the duality between planes and points. Planes and points in 3-space can be related by a geometric duality [9], which associates the coefficients of a plane equation and the coordinates of a point. Classically, the plane $a_1x + a_2y + a_3z + a_4 = 0$ is identified with the point (a_1, a_2, a_3, a_4) in projective 3-space, which is equivalent to the point $(\frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$ in Cartesian 3-space. Alternative dualities can be defined by mapping other coefficients to the projective coordinate.

Definition 3 *The plane* $a_1x_1+a_2x_2+a_3x_3+a_4 = 0$ *is* a_i**-dual** *(*$i = 1, \ldots, 4$*) to the point* $(\frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$ *with* a_i *and* a_4 *interchanged. That is, this plane is* a_1*-dual to the point* $(\frac{a_4}{a_1}, \frac{a_2}{a_1}, \frac{a_3}{a_1})$*,* a_2*-dual to the point* $(\frac{a_1}{a_2}, \frac{a_4}{a_2}, \frac{a_3}{a_2})$*,* a_3*-dual to the point* $(\frac{a_1}{a_3}, \frac{a_2}{a_3}, \frac{a_4}{a_3})$*, and* a_4*-dual to the point* $(\frac{a_1}{a_4}, \frac{a_2}{a_4}, \frac{a_3}{a_4})$*.*

Duality is a mutual relationship that preserves incidence [9]:

- If the dual of A is B, then the dual of B is A.
- Let A be a plane and A^* its dual point. Let B be a point and B^* its dual plane. A contains B if and only if B^* contains A^*.

We can geometrically characterize the planes mapped to infinity by a_i-duality.

Lemma 4 *Planes whose normals are orthogonal to the x_i-axis are mapped to infinity by a_i-duality for $i = 1, 2, 3$. Planes through the origin are mapped to infinity by a_4-duality.*

Proof: The plane $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ maps to infinity under a_i-duality ($i = 1, 2, 3$) if $a_i = 0$, in which case its normal (a_1, a_2, a_3) lies in the plane $x_i = 0$ and is orthogonal to the x_i-axis. The plane maps to infinity under a_4-duality if $a_4 = 0$, in which case the plane must contain the origin. ∎

In practice, none of the planes whose normals are close to orthogonal to the x_i-axis or close to the origin are robustly represented by a_i-duality and a_4-duality, respectively.

Definition 5 *Let $S(s, t)$ be a surface in 3-space. The* **tangential a_i-surface** *of S ($i = 1, \ldots, 4$) is the surface S_i^* where $S_i^*(s, t)$ is the a_i-dual of the tangent plane at $S(s, t)$.*

As a result of Lemma 4, a tangential a_i-surface typically contains points at infinity. In particular, all tangential a_i-surfaces ($i = 1, 2, 3$) of closed surfaces contain points at infinity, as do the a_4-surfaces of all surfaces whose kernels do not contain the origin.[1]

4 The tangential surface system

Since information is lost at infinity, a single tangential surface is not a robust representation of a surface's tangent space. A collection of tangential surfaces is necessary. Since a robust dual representation of tangent space depends on a robust dual representation of planes, we begin with planes. To robustly represent all planes in dual space, the space of planes must be decomposed into several parts, using a different duality to map each part. This decomposition should be complete (covers all planes), nonoverlapping (no plane is covered twice), minimal (as few parts as possible), and robust (each part is mapped finitely). a_i-dominance yields a decomposition that satisfies all of these properties. In this strategy, a_i-dominant planes are represented in a_i-dual space. A plane $a_1x_1 + a_2x_2 + a_3x_3 + a_4 = 0$ is **a_i-dominant** ($i = 1, 2, 3$) if $|a_i| = \max(|a_1|, |a_2|, |a_3|)$.

[1] If the origin lies in the kernel of S, no tangent plane of S will touch the origin and so no tangent plane will a_4-dualize to infinity.

This decomposition is complete and nonoverlapping (to break ties, we favour a_i-dominance over a_j-dominance if $i < j$). It is also minimal: one part is not enough to represent all planes robustly (Lemma 4); two parts are also insufficient, since a plane can be orthogonal to two axes simultaneously, or orthogonal to an axis and through the origin. Most importantly, a_i-dominance defines a robust decomposition, as we now show.

