MODEL EQUATIONS FOR WAVE PROPAGATION FROM DEEP TO SHALLOW WATER

by

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Model Equations for Wave Propagations From Deep to Shallow Water

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This paper reviews two different approaches for deriving shallow water equations. The Hamiltonian approach is first used to obtain the Boussinesq equations in terms of the horizontal velocity on the free surface. A direct perturbation method is introduced to derive a general nonlinear shallow water equation. Various forms of Boussinesq equations are discussed. Some of these Boussinesq equations are shown to be unstable subject to short wave disturbances. The dispersion characteristics, both linear and nonlinear, of the shallow water equations are compared with those of the Stokes' wave theory. As far as the linear dispersion is concerned, an optimal form of the Boussinesq equations is identified, which is applicable even in deep water. However, if the nonlinearity is important in deep water, none of the equations discussed in this paper can provide an adequate description of nonlinear dispersion in deep water.

1. Introduction

To design a coastal structure in nearshore region, engineers must have a means to estimate wave climate. Numerical models are often used to calculate wave propagation from an offshore location, where wave data are available, to the nearshore area of concern. Waves, approaching the surf zone from offshore, experience changes caused by combined effects of bathymetric variations, interference of man-made structures, and nonlinear interactions among wave trains. Inside the surf zone, where wave breaking is a dominating feature, waves undergo a much more rapid transformation. Early efforts to model this wave evolution process were based primarily on the geometrical ray theory, which ignores both diffraction and nonlinearity.

Significant research accomplishments were made in the '70s and '80s to overcome the shortcoming imbedded in the ray theory. To include the wave diffraction, mild-slope equations are derived from the linear wave theory by assuming that evanescent modes can be ignored and the bathymetric changes are small within a typical wavelength. Furthermore, the vertical structure of the velocity field is assumed to be the same as that for a progressive wave over a constant depth. Therefore, mild-slope equations are two-dimensional (in the horizontal space) partial differential equations of the elliptic type, which require boundary conditions along the entire boundaries of the computational domain. Numerical solutions have demonstrated that mild-slope equations give adequate description of combined refraction and diffraction for small amplitude waves. The general mild-slope equation for nonlinear waves is still not available.

In applying mild-slope equations to a large coastal region, one encounters the difficulty in defining the location of breaker line *a prioi*. One of the most important developments in wave modeling during the last decade was the application of the parabolic approximation to mild-slope equations. The parabolic approximation can be viewed as a modification of the ray theory. While waves propagate along "rays", wave energy is allowed to "diffuse" across the "rays". Therefore, the effects of diffraction are included approximately in the parabolic approximation. Although the parabolic approximation models have been used primarily for forward propagation, weakly backward propagation modes can be included by an iterative procedure (Liu and Tsay 1983, Chen and Liu 1993). Nonlinearity can also be included in the forward propagation mode. More detailed discussions on mild-slope equations and the associated parabolic approximations can be found in Liu (1990) and Mei and Liu (1993).

As waves approach the surf zone, wave amplitudes become large and the Stokes' wave theory is no longer valid. A more relevant approach is based on Boussinesq equations for weakly nonlinear and weakly dispersive waves. Peregrine (1967, 1972) provided several versions of the Boussinesq equations, written in terms of either the depth-averaged velocity or the velocity along the bottom or the velocity on the free surface. Because the dispersive terms in the Boussinesq equations are of higher order, they can be further manipulated by replacing the time derivative or the spatial derivatives by the lower order relations (e.g., see Mei 1989). Although all these different forms of Boussinesq equations have the same order of magnitude of accuracy, their dispersion relations and the associated phase velocity are different.

The major restriction of the Boussinesq equations is their depth limitation. The best forms of the Boussinesq equations, using the depth-averaged velocity, break down when the depth is larger than one-fifth of the equivalent deep water wavelength, corresponding to a five percent phase velocity error (McCowan 1987). For many engineering applications, a less restrictive depth limitation is desirable. Moreover, when the Boussinesq equations are solved numerically, high frequency numerical disturbances related to the grid size could cause instability. The search for a new set of two-dimensional governing equations which can describe the wave propagation from a deeper depth to a shallow depth is currently an active area of research (Witting 1984, McCowan and Blackman 1989, Murray 1989, Madsen, Murray and Sorensen 1991, Nwogu 1993).

In this paper, we first review the derivation of Boussinesq equations using the Hamiltonian approach and the direct perturbation procedure. In the direct perturbation approach, shallow water equations for highly nonlinear waves are also derived. The Boussinesq equations become a special case. The dispersive characteristics of nonlinear shallow water equations and Boussinesq equations are then compared with those of Stokes' waves. The possibility of extending the range of applicability of nonlinear shallow water equations into a deeper water is discussed. Different methods for improving the dispersive characteristics are also discussed.

2. Governing Equation and Boundary Conditions

In this section, the governing equation and boundary conditions for water waves propagating in a varying depth are summarized. Denoting $\mathbf{x}' = (x', y')$ as the horizontal coordinates and z' the vertical coordinate, we define the flow domain as a layer of water bounded by a free surface $z' = \eta'(\mathbf{x}', t)$ and a solid bottom $z' = -h'(\mathbf{x}')$. Using the characteristic wavelength, $(k')^{-1}$ as the horizontal length scale, the characteristic depth, h'_o, as the vertical length scale, and $(k'\sqrt{gh'_o})^{-1}$ as the time scale, we introduce the following dimensionless variables:

$$\mathbf{x} = k'\mathbf{x}', \quad z = z'/h'_o,$$

$$h = h'/h'_o, \quad t = k'\sqrt{gh'_o}\,t' \tag{2.1}$$

Assuming that the flow field is irrotational, we represent the velocity field by the gradient of a velocity potential, Φ'. Denoting a'_o as the characteristic wave amplitude, we normalize the free surface displacement η' and the associated potential function and obtain the following dimensionless variables:

$$\eta = \eta'/a'_o\,, \quad \Phi = k'\frac{\sqrt{gh'_o}}{ga'_o}\,\Phi' \tag{2.2}$$

