

## Cryptographically Resilient Functions

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**Abstract**—This correspondence studies resilient functions which have applications in fault-tolerant distributed computing, quantum cryptographic key distribution, and random sequence generation for stream ciphers. We present a number of new methods for synthesizing resilient functions. An interesting aspect of these methods is that they are applicable both to linear and nonlinear resilient functions. Our second major contribution is to show that every linear resilient function can be transformed into a large number of nonlinear resilient functions with the same parameters. As a result, we obtain resilient functions that are highly nonlinear and have a high algebraic degree.

**Index Terms**—Correlation-immune functions, cryptography, nonlinearity, resilient functions.

### I. INTRODUCTION

An  $(n, m, t)$ -resilient function is an  $n$ -input  $m$ -output function  $F$  with the property that it runs through every possible output  $m$ -tuple an equal number of times when  $t$  arbitrary inputs are fixed and the remaining  $n - t$  inputs run through all the  $2^{n-t}$  input tuples once. The concept of a resilient function was first introduced by Chor *et al.* [1] and independently, by Bennett, Brassard, and Robert in [2]. It turned out that *balanced* correlation immune functions introduced by Siegenthaler [3] are a special case of resilient functions. Areas where resilient functions find their applications include fault-tolerant distributed computing [1], quantum cryptographic key distribution [2], and random sequence generation for stream ciphers [4].

Manuscript received April 3, 1995; revised June 14, 1996. This work was supported in part by the Australian Research Council (ARC) under the reference number A49232172 and by the Australian Telecommunications and Electronics Research Board (ATERB) under the reference numbers C010/058 and N069/412. An early version of this correspondence was presented as "On Nonlinear Resilient Function" at EUROCRYPT'95, Saint-Malo, France, May 21–25, 1995.

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Publisher Item Identifier S 0018-9448(97)05220-6.

Researchers have concentrated on linear resilient functions, with one exception being the work by Stinson and Massey [5]. The aim in [5] was solely to disprove a conjecture posed in [1] (namely, if there exists a nonlinear resilient function then there exists a linear resilient function with the same parameters) rather than to explore cryptographic merits of nonlinear resilient functions. Recent advances in cryptanalysis, in particular the discovery of the linear cryptanalytic attack [6], have shown the vital importance of nonlinear functions in data encryption and one-way hashing algorithms. With the further revelation of the potential power of the linear attack, we might see its serious implications on the security of many other cryptographic routines, including those employing resilient functions. A relevant but earlier development is the best affine approximation (BAA) attack proposed by Ding, Xiao, and Shan in [7]. It has been shown that the BAA attack can successfully break a number of types of key stream generators that employ a combining or filtering function which, though correlation-immune, has a *low* nonlinearity. Success of these attacks clearly shows a need to investigate highly nonlinear resilient functions.

The rest of the correspondence is organized as follows: Section II introduces basic definitions. It also reviews important properties of resilient functions, as well as previous work in the area. Section III presents a number of methods for constructing new resilient functions from old. Some of these methods significantly generalize previously known methods. An exceptional feature of our methods is that they can be applied *both to linear and to nonlinear* resilient functions. Section IV shows how to turn a known resilient function into a new one. As a result, we can obtain a large number of highly nonlinear resilient functions from a linear one. Some miscellaneous results on resilient functions, including a discussion on algebraic degree, are included in Section V, and the paper is closed by some concluding remarks in Section VI.

### II. PRELIMINARIES

The vector space of  $n$ -tuples of elements from  $\text{GF}(2)$  is denoted by  $V_n$ . These vectors, in ascending lexicographical order, are denoted by  $\alpha_0, \alpha_1, \dots, \alpha_{2^n-1}$ . As vectors in  $V_n$  and integers in  $[0, 2^n - 1]$  have a natural one-to-one correspondence, it allows us to switch from a vector in  $V_n$  to its corresponding integer in  $[0, 2^n - 1]$ , and *vice versa*.

Let  $f$  be a function from  $V_n$  to  $\text{GF}(2)$  (or simply, a function on  $V_n$ ). The *sequence* of  $f$  is defined as

$$((-1)^{f(\alpha_0)}, (-1)^{f(\alpha_1)}, \dots, (-1)^{f(\alpha_{2^n-1})})$$

where each exponent is regarded as being real-valued, while the *truth table* of  $f$  is defined as

$$(f(\alpha_0), f(\alpha_1), \dots, f(\alpha_{2^n-1})).$$

$f$  is said to be *balanced* if its truth table assumes an equal number of zeros and ones. We call  $h(x) = a_1x_1 \oplus \dots \oplus a_nx_n \oplus c$  an *affine function*, where  $x = (x_1, \dots, x_n)$  and  $a_j, c \in \text{GF}(2)$ . In particular,  $h$  will be called a *linear function* if  $c = 0$ . The sequence of an affine (linear) function will be called an *affine (linear) sequence*.

Functions on  $V_n$  can be represented by polynomials of  $n$  coordinates. We are particularly interested in the so-called *algebraic normal form* representation in which a function is viewed as the sum of products of coordinates. The *algebraic degree*  $\text{deg}(f)$  of a function

$f$  is the number of coordinates in the longest product in the algebraic normal form representation of the function. The *Hamming weight* of a vector  $v$ , denoted by  $W(v)$ , is the number of ones in  $v$ . Let  $f$  and  $g$  be functions on  $V_n$ . Then

$$d(f, g) = \sum_{f(x) \neq g(x)} 1$$

where the addition is over the reals, is called the *Hamming distance* between  $f$  and  $g$ . Let  $\varphi_0, \dots, \varphi_{2^n+1-1}$  be the affine functions on  $V_n$ . Then

$$N_f = \min_{i=0, \dots, 2^n+1-1} d(f, \varphi_i)$$

is called the *nonlinearity* of  $f$ . It is well known that the nonlinearity of  $f$  on  $V_n$  satisfies  $N_f \leq 2^{n-1} - 2^{(1/2)n-1}$ . An extensive investigation of highly nonlinear balanced functions has been carried out in [8]–[10].