Lemma 6 *a_i-dominant planes are finitely represented in a_i-dual space ($i = 1, 2, 3$).*

Proof: Dominance can be characterized geometrically in terms of the plane's normal: a plane is a_i-dominant if the angle of its normal to the x_i-axis is less than or equal to its angle to any other axis. Thus, the normal of an a_i-dominant plane is not orthogonal to the x_i-axis, since it cannot be orthogonal to all three axes, and the plane is not mapped to infinity (Lemma 4). In fact, the normal of an a_i-dominant plane is far from orthogonal to the x_i-axis, so this plane is mapped far from infinity (Lemma 8). ∎

The next lemma establishes a relationship between a_i-dominance and a_i-boxes.

Definition 7 *The a_i-**box** ($i = 1, 2, 3$) is the infinite box in a_i-dual space restricted to $x_j \in (-1, 1)$ for $j < i$ and $x_j \in [-1, 1]$ for $j > i$. The open and closed intervals correspond to the breaking of ties in a_i-dominance as above.*

Lemma 8 *A plane is a_i-dominant if and only if its a_i-dual is a finite point lying inside the a_i-box ($i = 1, 2, 3$).*

Proof: Consider $i = 1$. Other cases are analogous. The plane $a_1x_1 + a_2x_2 + a_3x_3 + a_4 = 0$ a_1-dualizes to the point $(\frac{a_4}{a_1}, \frac{a_2}{a_1}, \frac{a_3}{a_1})$. This plane is a_1-dominant if and only if $|a_1| = \max(|a_1|, |a_2|, |a_3|)$ if and only if $|\frac{a_2}{a_1}| \le 1$ and $|\frac{a_3}{a_1}| \le 1$ if and only if the dual point lies in the a_1-box. The point is finite since $a_1 \ne 0$ (otherwise $a_1 = a_2 = a_3 = 0$ and the plane is undefined). ∎

Theorem 9 *The space of planes in 3-space is finitely and uniquely represented by the a_1-box in a_1-dual space, the a_2-box in a_2-dual space, and the a_3-box in a_3-dual space.*

This robust dual representation of a plane suggests a robust dual representation of a surface's tangent space.

Definition 10 *Let $S(s,t)$ be a surface in 3-space. S's* **tangential surface system** *is the tangential a_1-surface restricted to the a_1-box, the tangential a_2-surface restricted to the a_2-box, and the tangential a_3-surface restricted to the a_3-box.*

Theorem 11 *The tangent space of a surface S is represented finitely and uniquely by its tangential surface system.*

Proof: The a_i-dominant tangent planes of a surface S are represented by the tangential a_i-surface S_i^* inside the a_i-box. By Lemma 8, S_i^* contains no points at infinity inside the a_i-box. ■

Figure 1b illustrates a tangential surface system, with a surface in primal space on the left, its tangential a_1-surface on the top right, its tangential a_2-surface on the middle right, and its tangential a_3-surface on the bottom right. A portion of the a_i-box is also shown in each dual space. The tangential surfaces have been rotated independently for optimal viewing. All of the tangential surfaces have been clipped to remove infinite patches, as described in Section 5.

Notice that the tangential a_4-surface is not included in the tangential surface system. If we include the a_4-surface in any collection of tangential surfaces, three tangential surfaces are still not enough for a robust representation. For example, if we choose the a_1-, a_2- and a_4-surfaces, tangent planes near the origin with normals in the neighbourhood of the z-axis are still mapped near infinity on all three surfaces.

Notice that computation with a tangential surface can be restricted to a box without actually clipping to the box, using a form of lazy evaluation. However, we do need to remove points at infinity, as discussed next.

5 Conservative clipping

Points at infinity cause many problems in working with a tangential surface and must be removed. The tangential surface system frees us to clip each tangential a_i-surface in a neighbourhood about infinity, since only the part inside the a_i-box is of interest and this box does not contain any points at infinity (Lemma 8). Infinite patches of the tangential a_i-surface that overlap the a_i-box are the issue. Infinite patches that lie completely outside the a_i-box can simply be ignored. A naive solution would be to intersect the tangential surface with the a_i-box and clip along these intersection curves. However, this type of clipping is expensive. It is also excessive since the only requirement of the clipping is to remove points at infinity, not to clip exactly to the box.