The dimensionless continuity equation becomes

$$\mu^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0\,, \quad -h < z < \varepsilon\eta \tag{2.3}$$

where

$$\varepsilon = a'_o/h'_o\,, \quad \mu^2 = (k'\,h'_o)^2 \tag{2.4}$$

are parameters representing nonlinearity and frequency dispersion, respectively. The no-flux boundary condition along the bottom requires

$$\mu^2\,\nabla h \cdot \nabla\Phi + \frac{\partial\Phi}{\partial z} = 0\,,\; z = -h \tag{2.5}$$

in which $\nabla = (\partial/\partial x, \partial/\partial y)$ denotes the gradient vector on a horizontal plane. On the free surface, both the kinematic and the dynamic boundary conditions must be satisfied. The dynamic condition specifies the continuity of pressure across the free surface. Setting the atmospheric pressure at zero, the Bernoulli equation applied on the free surface becomes

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\varepsilon\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \mu^{-2}\left(\frac{\partial\Phi}{\partial z}\right)^2\right] + \eta = 0\,,\; z = \varepsilon\eta \tag{2.6}$$

The kinematic boundary condition states that the free surface is a material surface. Thus, following the free surface movement, the rate of change of the free surface, $F = z - \eta = 0$, must vanish. The dimensionless kinematic boundary condition is expressed as

$$\frac{\partial\eta}{\partial t} + \varepsilon\,\nabla\eta \cdot \nabla\Phi = \mu^{-2}\frac{\partial\Phi}{\partial z}\,, \quad z = \varepsilon\eta \tag{2.7}$$

3. Approximate Governing Equations on the Horizontal Plane

The main objective of deriving approximate equations is to reduce the three-dimensional governing equations and boundary conditions to two-dimensional forms so that lesser computational efforts are required in modeling wave propagation in a large domain. Several different methods can be used to achieve this goal. In this section, we will discuss two of them: a Hamiltonian approach and a direct perturbation approach.

3.1 *Hamiltonian Approach*

The total energy of a flow field is the sum of kinetic and potential energy. Denoting Ω as the projection of flow domain on the horizontal plane, we can express the total energy, which is also called Hamiltonian, in the following dimensionless form:

$$\mathcal{H} = \frac{1}{2} \int\int_{\Omega} \left\{ \int_{-h}^{\varepsilon\eta} \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \mu^{-2} \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] dz + \eta^2 \right\} dxdy \tag{3.1}$$

in which the total energy has been normalized by a factor $\rho g a_o'^2 / k'^2$. Because the total energy (Hamiltonian) must be finite, we assume that η, Φ and their derivatives vanish along the lateral boundaries of the horizontal domain.

The canonical theorem states that the free surface boundary conditions, (2.6) and (2.7), are equivalent to the following canonical equations (Broer 1974, Zakharov 1968, Miles 1977):

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \phi} \ , \quad \frac{\partial \phi}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \eta} \tag{3.2}$$

in which δ denotes a variational derivative and $\phi(\mathbf{x}, t)$ represents the potential evaluated on the free surface

$$\phi(\mathbf{x}, t) = \Phi(\mathbf{x}, \varepsilon\eta(\mathbf{x}, t), t) \tag{3.3}$$

To use the canonical equations, we need to know the relation between Φ and ϕ. In other words, the vertical distribution of the potential function must be derived first. This task can only be achieved approximately. In the following section, an approximated Hamiltonian will be obtained by adopting the Boussinesq approximation, i.e. $0(\varepsilon) \approx 0(\mu^2) << 1$.

3.1.1 *An approximate Hamiltonian*

The Hamiltonian can be rewritten as the sum of kinetic energy, E_k, and the potential energy E_p, i.e.,

$$\mathcal{H} = E_k + E_p \tag{3.4}$$

where

$$E_p = \frac{1}{2} \int\int_{\Omega} \eta^2 \, dxdy \tag{3.5}$$

$$E_k = E_{ko} + E_{k\eta} \tag{3.6}$$

with

$$E_{ko} = \frac{1}{2} \int\int_{\Omega}\int_{-h}^{0} \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \mu^{-2} \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] dz\, dxdy \tag{3.7}$$

$$E_{k\eta} = \frac{1}{2} \int\int_{\Omega}\int_{0}^{\varepsilon\eta} \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \mu^{-2} \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] dz\, dxdy \tag{3.8}$$

Applying the Green's theorem to the volume integral on the right-hand side of (3.7) and using the continuity equation (2.3) as well as the no-flux boundary condition on the bottom, (2.5), we obtain

$$E_{ko} = \frac{1}{2} \int\int_{\Omega} \left[\mu^{-2} \left(\Phi \frac{\partial \Phi}{\partial z} \right) \right]_{z=0} dxdy \tag{3.9}$$

where the integrand is evaluated on the still water level, $z = 0$. We remark here that up to this point, no approximation has been made.

The kinetic energy above the still water level, $z = 0$, given by (3.8) can be approximated by using the Taylor's series expansion

$$E_{k\eta} = \frac{1}{2} \int \int_{\Omega} \varepsilon \eta \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \mu^{-2} \left(\frac{\partial \Phi}{\partial z} \right)^2 \right]_{z=0} dxdy + 0(\varepsilon^2) \tag{3.10}$$

The approximate Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{2} \int \int_{\Omega} \left\{ \mu^{-2} \left(\Phi \frac{\partial \Phi}{\partial z} \right)_{z=0} + \varepsilon \eta \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 \right. \right.$$

$$\left. \left. + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \mu^{-2} \left(\frac{\partial \Phi}{\partial z} \right)^2 \right]_{z=0} + \eta^2 \right\} dxdy + 0(\varepsilon^2) \tag{3.11}$$

To continue the derivation of Boussinesq-type equations, we must find the vertical structure of the potential function, $\Phi(\mathbf{x}, z, t)$, such that the approximate Hamiltonian can be evaluated at the still water surface, $z = 0$.