Algebraic degree and nonlinearity can also be defined for mappings or tuples of functions. Let  $F = (f_1, \dots, f_m)$  be a function from  $V_n$  to  $V_m$  (where each  $f_i$  is a function on  $V_n$ ). The algebraic degree of  $F$ , denoted by  $\deg(F)$ , is defined as the minimum among the algebraic degrees of all nonzero linear combinations of the component functions of  $F$ , namely,

$$\deg(F) = \min_g \left\{ \deg(g) \mid g = \bigoplus_{j=1}^m c_j f_j, c_j \in \text{GF}(2), (c_1, c_2, \dots, c_m) \neq (0, 0, \dots, 0) \right\}.$$

Similarly, the nonlinearity of  $F$ , denoted by  $N_F$ , is defined as the minimum among the nonlinearities of all nonzero linear combinations of the component functions of  $F$

$$N_F = \min_g \left\{ N_g \mid g = \bigoplus_{j=1}^m c_j f_j, c_j \in \text{GF}(2), (c_1, c_2, \dots, c_m) \neq (0, 0, \dots, 0) \right\}.$$

This definition regarding  $N_F$  was first introduced by Nyberg in [11].

$F = (f_1, \dots, f_m)$  is said to be linear if all its component functions are linear, and to be nonlinear otherwise. If  $F$  is linear, then  $\deg(F) = 1$  and  $N_F = 0$ . The converse, however, is not always true.

### A. Properties of Resilient Functions

Now we summarize a number of facts regarding resilient functions. Though most of these results are either previously known from, for instance, [1], [2], and [12], or can be proven easily, they are collected here with the intent to help the reader to understand our results. We start with a formal definition of a resilient function.

*Definition 1:* Let  $F = (f_1, \dots, f_m)$  be a function from  $V_n$  to  $V_m$ , where  $n \geq m \geq 1$ , and let  $x = (x_1, \dots, x_n) \in V_n$ .

- 1)  $F$  is said to be *unbiased* with respect to a fixed subset  $T = \{j_1, \dots, j_t\}$  of  $\{1, \dots, n\}$ , if for every  $(a_1, \dots, a_t) \in V_t$

$$(f_1(x), \dots, f_m(x)) \Big|_{x_{j_1}=a_1, \dots, x_{j_t}=a_t}$$

runs through all the vectors in  $V_m$  each  $2^{n-m-t}$  times while  $(x_{i_1}, \dots, x_{i_{n-t}})$  runs through  $V_{n-t}$  once, where  $t \geq 0$

$$\{i_1, \dots, i_{n-t}\} = \{1, \dots, n\} - \{j_1, \dots, j_t\}$$

and  $i_1 < i_2 < \dots < i_{n-t}$ .

- 2)  $F$  is said to be a  $(n, m, t)$ -resilient function if  $F$  is unbiased with respect to every subset  $T$  of  $\{1, \dots, n\}$  with  $|T| = t$ . The parameter  $t$  is called the *resiliency* of the function.

Obviously,  $n - m \geq t$  holds for all  $(n, m, t)$ -resilient functions.

Resilient functions are closely related to correlation immune functions introduced by Siegenthaler [3]. As was noticed by Stinson, *et al.*, an  $(n, 1, t)$ -resilient function is the same as a *balanced*  $t$ th-order correlation immune function. We will come back to this issue shortly.

The following lemma is helpful in understanding the relationship between a resilient function and its component functions. It has been called *XOR Lemma* and expressed in terms of independence of random variables in [1] and [2]. It also appears as [13, Corollary 7.39]. Here we follow the version described in [14].

*Lemma 1:* Let  $F = (f_1, \dots, f_m)$  be a function from  $V_n$  to  $V_m$ , where  $n$  and  $m$  are integers with  $n \geq m \geq 1$  and each  $f_j$  is a function on  $V_n$ . Then  $F$  is unbiased, namely, it runs through all the vectors in  $V_m$  each  $2^{n-m}$  times while  $x$  runs through  $V_n$  once, if and only if each nonzero linear combination of  $f_1, \dots, f_m$  is balanced.

Hence, we have

*Lemma 2:* Let  $F = (f_1, \dots, f_m)$  be a function from  $V_n$  to  $V_m$ , where  $n$  and  $m$  are integers with  $n \geq m \geq 1$  and each  $f_j$  is a function on  $V_n$ . Then  $F$  is unbiased with respect to  $T = \{j_1, \dots, j_t\}$ , a fixed subset of  $\{1, \dots, n\}$ , if and only if every nonzero linear combination of  $f_1, \dots, f_m$

$$f(x) = \bigoplus_{j=1}^m c_j f_j(x)$$

is unbiased (i.e., balanced) with respect to  $T = \{j_1, \dots, j_t\}$ , where  $x = (x_1, \dots, x_n) \in V_n$ .

As an immediate consequence, we have

*Theorem 1:* Let  $F = (f_1, \dots, f_m)$  be a function from  $V_n$  to  $V_m$ , where  $n$  and  $m$  are integers with  $n \geq m \geq 1$  and each  $f_j$  is a function on  $V_n$ . Then  $F$  is an  $(n, m, t)$ -resilient function if and only if every nonzero linear combination of  $f_1, \dots, f_m$

$$f(x) = \bigoplus_{j=1}^m c_j f_j(x)$$

is a  $(n, 1, t)$ -resilient function, where  $x = (x_1, \dots, x_n) \in V_n$ .

It follows from Theorem 1 that if  $F = (f_1, \dots, f_m)$  is an  $(n, m, t)$ -resilient function, then  $G = (f_1, \dots, f_s)$  is an  $(n, s, t)$ -resilient function for each integer  $1 \leq s \leq m$ .

Theorem 1 shows that each  $(n, m, t)$ -resilient function gives  $2^m - 1$  distinct balanced  $t$ th-order correlation immune functions on  $V_n$ . It also indicates that we can study  $(n, m, t)$ -resilient functions, including their properties and constructions, through investigating the correlation immune characteristics of their component functions.

To facilitate our investigations, we introduce the following lemma.

*Lemma 3:* A function  $f$  on  $V_n$  is unbiased with respect to  $T = \{j_1, \dots, j_t\}$ , a fixed subset of  $\{1, \dots, n\}$ , if and only if for each linear function

$$\varphi(x) = c_{j_1} x_{j_1} \oplus \dots \oplus c_{j_t} x_{j_t}$$

on  $V_n$ , where

$$x = (x_1, \dots, x_n) f(x) \oplus \varphi(x)$$

is balanced. (Note that  $m = 1$  here.)