We now offer a method for clipping points at infinity from a parametric surface that uses efficient subdivision along isoparametric curves of the surface, followed by removal of certain patches. Care is taken not to remove any of the surface inside the a_i-box. As a preprocessing step, patches whose normals span all three axis directions can be subdivided to speed the clipping.

Definition 12 *An* **isocurve clipping** *of the surface S is a subdivision of S along a set of isocurves, followed by the removal of some of the resulting patches. An isocurve clipping of the tangential a_i-surface S is* **conservative** *if it removes all of the points at infinity on S without removing any of the surface inside the a_i-box.*

The a_i-box is bounded by $x_{i\oplus 1} = \pm 1$ and $x_{i\oplus 2} = \pm 1$ where $\oplus$ is modulo arithmetic (Definition 7). Rather than clipping to the a_i-box, we clip to a slightly larger box bounded by $x_{i\oplus 1} = \pm 2$ and $x_{i\oplus 2} = \pm 2$ to avoid any danger that some of the box will be removed by numerical imprecision.[2] Rather than clipping exactly to this larger box, we clip to it as well as we can with isocurves. The calculation of these isocurves is based on local extrema in parameter space, as follows. Let the tangential a_i-surface be $(x_1(s,t), x_2(s,t), x_3(s,t))$ in the following discussion.

We clip to each bounding plane of the box in turn. Consider the bounding plane $x_{i\oplus 1} = 2$. The curve $x_{i\oplus 1}(s,t) = 2$ is an implicit curve in the parameter space (s,t) of the tangential surface that represents the parameter values where the surface crosses the plane $x_{i\oplus 1} = 2$ (Figure 2a). It divides the surface's parameter space into regions that do not intersect the $x_{i\oplus 1} = 2$ plane. Subdivision along the local s-extrema and t-extrema of $x_{i\oplus 1}(s,t) = 2$ will successfully isolate $x_{i\oplus 1} = +\infty$ from the a_i-box (Figure 2e). Note that the global extrema of $x_{i\oplus 1}(s,t) = 2$ are insufficient: interior local extrema can be important. Subdividing at a local extremum in s (resp., t) of $x_{i\oplus 1}(s,t) = 2$ is equivalent to subdividing along an extremal horizontal (resp., vertical) isocurve that touches the $x_{i\oplus 1} = 2$ plane. This leads directly to an algorithm for the conservative isocurve clipping of a tangential a_i-surface S_i^*:

[2]This also allows a rougher search for isocurves, as discussed below.

1. Mark the infinite patches of S_i^* that overlap the a_i-box.

2. For every such infinite patch, find the local s-extrema and t-extrema of the curves $x_{i\oplus 1}(s,t) = \pm 2$ and $x_{i\oplus 2}(s,t) = \pm 2$.

3. Subdivide S_i^* at these parameter values.

4. Remove all patches of S_i^* that lie completely outside the a_i-box (which now includes all infinite patches).

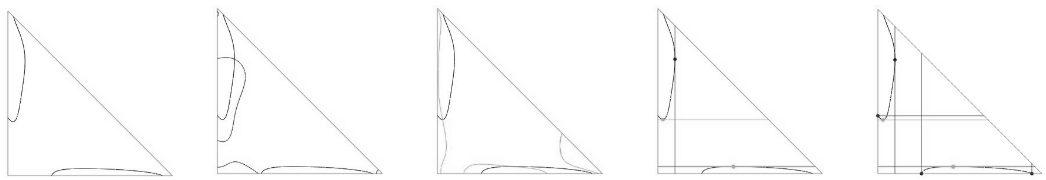


Figure 2: (a) The PAC $x_3(s,t) = -2$ for an infinite patch of the Utah teapot handle's tangential a_1-surface. (b) This PAC with its polar for vertical tangents. (c) This PAC with its polar for horizontal tangents. (d) Its horizontal and vertical tangents. (e) A partition of parameter space at local extrema.