3.1.2 *Vertical structure of the potential function*

Many approaches can be taken to find the vertical structure of the potential function. Here we follow the procedure originally developed by Lin and Clark (1959). Expanding the potential function in a power series in terms of $(z + h)$

$$\Phi(\mathbf{x}, z, t) = \sum_{n=0}^{\infty} (z + h)^n \phi^{(n)}(\mathbf{x}, t) \tag{3.12}$$

we can obtain recursive relations among $\phi^{(n)}(\mathbf{x}, t)$ by substituting (3.12) into the continuity equation, (2.3), and the bottom boundary condition (2.5)

$$\phi^{(1)} = - \frac{\mu^2 \nabla h \cdot \nabla \phi^{(0)}}{1 + \mu^2 |\nabla h|^2} \tag{3.13}$$

$$\phi^{(n+2)} = - \frac{\mu^2 \left[\nabla^2 \phi^{(n)} + 2(n+1) \nabla h \cdot \nabla \phi^{(n+1)} + (n+1) \nabla^2 h \phi^{(n+1)} \right]}{(n+1)(n+2)\left(1 + \mu^2 |\nabla h|^2\right)} \tag{3.14}$$

where $n = 0, 1, 2, \ldots$. From the recursive relations, $\phi^{(n)}$ can be expressed in terms of $\phi^{(0)}$, which is the potential along the bottom. For instance, up to $0(\mu^4)$ the potential function can be written as

$$\Phi(\mathbf{x}, z, t) = \phi^{(0)} - \mu^2 \Big[h\nabla h \cdot \nabla\phi^{(0)} + \frac{h^2}{2}\nabla^2\phi^{(0)} + z\,\nabla \cdot (h\,\nabla\phi^{(0)}) + \frac{z^2}{2}\,\nabla^2\phi^{(0)} \Big] + 0(\mu^4) \tag{3.15}$$

For later use, we introduce $\Phi_\alpha(\mathbf{x}, t)$ as the potential at an arbitrary elevation $z = z_\alpha(\mathbf{x})$. From (3.15) Φ_α can be expressed as

$$\Phi_\alpha = \phi^{(0)} - \mu^2 \Big[h\nabla h \cdot \nabla\phi^{(0)} + \frac{h^2}{2}\nabla^2\phi^{(0)} + z_\alpha\,\nabla \cdot (h\nabla\phi^{(0)}) + \frac{z_\alpha^2}{2}\nabla^2\phi^{(0)} \Big] + 0(\mu^4) \tag{3.16}$$

Subtracting (3.16) from (3.15) and using $\Phi_\alpha = \phi^{(0)} + 0(\mu^2)$, we can write the potential function in terms of Φ_α

$$\Phi(\mathbf{x}, z, t) = \Phi_\alpha(\mathbf{x}, t) - \mu^2 \Big[(z - z_\alpha)\,\nabla \cdot (h\nabla\Phi_\alpha) + \frac{(z^2 - z_\alpha^2)}{2}\,\nabla^2\Phi_\alpha \Big] + 0(\mu^4) \tag{3.17}$$

From (3.16) we can also express $\phi^{(0)}$ in terms of Φ_α:

$$\phi^{(0)} = \Phi_\alpha + \mu^2 \Big[h\nabla h \cdot \nabla\Phi_\alpha + \frac{h^2}{2}\nabla^2\Phi_\alpha + z_\alpha\,\nabla \cdot (h\nabla\Phi_\alpha) + \frac{z_\alpha^2}{2}\nabla^2\Phi_\alpha \Big] + 0(\mu^4) \tag{3.18}$$

Along the still water surface, $z_\alpha = 0$, and $\Phi_\alpha(\mathbf{x}, 0, t) = \Phi_o$. From (3.18) we obtain

$$\phi^{(0)} = M^{-1}\,\Phi_o \tag{3.19}$$

where

$$M^{-1} = 1 + \mu^2 \left[h\nabla h \cdot \nabla + \frac{h^2}{2}\nabla^2 \right] + 0(\mu^4) \qquad (3.20)$$

To evaluate the integrand in (3.11), we need to find the expression for the gradient of Φ on $z = 0$. From (3.12) $\sim$ (3.14) and (3.19), we have

$$\frac{\partial \Phi}{\partial z}\Big|_{z=0} = \sum_{n=0}^{\infty} (n + 1)h^n \phi^{(n+1)} = SM^{-1}\Phi_o + 0(\mu^6) \qquad (3.21)$$

where

$$S = -\mu^2 \Big\{ (1 - \mu^2 |\nabla h|^2) \nabla(h \cdot \nabla)$$

$$- \mu^2 \Big[2h\nabla h \cdot \nabla (\nabla h \cdot \nabla) + h\nabla^2 h(\nabla h \cdot \nabla) + h^2 \Big(\frac{1}{2} \nabla^2(\nabla h \cdot \nabla)$$

$$+ (\nabla h \cdot \nabla) \nabla^2 + \frac{1}{2}\nabla^2 h \nabla^2 \Big) + \frac{1}{6} h^3 \nabla^2 \nabla^2 \Big] \Big\} \qquad (3.22)$$

The first term in the Hamiltonian or (3.9) can be evaluated as

$$E_{ko} = \frac{\mu^{-2}}{2} \int\int_\Omega \Phi_o SM^{-1} \Phi_o \, dxdy$$

$$= \frac{1}{2} \int\int_\Omega \Big\{ h\nabla \Phi_o \cdot \nabla \Phi_o + \frac{\mu^2 h^2}{2} \nabla \cdot [\nabla \Phi_o (\nabla h \cdot \nabla \Phi_o)]$$

$$+ \frac{\mu^2 h^3}{3} \nabla \Phi_o \cdot (\nabla^2 \nabla \Phi_o) \Big\} \, dxdy + 0(\mu^4) \qquad (3.23)$$