*Proof:* First, we consider the simplest case where  $T = \{1, \dots, t\}$ . Let  $(a_1, \dots, a_t)$  be an arbitrary but fixed vector in  $V_t$ . Then

$$\begin{aligned} & (f(x) \oplus \varphi(x))|_{x_1=a_1, \dots, x_t=a_t} \\ &= f(a_1, \dots, a_t, x_{t+1}, \dots, x_n) \\ & \quad \oplus \varphi(a_1, \dots, a_t, x_{t+1}, \dots, x_n). \end{aligned}$$

Now suppose that  $f$  is unbiased with respect to  $T = \{1, \dots, t\}$ . Then

$$f(a_1, \dots, a_t, x_{t+1}, \dots, x_n)$$

is balanced. Note that

$$\varphi(a_1, \dots, a_t, x_{t+1}, \dots, x_n)$$

is a constant. Thus

$$(f(x) \oplus \varphi(x))|_{x_1=a_1, \dots, x_t=a_t}$$

is balanced. As  $(a_1, \dots, a_t)$  is arbitrary,  $f(x) \oplus \varphi(x)$  is a balanced function on  $V_n$ .

Conversely, suppose that  $f(x) \oplus \varphi(x)$  is balanced for an arbitrary  $\varphi(x) = c_1x_1 \oplus \dots \oplus c_tx_t$ .

Let  $\xi_{a_1 \dots a_t}$  be the sequence of

$$f(a_1, \dots, a_t, x_{t+1}, \dots, x_n).$$

By [8, Lemma 2]

$$\xi = \xi_0 \dots 0, \xi_0 \dots 1, \dots, \xi_1 \dots 1$$

is the sequence of  $f(x_1, \dots, x_n)$ .

Recall that a  $(1, -1)$  matrix  $H$  of order  $m$  is called a *Hadamard* matrix if  $HH^t = mI_m$ , where  $H^t$  is the transpose of  $H$  and  $I_m$  is the identity matrix of order  $m$  (see [15, ch. 2]). A Sylvester–Hadamard matrix of order  $2^n$ , denoted by  $H_n$ , is generated by the following recursive relation:

$$H_0 = 1, \quad H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, \quad n = 1, 2, \dots.$$

Now let  $L$  be the sequence of  $\varphi$ . Then  $L$  is a row of  $H_n$ . Since  $H_n = H_t \times H_{n-t}$ , where  $\times$  denotes the Kronecker product, we have  $L = \ell' \times \ell''$ , where  $\ell'$  is a row of  $H_t$  and  $\ell''$  is a row of  $H_{n-t}$ . Write  $\ell' = (d_0, \dots, d_{2^t-1})$ . Then  $L = (d_0\ell'', \dots, d_{2^t-1}\ell'')$  and hence

$$\begin{aligned} \langle \xi, L \rangle &= d_0 \langle \xi_0 \dots 0, \ell'' \rangle + d_1 \langle \xi_0 \dots 01, \ell'' \rangle \\ & \quad + \dots + d_{2^t-1} \langle \xi_1 \dots 11, \ell'' \rangle. \end{aligned} \quad (1)$$

Since  $f(x) \oplus \varphi(x)$  is balanced,  $\langle \xi, L \rangle = 0$ . Note that  $\ell' = (d_0, \dots, d_{2^t-1})$ , a row or column of  $H_t$ , is also the sequence of

$$\varphi'(x') = c_1x_1 \oplus \dots \oplus c_tx_t.$$

A fact with  $H_t$  is that the rows (columns) of  $H_t$  comprise all the linear sequences (see [8, Lemma 1]). Then from (1)

$$\begin{aligned} & (\langle \xi_0 \dots 0, \ell'' \rangle, \langle \xi_0 \dots 01, \ell'' \rangle, \dots, \langle \xi_1 \dots 11, \ell'' \rangle) H_t \\ &= (0, 0, \dots, 0). \end{aligned}$$

As  $H_t$  has an inverse, we have

$$\langle \xi_0 \dots 0, \ell'' \rangle = \langle \xi_0 \dots 01, \ell'' \rangle = \dots = \langle \xi_1 \dots 11, \ell'' \rangle = 0.$$

Rewrite  $\varphi(x) = \varphi'(x') \oplus \varphi''(x'')$ , where  $x' \in V_t$  and  $x'' \in V_{n-t}$ . Now  $\ell'$  is the sequence of  $\varphi'$  whereas  $\ell''$  is the sequence of  $\varphi''$ . Note that

$$\varphi'(x') = c_1x_1 \oplus \dots \oplus c_tx_t.$$

Thus  $\varphi'' = 0$  and  $\ell'' = (1, \dots, 1)$ . As a result

$$\langle \xi_{a_1 \dots a_t}, \ell'' \rangle = 0$$

which implies that  $\xi_{a_1 \dots a_t}$  is balanced and hence

$$f(a_1, \dots, a_t, x_{t+1}, \dots, x_n)$$

is balanced, where  $(a_1, \dots, a_t)$  is an arbitrary vector in  $V_t$ . This shows that  $f$  is unbiased with respect to  $T = \{1, \dots, t\}$ .

For the more general case where  $T = \{j_1, \dots, j_t\}$ , set

$$f(x_1, \dots, x_n) = g(x_{j_1}, \dots, x_{j_t}, x_{j_{t+1}}, \dots, x_{j_n})$$

where  $\{j_1, \dots, j_t\} = T$  and

$$\{x_{j_t}, x_{j_{t+1}}, \dots, x_{j_n}\} = \{1, \dots, n\} - T.$$

Also set

$$x_{j_1} = y_1, \dots, x_{j_n} = y_n.$$

Thus

$$g(x_{j_1}, \dots, x_{j_t}, x_{j_{t+1}}, \dots, x_{j_n}) = g(y_1, \dots, y_t, y_{t+1}, \dots, y_n).$$

Now write

$$\psi(y) = \psi(y_1, \dots, y_n) = c_1y_1 \oplus \dots \oplus c_ty_t$$

where  $y = (y_1, \dots, y_n)$ . Obviously

$$\psi(y_1, \dots, y_n) = \varphi(x_1, \dots, x_n).$$

Hence

$$f(x) \oplus \varphi(x) = g(y) \oplus \psi(y).$$

Clearly,  $f$  is unbiased with respect to  $\{j_1, \dots, j_t\}$  if and only if  $g$  is unbiased with respect to  $\{1, \dots, t\}$ , and by the above discussions, if and only if  $g(y) \oplus \psi(y) = f(x) \oplus \varphi(x)$  is balanced.  $\square$

A corollary of Lemma 3 is

*Corollary 1:*  $f$  is an  $(n, 1, t)$ -resilient function if and only if for each linear function

$$\varphi(x) = c_1x_1 \oplus \dots \oplus c_nx_n$$

with  $W(c_1, \dots, c_n) \leq t$ ,  $f(x) \oplus \varphi(x)$  is balanced.