We have reduced the clipping problem to finding the local extrema of an implicit curve (Step 2), for which Sederberg's piecewise algebraic curves (PAC) offer a natural solution [12, 13]. Piecewise algebraic curves are an elegant representation for algebraic curves, allowing an algebraic curve to be studied inside an arbitrary triangle by expressing it as a planar section of a triangular Bezier patch, with all of the benefits of this representation such as subdivision and simple computation of polars (see [12, 13] for details). Sederberg has shown how to use piecewise algebraic curves to find the horizontal and vertical tangents of an algebraic curve, through intersection of the PAC with one of its polars (Figure 2b-d). This solves our problem, since the local extrema of an implicit curve either have horizontal or vertical tangents or lie on the boundary (Figure 2e).

Sederberg's intersection algorithm for PAC's is subdivision-based, and at every stage a PAC is either shrunk or split. For the datasets generated by the tangential surface problem, we have found that shrinking is rarely as good an option as splitting, so we advise automatically splitting at each stage. This avoids the expense of measuring shrink distance (3 times per triangle) and speeds up the intersection. For additional efficiency, we suggest using a crude epsilon value as stopping condition for the PAC intersection, since conservative clipping does not demand a perfect clipping, simply a clipping that slashes away the points at infinity without touching the box. In a similar spirit, parameter values within epsilon can be merged, to reduce the number of subdivision values in Step 2 of the algorithm. The tangential surfaces of Figure 1b are clipped using this algorithm.

6 Bézier tangential surfaces

We now consider the computation of tangential surfaces, including how to express the tangential surface of a Bézier surface as a rational Bézier surface. This significantly simplifies the incorporation of the tangential surface into modeling systems.

The tangential surface is computed from the implicit equation of the tangent plane (Definitions 3 and 5). Consider the parametric surface $S(s,t) = (S_0(s,t), S_1(s,t), S_2(s,t))$. Since the implicit equation of the plane through P with normal N is $(X-P)\cdot N = 0$, the implicit equation of the tangent plane at $S(s,t)$ is $(x - S_0, y - S_1, z - S_2)\cdot(S^s \times S^t) = 0$ or, expanding, $x\ (S_1^s S_2^t - S_2^s S_1^t) + y\ (S_2^s S_0^t - S_0^s S_2^t) + z\ (S_0^s S_1^t - S_1^s S_0^t) - S_0(S_1^s S_2^t - S_2^s S_1^t) + S_1(S_2^s S_0^t - S_0^s S_2^t) + S_2(S_0^s S_1^t - S_1^s S_0^t) = 0$ where S_i^s (resp., S_i^t) is the derivative of $S_i(s,t)$ with respect to s (resp., t). The tangential a_i-surface of S is the a_i-dual of this parameterized plane.

Consider the Bézier representation of the tangential a_1-surface. Bézier representations of the other tangential surfaces are analogous, and indeed can be derived directly from the Bézier representation of the a_1-surface. In projective 3-space, the tangential a_1-surface of $S(s,t)$ is

$$\begin{pmatrix} S_0(S_2^s S_1^t - S_1^s S_2^t) + S_1(S_0^s S_2^t - S_2^s S_0^t) + S_2(S_1^s S_0^t - S_0^s S_1^t) \\ S_2^s S_0^t - S_0^s S_2^t \\ S_0^s S_1^t - S_1^s S_0^t \\ S_1^s S_2^t - S_2^s S_1^t \end{pmatrix} \tag{1}$$

Using the abbreviation $d_{\alpha\beta} = S_\alpha^s S_\beta^t$ for the product of two derivatives S_α^s and S_β^t and $\widehat{aA}$ for the product of the 1-dimensional Bézier surfaces a and A, (1) simplifies to

$$\begin{pmatrix} \widehat{S_0 d_{21}} - \widehat{S_0 d_{12}} + \widehat{S_1 d_{02}} - \widehat{S_1 d_{20}} + \widehat{S_2 d_{10}} - \widehat{S_2 d_{01}} \\ d_{20} - d_{02} \\ d_{01} - d_{10} \\ d_{12} - d_{21} \end{pmatrix} \tag{2}$$

The following two lemmas show how to compute the components of this formula. Proofs are omitted for lack of space (see the technical report).