Finally, the approximated Hamiltonian, (3.11), can be expressed in terms of Φ_o and η in the following form:

$$\mathcal{H} = \frac{1}{2} \int\int_\Omega \Big\{ (h + \varepsilon\eta) |\nabla \Phi_o|^2 + \frac{\mu^2 h^2}{2} \nabla \cdot [\nabla \Phi_o (\nabla h \cdot \nabla \Phi_o)]$$

$$+ \frac{\mu^2 h^3}{3} \nabla\Phi_o \cdot \nabla^2 (\nabla \Phi_o) + \eta^2 \Big\} \, dxdy + 0(\mu^4, \varepsilon\mu^2, \varepsilon^2) \qquad (3.24)$$

We note that from (3.17) with $z_\alpha = 0$,

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi_o}{\partial x} + 0(\mu^2), \; \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi_o}{\partial y} + 0(\mu^2), \; \frac{\partial \Phi}{\partial z} = 0(\mu^2)$$

which have been used in deriving the Hamiltonian (3.24).

To apply the canonical theorem, we first recognize that the potential on the actual free surface ($z = \varepsilon\eta$) , ϕ, is different from that on the still water level, Φ. However, the difference is small and can be shown by using the Taylor's series expansion

$$\phi = \Phi(x, \varepsilon\eta, t) = \Phi(x, 0, t) + \frac{\partial \Phi}{\partial z}\Big|_{z=0} \varepsilon\eta + \cdots$$

Hence

$$\Phi_o = \phi + 0(\varepsilon\mu^2) \qquad (3.25)$$

Therefore, within the limit of accuracy for the Hamiltonian ϕ and Φ_o are exchangeable.

Applying the canonical theorem, (3.2), to the Hamiltonian, (3.24), yields

$$\frac{\partial \eta}{\partial t} = - \nabla \cdot [(h + \varepsilon\eta) \nabla\Phi_o]$$

$$- \frac{1}{2} \mu^2 \nabla \cdot \left[h^2 \nabla (\nabla \cdot (h \nabla\Phi_o)) - \frac{h^3}{3} \nabla (\nabla \cdot \nabla\Phi_o) \right] \qquad (3.26)$$

$$\frac{\partial \Phi_o}{\partial t} = - \frac{\varepsilon}{2} | \nabla\Phi_o|^2 - \eta \qquad (3.27)$$

Introducing the horizontal velocity vector on the still water surface as

$$\mathbf{u}_o = \nabla\Phi_o \qquad (3.28)$$

we can rewrite (3.26) in the following form

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(h + \varepsilon\eta)\mathbf{u}_o] + \frac{\mu^2}{2} \nabla \cdot \Big[h^2 \nabla(\nabla \cdot h\,\mathbf{u}_o)$$

$$- \frac{h^3}{3} \nabla(\nabla \cdot \mathbf{u}_o) \Big] = 0(\varepsilon\mu^3 , \mu^4 , \varepsilon^2) \tag{3.29}$$

which represents the continuity equation. Taking the gradient of (3.27), we obtain the momentum equation in terms of the velocity $\mathbf{u}_o$

$$\frac{\partial \mathbf{u}_o}{\partial t} + \varepsilon\, \mathbf{u}_o \cdot \nabla \mathbf{u}_o + \nabla \eta = 0(\varepsilon\mu^3 , \mu^4 , \varepsilon^2) \tag{3.30}$$

Equations (3.29) and (3.30) are the conventional Boussinesq equations expressed in terms of the horizontal velocity on the still water surface. These equations can be rewritten in terms of horizontal velocity on the bottom or the depth-averaged velocity. We will discuss these alternative forms in section 3.2.2.

3.2 *A Direct Perturbation Approach*

3.2.1 *Governing equations for finite amplitude waves*

In the Hamiltonian approach, we have to employ the Boussinesq approximation, i.e., $0(\varepsilon) \approx 0(\mu^2) << 1$. Moreover, the knowledge of the vertical structure of the potential function is essential and is obtained via a perturbation method. In this section, we present a direct perturbation approach, which allows the parameter ε, representing the nonlinearity, to be arbitrary.

To facilitate the perturbation procedure efficiently, we integrate the continuity equation (2.3) from the bottom, $z = -h$, to the free surface, $z = \varepsilon\eta$. Using the kinematic boundary conditions (2.5) and (2.7), one may obtain the depth-integrated continuity equation

$$\nabla \cdot \left[\int_{-h}^{\varepsilon\eta} \nabla\, \Phi\, dz \right] + \frac{\partial \eta}{\partial t} = 0 \tag{3.31}$$

which is an exact equation. Following the perturbation procedure given in section 3.1.2 and substituting (3.17) into the continuity equation (3.31), we obtain

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(\varepsilon\eta + h)\, \nabla \Phi_\alpha]$$

$$+ \mu^2\, \nabla \cdot \Big\{ (\varepsilon\eta + h) \Big[\nabla (z_\alpha\, \nabla \cdot (h \nabla \Phi_\alpha)) + \frac{1}{2}\, (h$$

$$- \varepsilon\eta) \nabla\, (\nabla \cdot (h\, \nabla \Phi_\alpha)) + \frac{1}{2}\, \nabla (z_\alpha^2\, \nabla^2 \Phi_\alpha)$$

$$- \frac{1}{6}\, (\varepsilon^2 \eta^2 - \varepsilon\eta h + h^2) \nabla\, \nabla^2\, \Phi_\alpha \Big] \Big\} = 0(\mu^4) \qquad (3.32)$$