From this corollary and Theorem 1, it follows

*Corollary 2:*  $F$  is an  $(n, m, t)$ -resilient function if and only if it is an  $(n, m, s)$ -resilient function for each  $0 \leq s \leq t$ .

Note that Corollary 2 also follows immediately from the orthogonal array characterization in [16]. Now we go back to correlation-immune functions. Work by Xiao and Massey provides us with an equivalent definition of the concept [17].

*Definition 2:* A function  $f$  on  $V_n$  is said to be  $t$ th-order correlation-immune if for each linear function

$$\varphi(x) = c_1x_1 \oplus \cdots \oplus c_nx_n$$

with  $1 \leq W(c_1, \dots, c_n) \leq t$ ,  $f(x) \oplus \varphi(x)$  is balanced.

As  $W(c_1, \dots, c_n) = 0$  is excluded, the definition covers both balanced and nonbalanced correlation-immune functions, although stream ciphers prefer balanced to nonbalanced functions.

Comparing the definition with Corollary 1, it becomes clear that a balanced  $t$ th-order correlation-immune function is indeed identical to an  $(n, 1, t)$ -resilient function.

Having presented essential facts on resilient functions, next we consider transformations on the coordinates of a resilient function. Unlike nonlinearity and algebraic degree, the resiliency of functions is not invariant under a nonsingular linear transformation on the coordinates. This can be seen from the following example.

Let

$$f(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_n$$

where  $x = (x_1, \dots, x_n)$ . Then  $f$  is an  $(n, 1, n - 1)$ -resilient function. Now let  $B$  be a matrix of order  $n$  over  $\text{GF}(2)$  satisfying

$$(x_1, x_2, \dots, x_{n-1}, x_n)B = \left( x_2, x_3, \dots, x_{n-1}, \bigoplus_{j=1}^n x_j \right).$$

Set  $g(x) = f(xB^{-1})$ . Then  $g(x) = x_n$  whose resiliency is zero.

Another issue is in relation to the transformation of the component functions, namely output, of a resilient function. This will be discussed in detail in Section IV, where we show an important result regarding invariant properties of resilient functions under transformations of (output) component functions.

### B. Related Work

The concept of a resilient function was introduced in [1] and [2]. The equivalence between linear resilient functions and linear error-correcting codes was also established in [1] and [2], while the equivalence between resilient functions and large sets of orthogonal arrays was proved in [16]. Two upper bounds on resiliency which are the best known so far were derived in [12] and [18]. In [5], Stinson and Massey disproved the conjecture that if there exists a nonlinear resilient function then there exists a linear resilient function with the same parameters. The nonlinear resilient functions they constructed were based on the (nonlinear) Kerdock and Preparata codes [15]. Some linear resilient functions achieving an upper bound on resiliency can be found in [12] and [18]. Resilient functions which are symmetric were studied in [1] and [19], while nonbinary resilient functions were examined in [20].

Soon after the concept of a correlation-immune function was introduced by Siegenthaler [3], Xiao and Massey gave an equivalent definition in [17]. These were followed in [21] and [22], where various methods for constructing correlation-immune functions were presented.

### III. CONSTRUCTING NEW RESILIENT FUNCTIONS FROM OLD

Constructing new resilient functions from old ones is an interesting problem that has many practical applications. There are two opposite directions in relation to this problem, these being constructing “large” functions from “small” ones and “small” functions from “large” ones. Due to a close relationship between resilient functions and error-correcting codes (in particular, the equivalence between linear codes and linear resilient functions as was revealed in [1] and [2]), numerous techniques can be borrowed from the theory of error-correcting codes

to construct new resilient functions from old. These techniques have been further enriched by Stinson’s work on the equivalence between resilient functions and large sets of orthogonal arrays [16]. Some concrete examples on constructing new functions from old can be found in [12].

The main purpose of this section is to present a number of methods for directly synthesizing large resilient functions from small ones. A distinctive feature of these methods is that they are applicable both to linear and nonlinear resilient functions.

We start with (balanced) correlation immune functions. Let  $f_i$  be an  $(n_i, 1, t_i)$ -resilient function,  $i = 1, 2$ . Then  $f_1(x) \oplus f_2(y)$  is an  $(n_1 + n_2, 1, t_1 + t_2 + 1)$ -resilient function, where  $x \in V_{n_1}$  and  $y \in V_{n_2}$ . To show that this is correct, let  $\varphi$  be a linear function on  $V_{n_1+n_2}$  defined by

$$\varphi(x, y) = c_1x_1 \oplus \cdots \oplus c_{n_1}x_{n_1} \oplus d_1y_1 \oplus \cdots \oplus d_{n_2}y_{n_2}$$

where  $x = (x_1, \dots, x_{n_1})$ ,  $y = (y_1, \dots, y_{n_2})$ ,  $c_j, d_i \in \text{GF}(2)$ . Suppose that

$$W(c_1, \dots, c_{n_1}, d_1, \dots, d_{n_2}) \leq t_1 + t_2 + 1.$$

Then either  $W(c_1, \dots, c_{n_1}) \leq t_1$  or  $W(d_1, \dots, d_{n_2}) \leq t_2$ . By Corollary 1, either  $f_1(x) \oplus \varphi_1(x)$  or  $f_2(y) \oplus \varphi_2(y)$  is balanced, where

$$\varphi_1(x) = c_1x_1 \oplus \cdots \oplus c_{n_1}x_{n_1}$$

and

$$\varphi_2(y) = d_1y_1 \oplus \cdots \oplus d_{n_2}y_{n_2}.$$

Note that the sum of two functions with disjoint variables is balanced if one of the two functions is balanced. Hence

$$f_1(x) \oplus f_2(y) \oplus \varphi(x, y) = [f_1(x) \oplus \varphi_1(x)] \oplus [f_2(y) \oplus \varphi_2(y)]$$

is balanced. Again by Corollary 1,  $f_1(x) \oplus f_2(y)$  is an  $(n_1 + n_2, 1, t_1 + t_2 + 1)$ -resilient function.