Lemma 13 *Let $S(s,t) = (S_0(s,t), S_1(s,t), S_2(s,t))$ be a tensor product Bézier surface of degree (m,n) over the parameter interval $[s_1,s_2] \times [t_1,t_2]$ with control points $b_{i,j} = (b^0_{i,j}, b^1_{i,j}, b^2_{i,j})$. The* **derivative product** $d_{\alpha\beta} = S^s_\alpha S^t_\beta$, *where $0 \le \alpha \le 2$, $0 \le \beta \le 2$, is a 1-dimensional Bézier surface of degree $(2m-1, 2n-1)$ over the parameter interval $[s_1,s_2]\times[t_1,t_2]$ with control points $d_{k,l}$ where*

$$d_{k,l} = \gamma_{k,l} \sum_{\substack{0 \le i \le m-1 \\ 0 \le I \le m \\ i+I=k}} \sum_{\substack{0 \le j \le n \\ 0 \le J \le n-1 \\ j+J=l}} \delta_{iIjJ} \; \Delta^{1,0} b^\alpha_{i,j} \; \Delta^{0,1} b^\beta_{I,J}$$

and $\gamma_{k,l} = \frac{mn}{(s_2-s_1)(t_2-t_1)\binom{2m-1}{k}\binom{2n-1}{l}}$, $\delta_{iIjJ} = \binom{m-1}{i}\binom{m}{I}\binom{n}{j}\binom{n-1}{J}$, $\Delta^{1,0} b^\alpha_{i,j} = b^\alpha_{i+1,j} - b^\alpha_{i,j}$, $\Delta^{0,1} b^\beta_{I,J} = b^\beta_{I,J+1} - b^\beta_{I,J}$.

Lemma 14 *The product of the 1-dimensional Bézier surface a of degree (m,n) with control points $a_{i,j}$ and the 1-dimensional Bézier surface A of degree (M,N) with control points $A_{I,J}$ (defined over the same parameter interval) is the 1-dimensional Bézier surface $\widehat{\mathbf{aA}}$ of degree $(m+M, n+N)$ with control points $\widehat{aA}_{kl}$ where*

$$\widehat{aA}_{kl} = \sum_{\substack{0 \le i \le m \\ 0 \le I \le M \\ i+I=k}} \sum_{\substack{0 \le j \le n \\ 0 \le J \le N \\ j+J=l}} \frac{\binom{m}{i}\binom{M}{I}\binom{n}{j}\binom{N}{J}}{\binom{m+M}{k}\binom{n+N}{l}} a_{i,j} A_{I,J}$$

Lemmas 13 and 14 show how to implement the formula (2) for the tangential a_1-surface. Since the first coordinate of (2) is of degree $(3m-1, 3n-1)$ while the last 3 coordinates are of degree $(2m-1, 2n-1)$, the last 3 coordinates must be degree elevated to $(3m-1, 3n-1)$ [4]. This Bézier surface in projective 3-space is then translated to a rational Bézier surface in 3-space, using the fact that a Bézier surface in projective 3-space with control points $(b^0_{ij}, b^1_{ij}, b^2_{ij}, b^3_{ij})$ is equivalent to a rational Bézier surface in 3-space with weights b^3_{ij} and control points $(\frac{b^0_{ij}}{b^3_{ij}}, \frac{b^1_{ij}}{b^3_{ij}}, \frac{b^2_{ij}}{b^3_{ij}})$. This yields the rational Bézier representation of the tangential a_1-surface.

Theorem 15 *The tangential a_1-surface of a Bézier surface of degree (m, n) can be represented as a rational Bézier surface of degree $(3m-1, 3n-1)$, applying Lemmas 13-14 and degree elevation to (2).*

Once the tangential a_1-surface is known, the tangential a_2-surface and a_3-surface can be derived directly from it, through permutation of the coordinates in (2). Notice that points at infinity on the tangential surface are marked by patches that contain both positive and negative weights. This mixing of weights is easy to achieve, since there are no constraints on the sign of any of the coordinates of (1).

7 Conclusions

This paper has developed a robust dual representation of a surface's tangent space: the tangential surface system. The representation was made robust by sharing the representation across three dual spaces and clipping points at infinity, while the representation was made practical by computing its Bézier formulation. This dual representation of tangent space simplifies the analysis of tangential interactions between surfaces, as required in visibility analysis. The larger goal of this work is the development of a theory of visibility, rendering, lighting and motion for a smooth scene, where obstacles and lights are bounded by parametric surfaces.

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