We reiterate here that the parameter ε is assumed to be an arbitrary constant. Substitution of (3.17) into the dynamic free surface boundary condition, (2.6), yields

$$\frac{\partial \Phi_\alpha}{\partial t} + \eta + \frac{\varepsilon}{2}\, |\nabla \Phi_\alpha|^2 + \mu^2 \Big[(z_\alpha - \varepsilon\eta)\, \nabla \cdot\, (h \nabla\, \frac{\partial \Phi_\alpha}{\partial t})$$

$$+ \frac{1}{2}\, (z_\alpha^2 - \varepsilon^2 \eta^2)\, \nabla^2\, (\frac{\partial \Phi_\alpha}{\partial t}) \Big] - \varepsilon \mu^2\, \nabla \Phi_\alpha \cdot \Big[- \nabla z_\alpha (\nabla \cdot (h\, \nabla \Phi_\alpha))$$

$$+ (\varepsilon\eta - z_\alpha) \nabla\, (\nabla \cdot (h \nabla \Phi_\alpha)) - z_\alpha \nabla z_\alpha\, \nabla^2 \Phi_\alpha$$

$$+ \frac{1}{2} (\varepsilon^2 \eta^2 - z_\alpha^2) \nabla (\nabla^2 \Phi_\alpha) \Big] + \frac{\varepsilon \mu^2}{2}\, [(\nabla \cdot (h \nabla \Phi_\alpha))^2$$

$$+ 2\varepsilon\eta \nabla \cdot (h \nabla \Phi_\alpha)\, \nabla^2 \Phi_\alpha + \varepsilon^2 \eta^2\, (\nabla^2 \Phi_\alpha)^2] = 0(\mu^4) \qquad (3.33)$$

From (3.17) the horizontal velocity at $z = z_\alpha$ can be defined as

$$\mathbf{u}_\alpha = (\nabla \Phi) \Big|_{z = z_\alpha} = \nabla \Phi_\alpha$$

$$+ \mu^2\, [\nabla\, z_\alpha\, \nabla \cdot (h\, \nabla \Phi_\alpha) + z_\alpha\, \nabla z_\alpha\, \nabla^2 \Phi_\alpha] + 0(\mu^4) \qquad (3.34)$$

Equivalently,

$$\nabla \Phi_\alpha = \mathbf{u}_\alpha - \mu^2 [\nabla\, z_\alpha\, \nabla \cdot (h\, \nabla \mathbf{u}_\alpha)$$

$$+ z_\alpha \nabla z_\alpha \nabla \cdot \mathbf{u}_\alpha] + 0(\mu^4) \tag{3.35}$$

Replacing $\nabla \Phi_\alpha$ by the right-hand side member in (3.35), we can rewrite the continuity equation (3.32) as

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(\varepsilon \eta + h)\, \mathbf{u}_\alpha] + \mu^2 \nabla \cdot \left\{ \left(\frac{z_\alpha^2}{2} - \frac{h^2}{6} \right) h \nabla(\nabla \cdot \mathbf{u}_\alpha) \right.$$

$$\left. + \left(z_\alpha + \frac{h}{2} \right) h \nabla[\nabla \cdot (h\, \mathbf{u}_\alpha)] \right\} + NL1 = 0(\mu^4) \tag{3.36}$$

where

$$NL1 = \mu^2 \nabla \cdot \left\{ \varepsilon\eta \left[\left(z_\alpha - \frac{1}{2} \varepsilon \eta \right) \nabla (\nabla \cdot h\, \mathbf{u}_\alpha) \right. \right.$$

$$\left. \left. + \frac{1}{2} \left(z_\alpha^2 - \frac{1}{3} \varepsilon^2 \eta^2 \right) \nabla (\nabla \cdot \mathbf{u}_\alpha) \right] \right\} \tag{3.37}$$

The terms in $NL1$, (3.37), are of the order of magnitude of $0(\mu^2)$. Notice that these terms are the combination of cubic and quadratic nonlinearity.

Taking the gradient of (3.33) and using (3.35) in the resulting equation, we obtain

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} + \nabla \eta + \varepsilon \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \mu^2 \left\{ z_\alpha \nabla \left[\nabla \cdot (h \frac{\partial \mathbf{u}_\alpha}{\partial t}) \right] \right.$$

$$\left. + \frac{1}{2} z_\alpha^2 \nabla (\nabla \cdot \frac{\partial \mathbf{u}_\alpha}{\partial t}) \right\} + NL2 = 0(\mu^4) \tag{3.38}$$

where

$$NL2 = \mu^2 \varepsilon \left\{ (\mathbf{u}_\alpha \cdot \nabla z_\alpha) \nabla(\nabla \cdot h\mathbf{u}_\alpha) + z_\alpha \nabla [\mathbf{u}_\alpha \cdot \nabla(\nabla \cdot h\mathbf{u}_\alpha)] \right.$$

$$+ z_\alpha (\mathbf{u}_\alpha \cdot \nabla z_\alpha) \nabla (\nabla \cdot \mathbf{u}_\alpha) + \frac{z_\alpha^2}{2} \nabla [\mathbf{u}_\alpha \cdot \nabla(\nabla \cdot \mathbf{u}_\alpha)]$$

$$\left. + \nabla \cdot (h\, \mathbf{u}_\alpha) \nabla(\nabla \cdot h\mathbf{u}_\alpha) - \nabla \left[\eta \nabla \cdot \left(h \frac{\partial \mathbf{u}_\alpha}{\partial t} \right) \right] \right\}$$

$$- \mu^2\varepsilon^2 \left\{ \frac{1}{2} \nabla \left[\eta^2 \left(\nabla \cdot \frac{\partial \mathbf{u}_\alpha}{\partial t} \right) \right] + \nabla \left[\mathbf{u}_\alpha \cdot (\eta \nabla (\nabla \cdot h\mathbf{u}_\alpha) \right] \right.$$

$$\left. - \nabla \left[\eta (\nabla \cdot h\,\mathbf{u}_\alpha) \nabla \cdot \mathbf{u}_\alpha \right] \right\}$$

$$- \mu^2\varepsilon^3 \nabla \left\{ \frac{\eta^2}{2} \left[\nabla(\nabla \cdot \mathbf{u}_\alpha) - (\nabla \cdot \mathbf{u}_\alpha)^2 \right] \right\} \tag{3.39}$$

We reiterate that similar to those terms in $NL1$, the terms in $NL2$ are nonlinear in both quadratic and cubic forms. By allowing the wave parameter ε to be arbitrary, we permit finite amplitude waves in very shallow water.