By induction, we have the following result.

*Lemma 4:* Let  $f_i$  be an  $(n_i, 1, t_i)$ -resilient function,  $i = 1, \dots, s$ . Then  $f_1(x) \oplus \cdots \oplus f_s(y)$  is a  $(\sum_{j=1}^s n_j, 1, s - 1 + \sum_{j=1}^s t_j)$ -resilient function, where  $x \in V_{n_1}, \dots, y \in V_{n_s}$ .

As an application of Lemma 4, we can combine known resilient functions to obtain a new one. First we show that if  $F = (f_1, \dots, f_m)$  is an  $(n, m, t)$ -resilient function, then

$$G(x, y, z) = (F(x) \oplus F(y), F(y) \oplus F(z))$$

is an  $(3n, 2m, 2t + 1)$ -resilient function, where  $x, y, z \in V_n$ .

To prove that  $G$  is a  $(3n, 2m, 2t + 1)$ -resilient function, we first note that

$$\begin{aligned} &f_1(x) \oplus f_1(y), \dots, f_m(x) \oplus f_m(y) \\ &f_1(y) \oplus f_1(z), \dots, f_m(y) \oplus f_m(z) \end{aligned}$$

comprise all the  $2m$  component functions of  $G$ . Consider a nonzero linear combination of these  $2m$  component functions

$$f(x, y, z) = \bigoplus_{j=1}^m c_j(f_j(x) \oplus f_j(y)) \oplus \bigoplus_{j=1}^m d_j(f_j(y) \oplus f_j(z))$$

where either  $(c_1, \dots, c_m) \neq (0, \dots, 0)$  or  $(d_1, \dots, d_m) \neq (0, \dots, 0)$ .

Note that

$$f(x, y, z) = \bigoplus_{j=1}^m c_j f_j(x) \oplus \bigoplus_{j=1}^m (c_j \oplus d_j) f_j(y) \oplus \bigoplus_{j=1}^m d_j f_j(z).$$

By Theorem 1,  $\bigoplus_{j=1}^m c_j f_j(x)$  is an  $(n, 1, t)$ -resilient function when

$$(c_1, \dots, c_m) \neq (0, \dots, 0).$$

Similarly,  $\bigoplus_{j=1}^m d_j f_j(z)$  is an  $(n, 1, t)$ -resilient function when

$$(d_1, \dots, d_m) \neq (0, \dots, 0)$$

and  $\bigoplus_{j=1}^m (c_j \oplus d_j) f_j(y)$  is an  $(n, 1, t)$ -resilient function when

$$(c_1 \oplus d_1, \dots, c_m \oplus d_m) \neq (0, \dots, 0).$$

Since either  $(c_1, \dots, c_m) \neq (0, \dots, 0)$  or  $(d_1, \dots, d_m) \neq (0, \dots, 0)$ , at least two among the following three inequalities hold:

$$\begin{aligned} (c_1, \dots, c_m) &\neq (0, \dots, 0) \\ (d_1, \dots, d_m) &\neq (0, \dots, 0) \end{aligned}$$

and

$$(c_1 \oplus d_1, \dots, c_m \oplus d_m) \neq (0, \dots, 0).$$

By Lemma 4, when two hold  $f(x, y, z)$  is a  $(3n, 1, 2t+1)$ -resilient function, while when all three hold it is a  $(3n, 1, 3t+2)$ -resilient function. By Theorem 1,  $G(x, y, z)$  is indeed a  $(3n, 2m, 2t+1)$ -resilient function.

It was first observed in [1] that

$$g(x_1, \dots, x_{3h}) = (x_1 \oplus \dots \oplus x_{2h}, x_{h+1} \oplus \dots \oplus x_{3h})$$

is a linear  $(3h, 2, 2h-1)$ -resilient function. We can view this function as being obtained from

$$f(x_1, \dots, x_h) = x_1 \oplus \dots \oplus x_h$$

which is an  $(h, 1, h-1)$ -resilient function, by using the technique described above. Conversely, we can also regard our technique as a significant generalization of the idea underlying the construction of

$$g(x_1, \dots, x_{3h}) = (x_1 \oplus \dots \oplus x_{2h}, x_{h+1} \oplus \dots \oplus x_{3h}).$$

Now applying the same technique to the resulting  $(3n, 2m, 2t+1)$ -resilient function  $G$  itself, we obtain a  $(3^2n, 2^2m, 2^2(1+t)-1)$ -resilient function. In general, repeating the technique for  $k$  times,  $k = 1, 2, \dots$ , we obtain a  $(3^k n, 2^k m, 2^k(1+t)-1)$ -resilient function from an  $(n, m, t)$ -resilient function.

The technique can also be generalized in other directions. In particular, it is easy to prove that if  $F = (f_1, \dots, f_m)$  is an  $(n, m, t)$ -resilient function, then

$$G(x, y, z, u) = (F(x) \oplus F(y), F(y) \oplus F(z), F(z) \oplus F(u))$$

is a  $(4n, 3m, 2t+1)$ -resilient function, where  $x, y, z, u \in V_n$ . Again by iterating the technique, we can construct from an  $(n, m, t)$ -resilient function a  $(4^k n, 3^k m, 2^k(1+t)-1)$ -resilient function for all  $k = 1, 2, \dots$ .

To summarize the discussions, we have

**Lemma 5:** Given an  $(n, m, t)$ -resilient function, there is an iterative method to construct an  $((h+1)^k n, h^k m, 2^k(1+t)-1)$ -resilient function for all  $h = 2, 3, \dots$  and  $k = 1, 2, \dots$ .

As another application of Lemma 4, we give the following result.