3.2.2 *Summary of conventional Boussinesq equations*

In the Boussinesq approximation, the Ursell number is assumed to be of the order of magnitude of one. In other words, $0(\varepsilon) \approx 0(\mu^2) << 1$. Consequently, the members of $NL1$ and $NL2$ in (3.37) and (3.39) are in the same order of magnitude as or smaller than $0(\mu^4)$. Therefore, the Boussinesq equations written in terms of the horizontal velocity components, $\mathbf{u}_\alpha$, and free surface displacement, η, are given in (3.36) and (3.38) without $NL1$ and $NL2$. These equations are accurate up to $0(\mu^4, \mu^2\varepsilon)$. When the velocity is evaluated on the still water surface, $z_\alpha = 0$ and $\mathbf{u}_\alpha = \mathbf{u}_o$, the Boussinesq equations can be reduced to (3.29) and (3.30) which were derived from the Hamiltonian approach.

The limitations of the Boussinesq equations are two-folds: In very shallow water where waves are close to breaking, the nonlinearity is important and the nonlinearity parameter ε could become as large as $0.3 \sim 0.4$. At the same time the dispersion parameter μ^2 becomes smaller as depth decreases. Therefore, free surface profiles for a near breaking wave obtained from the Boussinesq equation are usually more symmetric with respect to the wave crest than that observed in the laboratory (Liu, 1990). This shortcoming can be overcome by relaxing the restriction on nonlinearity parameter ε. In other words, some terms in $NL1$ and $NL2$ given in (3.37) and (3.39), such as $0(\mu^2\varepsilon)$ terms, can be included in the Boussinesq equations. The second limitation of the Boussinesq equations is their inadequate behavior in the intermediate depth region. We will illustrate this point by examining the dispersion relations corresponding to different forms of the Boussinesq equations in the following section.

The conventional Boussinesq equations appear in different forms. They can be expressed in terms of the velocity on the free surface, as shown in (3.29) and (3.30). They can also be written in terms of velocity vectors on the bottom or the depth-averaged velocity (Peregrine, 1972). For later uses, we present these well-known Boussinesq equations here. In terms of the velocity along the bottom, $z_\alpha = -h$, (3.36) and (3.38) becomes

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(\varepsilon\eta + h)\mathbf{u}_b] + \mu^2 \left\{ \nabla \cdot \frac{1}{3} h^3 \nabla(\nabla \cdot \mathbf{u}_b) - \frac{1}{2} h^2 \nabla [\nabla \cdot (h\mathbf{u}_b)] \right\} = 0(\mu^4) \tag{3.40}$$

$$\frac{\partial \mathbf{u}_b}{\partial t} + \nabla\eta + \varepsilon \mathbf{u}_b \cdot \nabla \mathbf{u}_b + \mu^2 \left\{ -h \nabla \left(\nabla \cdot h \frac{\partial \mathbf{u}_b}{\partial t} \right) + \frac{1}{2} h^2 \nabla \left(\nabla \cdot \frac{\partial \mathbf{u}_b}{\partial t} \right) \right\} = 0(\mu^4) \tag{3.41}$$

where $\mathbf{u}_b$ denotes the velocity on the sea bottom. To write the Boussinesq equations in terms of the depth-averaged velocity

$$\bar{\mathbf{u}} = \frac{1}{h + \varepsilon\eta} \int_{-h}^{\varepsilon\eta} \nabla \Phi \, dz \tag{3.42}$$

We first rewrite the continuity equation (3.31) as

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(h + \varepsilon\eta)\, \bar{\mathbf{u}}] = 0 \tag{3.43}$$

which is exact. Secondly, from (3.17), we obtain

$$\nabla \Phi_\alpha = \frac{\bar{\mathbf{u}}}{h} - \frac{\varepsilon\eta}{h^2}\bar{\mathbf{u}} + \mu^2 \left\{ \frac{h}{6}(h^2 - 3z_\alpha^2) \nabla \cdot \left(\frac{\bar{\mathbf{u}}}{h} \right) - \frac{1}{2} h(h + 2z_\alpha \nabla \cdot \bar{\mathbf{u}}) \right\} + 0(\mu^4, \varepsilon^2) \tag{3.44}$$

Substituting (3.34) and (3.44) into (3.38), we obtain after a lengthy manipulation,

$$\frac{\partial \bar{u}}{\partial t} + \nabla\eta + \varepsilon\,\bar{u} \cdot \nabla\bar{u} + \mu^2 \left[\frac{h^2}{6}\,\nabla\left(\nabla \cdot \frac{\partial \bar{u}}{\partial t}\right) - \frac{h}{2}\,\nabla\nabla \cdot \left(h\,\frac{\partial \bar{u}}{\partial t}\right)\right] = 0(\mu^4\,,\,\varepsilon^2) \qquad (3.45)$$

We remark here that different forms of the Boussinesq equations can be further deduced by replacing the higher order time derivative terms in the momentum equations, (3.41) and (3.45) by the spatial derivative of η.

3.3 *Dispersion Relation and Phase Velocity of Nonlinear Shallow-water Equations*

The major difference among the conventional Boussinesq equations is in the higher order derivative terms. These higher order derivative terms affect the dispersive properties and the stability of the equations. In this section, we examine the dispersion relations for the nonlinear shallow-water equations as well as the conventional Boussinesq equations. To simplify the discussion, we only investigate one-dimensional constant depth case. From (3.36) and (3.38), we can write the continuity and the momentum equation as

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(\varepsilon\eta + h)u_\alpha\right] + \mu^2\left(\alpha + \frac{1}{3}\right)h^3\,\frac{\partial^3 u_\alpha}{\partial x^3} + \beta\left\{\mu^2\varepsilon\alpha\,h^2\,\frac{\partial}{\partial x}\left(\eta\,\frac{\partial^2 u_\alpha}{\partial x^2}\right) - \frac{\mu^2\varepsilon^2}{2}\,h\,\frac{\partial}{\partial x}\left(\eta^2\,\frac{\partial^2 u_\alpha}{\partial x^2}\right) + \frac{\mu^2\varepsilon^3}{6}\,\frac{\partial}{\partial x}\left(\eta^3\frac{\partial^2 u_\alpha}{\partial x^2}\right)\right\} = 0 \qquad (3.46)$$

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial \eta}{\partial x} + \mu^2\,\alpha\,h^2\,\frac{\partial^3 u_\alpha}{\partial x^2 \partial t} + \varepsilon u_\alpha\,\frac{\partial u_\alpha}{\partial x}$$

$$+\,\beta\mu^2\varepsilon\left\{h^2\,\alpha\frac{\partial}{\partial x}\left(u_\alpha\,\frac{\partial^2 u_\alpha}{\partial x^2}\right) + h^2\,\frac{\partial u_\alpha}{\partial x}\,\frac{\partial^2 u_\alpha}{\partial x^2} - h\frac{\partial}{\partial x}\left(\eta\,\frac{\partial^2 u_\alpha}{\partial x \partial t}\right)\right\}$$

$$-\,\beta\mu^2\,\varepsilon^2\,\frac{\partial}{\partial x}\left[\frac{\eta^2}{2}\,\frac{\partial^2 u_\alpha}{\partial x \partial t} + u_\alpha\eta h\,\frac{\partial^2 u_\alpha}{\partial x^2} - \eta h\left(\frac{\partial u_\alpha}{\partial x}\right)^2\right]$$

$$-\beta\mu^2\varepsilon^3\,\frac{\partial}{\partial x}\left\{\frac{\eta^2}{2}\left[\frac{\partial^2 u_\alpha}{\partial x^2}-\left(\frac{\partial u_\alpha}{\partial x}\right)^2\right]\right\} = 0 \tag{3.47}$$

where $\alpha = (z_\alpha/h)^2/2 + (z_\alpha/h)$ and $\beta = 1$. If the terms in the order of $\mu^2\varepsilon$, $\mu^2\varepsilon^2$ and $\mu^2\varepsilon^3$ are ignored (i.e. $\beta = 0$), the above equations are reduced to the Boussinesq equations.

We look for a solution of (3.46) and (3.47) in the expansion

$$\eta = \eta_1(\theta) + \varepsilon\eta_2(\theta) + \varepsilon^2\eta_3(\theta) + \ldots \tag{3.48}$$

$$u_\alpha = u_1(\theta) + \varepsilon u_2(\theta) + \varepsilon^2 u_3(\theta) + \ldots \tag{3.49}$$

$$\theta = kx - wt \tag{3.50}$$

$$\omega = \omega_0 + \varepsilon^2\omega_2 + \ldots \tag{3.51}$$

where $\varepsilon = a/h$ denotes the small parameter for nonlinearity. Substituting (3.48) $\sim$ (3.51) into (3.46) and (3.47), we obtain the hierarchy

$$-\omega_o\eta_1' + khu_1' + \mu^2(\alpha+\frac{1}{3})k^3h^3u_1''' = 0 \tag{3.52}$$

$$-\omega_o u_1' + k\eta_1' - \mu^2\alpha k^2h^2\omega_o u_1''' = 0 \tag{3.53}$$

$$-\omega_o\eta_2' + khu_2' + \mu^2(\alpha+\frac{1}{3})k^3h^3u_2''' = -k(\eta_1u_1)' - \beta\mu^2\alpha k^3h^2(\eta_1u_1'')' \tag{3.54}$$

$$-\omega_o u_2' + k\eta_2' - \mu^2\alpha k^2h^2\omega_o u_2''' = -ku_1u_1'$$

$$-\beta\mu^2\left\{\alpha k^3h^2(u_1u_1'')' + k^3h^2u_1'u_1'' + \omega_o k^2h(\eta_1u_1'')'\right\} \tag{3.55}$$

$$-\omega_o\eta_3' + khu_3' + \mu^2(\alpha + \frac{1}{3})k^3h^3u_3''' = -k(\eta_1 u_2 + \eta_2 u_1)' + \omega_2\eta_1'$$

$$-\beta\mu^2\alpha k^3h^2[(\eta_1 u_2'')' + (\eta_2 u_1'')'] + \beta\frac{\mu^2}{2}k^3h(\eta_1^2 u_1'')' \qquad (3.56)$$

$$-\omega_o u_3' + k\eta_3' - \mu^2\alpha k^2h^2\omega_o u_3''' = -k(u_1 u_2)' + \omega_2 u_1'$$

$$-\beta\mu^2\Big\{k^3h^2\alpha[(u_1u_2'')' + u_2u_1''] + k^3h^2[u_1'u_2'' + u_2'u_1''] + \omega_o k^2h[(\eta_1 u_2'')'$$

$$+ (\eta_2 u_1'')'] + \frac{1}{2}\omega_o k^2(\eta_1^2 u_1'')' - k^3h(u_1\eta_1 u_1'')' + [\eta_1(u_1')^2]'\Big\} \qquad (3.57)$$