**Corollary 3:** Let  $F = (f_1, \dots, f_m)$  be a  $(n_1, m, t_1)$ -resilient function and  $G = (g_1, \dots, g_m)$  a  $(n_2, m, t_2)$ -resilient function. Then

$$P(z) = F(x) \oplus G(y) = (f_1(x) \oplus g_1(y), \dots, f_m(x) \oplus g_m(y))$$

is an  $(n_1+n_2, m, t_1+t_2+1)$ -resilient function, where  $z = (x, y)$ ,  $x \in V_{n_1}$ , and  $y \in V_{n_2}$ .

*Proof:* Consider an arbitrary nonzero linear combination of the component functions of  $P(z)$ , say

$$\begin{aligned} p(z) &= \bigoplus_{j=1}^m c_j [f_j(x) \oplus g_j(y)] \\ &= \bigoplus_{j=1}^m c_j f_j(x) \oplus \bigoplus_{j=1}^m c_j g_j(y). \end{aligned}$$

By Theorem 1,  $\bigoplus_{j=1}^m c_j f_j(x)$  is a  $t_1$ -resilient function, while  $\bigoplus_{j=1}^m c_j g_j(y)$  is a  $t_2$ -resilient function. Hence, by Lemma 4,  $p(z)$  is a  $t_1+t_2+1$ -resilient function. As  $p(z)$  is arbitrary, again by Theorem 1,  $P(z)$  is an  $(n_1+n_2, m, t_1+t_2+1)$ -resilient function.  $\square$

A special case of the technique indicated in Corollary 3, namely, when both  $F$  and  $G$  are linear, has been employed by Bierbrauer, Gopalakrishnan, and Stinson in proving [12, Theorem 7].

The following result is concerned with placing resilient functions in parallel.

**Corollary 4:** Let  $F = (f_1, \dots, f_{m_1})$  be a  $(n_1, m_1, t_1)$ -resilient function and  $G = (g_1, \dots, g_{m_2})$  be a  $(n_2, m_2, t_2)$ -resilient function. Then

$$P(z) = (f_1(x), \dots, f_{m_1}(x), g_1(y), \dots, g_{m_2}(y))$$

is an  $(n_1+n_2, m_1+m_2, \rho)$ -resilient function, where  $z = (x, y)$ ,  $x \in V_{n_1}$ ,  $y \in V_{n_2}$ , and  $\rho = \min\{t_1, t_2\}$ .

*Proof:* Consider an arbitrary nonzero linear combination of the component functions of  $P(z)$

$$p(z) = \bigoplus_{j=1}^{m_1} c_j f_j(x) \oplus \bigoplus_{j=1}^{m_2} d_j g_j(y).$$

As  $(c_1, \dots, c_{m_1}, d_1, \dots, d_{m_2})$  is a nonzero vector, without loss of generality, we can assume that  $(c_1, \dots, c_{m_1}) \neq (0, \dots, 0)$ . Now consider

$$\bigoplus_{j=1}^{m_1} c_j f_j(x) |_{x_{j_1}=a_1, \dots, x_{j_{\lambda_1}}=a_{\lambda_1}}$$

for an arbitrary  $\lambda_1$ -subset

$$\{j_1, \dots, j_{\lambda_1}\} \subseteq \{1, \dots, m_1\}$$

and an arbitrary  $\lambda_2$ -subset

$$\{i_1, \dots, i_{\lambda_2}\} \subseteq \{1, \dots, m_2\}$$

where  $\lambda_1+\lambda_2=\rho$ , and arbitrary  $a_1, \dots, a_{\lambda_1}, b_1, \dots, b_{\lambda_2} \in \text{GF}(2)$ . By Theorem 1, and the fact that the sum of two functions with disjoint variables is balanced if one of the two functions is balanced

$$\bigoplus_{j=1}^{m_1} c_j f_j(x) |_{x_{j_1}=a_1, \dots, x_{j_{\lambda_1}}=a_{\lambda_1}}$$

is balanced. Thus

$$\begin{aligned} \bigoplus_{j=1}^{m_1} c_j f_j(x) |_{x_{j_1}=a_1, \dots, x_{j_{\lambda_1}}=a_{\lambda_1}} &= a_{\lambda_1} \oplus \bigoplus_{j=1}^{m_2} d_j g_j(y) |_{y_{i_1}} \\ &= b_1, \dots, b_{i_{\lambda_2}} = b_{\lambda_2} \end{aligned}$$

is balanced. It follows from Theorem 1 that

$$P(z) = (f_1(x), \dots, f_{m_1}(x), g_1(y), \dots, g_{m_2}(y))$$

is an  $(n_1+n_2, m_1+m_2, \rho)$ -resilient function.  $\square$

IV. TRANSFORMING LINEAR RESILIENT FUNCTIONS TO NONLINEAR ONES

Recall that a resilient function is said to be *linear* if its component functions are all linear, and said to be *nonlinear* otherwise. When the concept of resilient functions was introduced, it was conjectured that if there exists a *nonlinear* resilient function with certain parameters, then there exists a *linear* resilient function with the same parameters [1], [2]. This conjecture was disproved by Stinson and Massey [5]. In particular, they showed that there exists an infinite class of nonlinear resilient functions for which there do not exist linear resilient functions with the same parameters. They used nonlinear error-correcting codes in their proof. In this section, we investigate this topic from a different point of view. In particular, we show that by permuting the output  $m$ -tuples (i.e., all  $2^m$  vectors in  $V_m$ ), instead of only reordering the  $m$  component functions of an  $(n, m, t)$ -resilient function, we can obtain  $2^m!$  distinct  $(n, m, t)$ -resilient functions. A consequence of this result is that the *converse* of the conjecture in [1] and [2] is true, namely, if there exists a *linear* resilient function with certain parameters, then there exists a *nonlinear* resilient function with the same parameters.

Here is the main result in this section.

*Theorem 2:* Let  $F$  be an  $(n, m, t)$ -resilient function and  $G$  be a permutation on  $V_m$ . Then  $P = G \circ F$ , namely,  $P(x) = G(F(x))$ , is also an  $(n, m, t)$ -resilient function.

*Proof:* Since  $F$  is an  $(n, m, t)$ -resilient function, for each  $\{j_1, \dots, j_t\} \subseteq \{1, \dots, n\}$  and  $a_1, \dots, a_t \in \text{GF}(2)$ ,

$$F(x)|_{x_{j_1}=a_1, \dots, x_{j_t}=a_t}$$

runs through all the vectors in  $V_m$  each  $2^{n-m-t}$  times while  $(x_{i_1}, \dots, x_{i_{n-t}})$  runs through  $V_n$  once, where

$$\{i_1, \dots, i_{n-t}\} = \{1, \dots, n\} - \{j_1, \dots, j_t\}$$

and  $i_1 < \dots < i_{n-t}$ . As  $G$  is a permutation on  $V_m$

$$P(x)|_{x_{j_1}=a_1, \dots, x_{j_t}=a_t} = G(F(x))|_{x_{j_1}=a_1, \dots, x_{j_t}=a_t}$$

runs through all the vectors in  $V_m$  each  $2^{n-m-t}$  times while  $(x_{i_1}, \dots, x_{i_{n-t}})$  runs through  $V_n$  once. It immediately follows that  $P$  is an  $(n, m, t)$ -resilient function.

Note that the total number of different permutations on  $V_m$  is  $2^m!$  which is far larger than  $m!$ . The latter is the number of ways to reorder the  $m$ -component functions. New resilient functions generated using these permutations are all different. To prove this, let  $G_1$  and  $G_2$  be two different permutations on  $V_m$ . We want to prove that  $G_1 \circ F \neq G_2 \circ F$ . Suppose for contradiction that  $G_1 \circ F = G_2 \circ F$ . Then  $F = G_1^{-1} \circ G_2 \circ F$ . As  $F$  is unbiased, for each  $\beta \in V_m$ , there exist  $2^{n-m}$  different vectors  $\alpha \in V_n$  such that  $F(\alpha) = \beta$ . This causes  $\beta = G_1^{-1} \circ G_2(\beta)$ . As  $\beta$  is arbitrary,  $G_1^{-1} \circ G_2$  must be the identity permutation on  $V_m$ , which contradicts the fact that  $G_1 \neq G_2$ . Thus we have proved the following:

*Corollary 5:* Given an  $(n, m, t)$ -resilient function, Theorem 2 produces  $2^m!$  distinct  $(n, m, t)$ -resilient function.

Now we describe an example to show applications of Theorem 2. It is easy to verify that

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 \oplus x_2 \oplus x_3, x_3 \oplus x_4 \oplus x_5, x_5 \oplus x_6 \oplus x_1)$$

is a linear  $(6, 3, 2)$ -resilient function. Consider a permutation  $G$  on  $V_3$  defined by

$$G(u_1, u_2, u_3) = (u_1 \oplus u_3 \oplus u_2 u_3, u_1 \oplus u_2 \oplus u_1 u_3, u_2 \oplus u_3 \oplus u_1 u_2).$$

Then  $P = G \circ F$  takes the form of

$$P(x) = G(F(x)) = (p_1(x), p_2(x), p_3(x))$$

where  $x = (x_1, \dots, x_6)$  and

$$\begin{aligned} p_1(x) &= x_2 \oplus x_3 \oplus x_6 \oplus x_1 x_3 \oplus x_1 x_4 \oplus x_1 x_5 \oplus \\ &\quad x_3 x_5 \oplus x_3 x_6 \oplus x_4 x_5 \oplus x_4 x_6 \oplus x_5 x_6 \\ p_2(x) &= x_2 \oplus x_4 \oplus x_5 \oplus x_1 x_2 \oplus x_1 x_3 \oplus x_1 x_5 \oplus \\ &\quad x_1 x_6 \oplus x_2 x_5 \oplus x_2 x_6 \oplus x_3 x_5 \oplus x_3 x_6 \\ p_3(x) &= x_1 \oplus x_4 \oplus x_6 \oplus x_1 x_3 \oplus x_1 x_4 \oplus x_1 x_5 \oplus \\ &\quad x_2 x_3 \oplus x_2 x_4 \oplus x_2 x_5 \oplus x_3 x_4 \oplus x_3 x_5. \end{aligned}$$

By Theorem 2,  $P = G \circ F$  is also a  $(6, 3, 2)$ -resilient function.

Note that all component functions of the resulting resilient function  $P$  are quadratic. The rest of this section is devoted to this direction, namely, converting linear resilient functions to nonlinear ones. We also show how to calculate the nonlinearity of a resulting nonlinear resilient function. The following lemma will be used in the discussions.

*Lemma 6:* Let  $g$  be a function on  $V_m$  whose nonlinearity is  $N_g$ . Let  $n \geq m$  and  $B$  be an  $n \times m$  matrix over  $\text{GF}(2)$  whose rank is  $m$ . Set

$$h(x_1, \dots, x_n) = g((x_1, \dots, x_m)B).$$

Then the nonlinearity  $N_h$  of  $h$ , a function on  $V_n$ , satisfies  $N_h = 2^{n-m} N_g$ , and the algebraic degree of  $h$  is the same as that of  $g$ .

*Proof:* First we note that this lemma is a generalization of the following result: Let  $h(x_1, \dots, x_n) = g(x_1, \dots, x_k)$ . Then  $h$ , a function on  $V_n$ , satisfies  $N_h = 2^{n-m} N_g$ . A proof for this special case can be found in, for instance, [22].

To prove this lemma, we append to  $B$  an  $n \times (n - m)$  matrix  $C$  so that  $A = [B, C]$  is a nonsingular matrix of order  $n$  over  $\text{GF}(2)$ . Set  $(u_1, \dots, u_n) = (x_1, \dots, x_n)A$ . Now define a function on  $V_n$ , say  $g^*$ , as follows:

$$g^*(u_1, \dots, u_n) = g(u_1, \dots, u_m).$$

Then  $N_{g^*} = 2^{n-m} N_g$ , and  $g^*$  and  $g$  share the same algebraic degree. On the other hand, from the construction of  $h$

$$\begin{aligned} h(x_1, \dots, x_n) &= g((x_1, \dots, x_n)B) \\ &= g^*((x_1, \dots, x_n)A). \end{aligned}$$

By noting the fact that the nonlinearity and algebraic degree of a function are invariant under a nonsingular linear transformation on coordinates, we have  $N_h = N_{g^*} = 2^{n-m} N_g$ , and that  $h$  has the same algebraic degree as that of  $g^*$ , which is the same as that of  $g$ .

Now we prove a significant result on constructing new resilient functions from old, linear ones.

*Theorem 3:* Let  $F$  be a linear  $(n, m, t)$ -resilient function and  $G$  be a permutation on  $V_m$  whose nonlinearity is  $N_G$ . Then  $P = G \circ F$  is an  $(n, m, t)$ -resilient function and

- 1) the nonlinearity  $N_P$  of  $P$  satisfies  $N_P = 2^{n-m} N_G$ ,
- 2) the algebraic degree of  $P$  is the same as that of  $G$ .

*Proof:* As  $F$  is a linear resilient function, it can be written as

$$F(x_1, \dots, x_n) = (x_1, \dots, x_n)B$$

where  $B$  is an  $n \times m$  matrix of rank  $m$  over  $\text{GF}(2)$  and  $(x_1, \dots, x_n) \in V_n$ . The theorem follows immediately from Lemma 6.

We turn our attention back to the nonlinear  $(6, 3, 2)$ -resilient function constructed above. It is easy to verify that the nonlinearity of each nonzero linear combination of the component functions of

$G$  is 2. By Theorem 3, the nonlinearity of  $P$  is 16, and as we have seen, the algebraic degree of  $P$  is indeed 2.

Theorem 3 implies that highly nonlinear resilient functions can be constructed from linear resilient functions by applying highly nonlinear permutations in the transforming process. A number of highly nonlinear permutations which are based on polynomials on a finite field have been shown in [23] and [24]. In particular, it is shown in [23] that the nonlinearity of a permutation  $G$  based on the inverse function on  $\text{GF}(2^m)$  satisfies  $N_G \geq 2^{m-1} - 2^{(1/2)^m}$  and the algebraic degree of  $G$  is  $m - 1$ . Hence, the following is proved:

*Corollary 6:* If there exists a linear  $(n, m, t)$ -resilient function, then there exists a nonlinear  $(n, m, t)$ -resilient function  $P$  whose nonlinearity satisfies  $N_P \geq 2^{n-1} - 2^{n-(1/2)^m}$  and whose algebraic degree is  $m - 1$ .

Another important implication of Theorem 3 is that from each linear resilient function, we can derive a large number of nonlinear resilient functions with the same parameters. This, together with the result by Stinson and Massey [5], shows that we have far more freedom in choosing nonlinear resilient functions than in linear resilient functions, both in terms of the numbers and the parameters.

## V. REMARKS ON ALGEBRAIC DEGREE

In his pioneering work [3], Siegenthaler showed, by a lengthy argument, that the algebraic degree of a balanced correlation-immune function, i.e., an  $(n, 1, t)$ -resilient function, is at most  $n - t - 1$ , except for the case when  $t = n - 1$ . Here we show that the proof can be substantially shortened by employing [15, p. 372, Theorem 1]. A short proof for the same result was also given in [16], in a different approach.

Let  $f$  be an  $(n, 1, t)$ -resilient function. As  $f$  is a function on  $V_n$ , by [15, p. 372, Theorem 1], it can be expressed in the algebraic normal form, namely,

$$f(x_1, \dots, x_n) = \bigoplus_{a_1, \dots, a_n \in \text{GF}(2)} g(a_1, \dots, a_n) x_1^{a_1} \dots x_n^{a_n}$$

where

$$g(a_1, \dots, a_n) = \bigoplus_{(b_1, \dots, b_n) \subset (a_1, \dots, a_n)} f(b_1, \dots, b_n)$$

and by  $(b_1, \dots, b_n) \subset (a_1, \dots, a_n)$  we mean that if  $b_j = 1$  then  $a_j = 1$ .

Consider the coefficient of the term  $x_1 \dots x_{n-t}$ , that is,

$$\bigoplus_{b_1, \dots, b_{n-t} \in \text{GF}(2)} f(b_1, \dots, b_{n-t}, 0, \dots, 0). \quad (2)$$

Since  $f$  is an  $(n, 1, t)$ -resilient function, (2) becomes zero, except for  $n - t = 1$  in which case (2) becomes one. By the same reasoning, we can see that the coefficient of every term of algebraic degree  $n - t$  is zero. This proves that the algebraic degree of  $f$  is at most  $n - t - 1$ .

By noting our Theorem 1, we have

*Corollary 7:* The algebraic degree of an  $(n, m, t)$ -resilient function is at most  $n - t - 1$ , except for the case when  $t = n - 1$ .

Recall that it is easy to construct linear  $(n, n - 1, 1)$ -resilient functions from linear error-correcting codes. Using Corollaries 5 and 6, we obtain  $2^{n-1}!$  distinct  $(n, n - 1, 1)$ -resilient functions, a large number of which have a nonlinearity of at least  $2^{n-1} - 2^{(n+1/2)}$  and whose algebraic degree is  $n - 2$ .

It should be noted, however, that due to Corollary 7, applying Theorem 3 to a nonlinear  $(n, n - 1, 1)$ -resilient function does not always yield a function that has a higher algebraic degree.

In [18], Friedman proved that the resiliency  $t$  of an  $(n, m, t)$ -resilient function is bounded from above by

$$B_1 = \left\lfloor \frac{2^{m-1}n}{2^m - 1} \right\rfloor - 1. \quad (3)$$

Bierbrauer *et al.* [12, Theorem 3] give another upper bound

$$B_2 = 2 \left\lfloor \frac{2^{m-2}(n+1)}{2^m - 1} \right\rfloor - 1. \quad (4)$$

As shown in [12], a linear  $(2^m - 1, m, 2^{m-1} - 1)$ -resilient function can be obtained from a simplex code. This function achieves the upper bound on resiliency (3). Applying Corollaries 5 and 6 to this resilient function, we obtain  $2^m!$  distinct  $(2^m - 1, m, 2^{m-1} - 1)$  resilient functions, some of which have a nonlinearity of at least  $2^{2^m-2} - 2^{2^{m-1}-(1/2)^m}$ , and whose algebraic degree is  $m - 1$ . All the resulting functions achieve the upper bound on resiliency indicated in (3).

## VI. CONCLUSION

The main results of this correspondence are related to the construction of nonlinear resilient functions. Of particular importance to practical applications is the method for transforming linear resilient functions into nonlinear ones.

## ACKNOWLEDGMENT

The authors wish to thank D. Stinson for kindly providing them with his papers. Thanks are also extended to A. Mich and the anonymous referees whose comments have improved the presentation.

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