The solution for the leading order equations, (3.52) and (3.53) represents a periodic wave

$$\eta_1 = \cos\theta \quad , \quad u_1 = \frac{k}{\omega_o}\left(\frac{1}{1-\mu^2\alpha k^2h^2}\right)\cos\theta \qquad (3.58)$$

with the linear dispersion relation

$$C^2 = \frac{\omega_o^2}{k^2} = h\left[\frac{1-\mu^2(\alpha+\frac{1}{3})k^2h^2}{1-\mu^2\alpha k^2h^2}\right] \qquad (3.59)$$

If the above equation becomes negative, the frequency and the phase velocity become imaginary. This implies that the solution grows in time and becomes unstable. Hence, the instability condition requires

$$\left[1 - \mu^2\left(\alpha+\frac{1}{3}\right)k^2h^2\right](1 - \mu^2\alpha k^2h^2) > 0$$

However, because $-\frac{1}{2} < \alpha < 0$ the above condition becomes

$$1 - \mu^2\left(\alpha+\frac{1}{3}\right)k^2h^2 > 0 \qquad (3.60)$$

For $\alpha \leq -1/3$ the stability condition is always satisfied for all kh. On the other hand, if $0 > \alpha > -1/3$, the relative depth is limited by the following condition

$$\mu\, kh \,<\, \frac{1}{\sqrt{\alpha + \frac{1}{3}}} \tag{3.61}$$

for the stability requirement. For example, the conventional Boussinesq equations using the velocity on the free surface, $u_o(\alpha = 0)$, become unstable when $\mu(kh)$ is greater than $\sqrt{3}$.

As we mentioned before, the higher derivative terms (dispersive terms) in the Boussinesq equations can appear in different forms; we can replace the spatial derivative by the time derivative vice versa through the leading order approximation. For instance, (3.47) can be rewritten as

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial \eta}{\partial x} - \mu^2\, \alpha\, h^2\, \frac{\partial^3 \eta}{\partial x^3} + \varepsilon u_\alpha\, \frac{\partial u_\alpha}{\partial x} = 0 \tag{3.62}$$

The dispersive properties of the new set of Boussinesq equations, (3.46) with $\beta = 0$ and (3.62) are different from those of (3.46) and (3.47). Substituting (3.48) $\sim$ (3.51) into (3.48) and (3.62), we find the following linear dispersion relation

$$C^2 = \frac{\omega^2}{k^2} = h\left[1 - \mu^2\left(\alpha + \frac{1}{3}\right)k^2h^2\right](1 + \mu^2\alpha k^2h^2) \tag{3.63}$$

Comparing (3.59) and (3.63), we observe that (3.63) is the truncated binomial expansion of (3.59) for small k^2h^2. To ensure that the system is stable, the kh value must satisfy the following condition

$$\left[1 - \mu^2\left(\alpha + \frac{1}{3}\right)k^2h^2\right](1 + \mu^2\alpha\, k^2h^2) > 0 \tag{3.64}$$

This is a more restrictive condition than (3.60). With $\alpha = -1/3$, the conventional Boussinesq equations using the depth-averaged velocity also become unstable when μkh is greater than $\sqrt{3}$. Hence, the form of Boussinesq equations expressed in (3.46) and (3.47) are preferred as far as the instability is concerned.

Substituting (3.58) into (3.54) and (3.55), we find that the right-hand side terms are proportional to $\sin 2\theta$. Therefore, the solution for (3.54) and (3.55) can be written as

$$\eta_2 = B\cos 2\theta \quad , \quad u_2 = D\cos 2\theta \tag{3.65}$$

where

$$B = \frac{\omega_o}{k}(1 - 4\mu^2\alpha k^2h^2)D - \frac{1}{4}\frac{k^2}{\omega_o^2}\frac{1}{(1-\mu^2\alpha k^2h^2)^2}$$

$$\left\{1 - \beta\mu^2\left[(2\alpha+1)k^2h^2 + 2\omega_o^2 h(1 - \mu^2\alpha k^2h^2)\right]\right\} \tag{3.66}$$

$$D = \frac{1}{4\mu^2k^3h^3}\frac{k^2}{\omega_o}\frac{1}{1-\mu^2\alpha k^2h^2}\left\{3 - 2\mu^2\alpha k^2h^2\right.$$

$$\left. - \beta\mu^2(4\alpha + 3)k^2h^2 + 2\beta\mu^4\left(\frac{1}{3} + \alpha + 4\alpha^2\right)k^4h^4\right\} \tag{3.67}$$

The right-hand side of the equations for η_3 and u_3 can be written as

$$RHS\,\text{of}\,(3.56) = \left\{-\omega_2 + \frac{1}{2}k(D + BU) + \beta\mu^2\left[\frac{k^3h}{4}U\right.\right.$$

$$\left.\left. - \frac{k}{2}\left(4\alpha k^2h^2D + \alpha k^2h^2BU - \frac{1}{4}k^2hU\right)\right]\right\}\sin\theta + L_1\sin 3\theta \tag{3.68}$$

$$RHS\,\text{of}\,(3.57) = \left\{-\omega_2 U + \frac{1}{2}UDk - \beta\mu^2\left[\frac{1}{2}k^2h\omega_oBU\right.\right.$$

$$+ \left(\frac{k^4h^2}{2\omega_o}\frac{5\alpha-2}{1-\mu^2\alpha k^2h^2} + 2k^2h\omega_o\right)D + \frac{3}{8}k^3\frac{1}{1-\mu^2\alpha k^2h^2}$$

$$\left.\left. - \frac{1}{2}\frac{k^5h}{\omega_o^2}\frac{1}{(1-\mu^2\alpha k^2h^2)^2}\right]\right\}\sin\theta + L_2\sin 3\theta \tag{3.69}$$

where

$$U = \frac{k}{\omega_o} \frac{1}{1 - \mu^2 \alpha k^2 h^2} \tag{3.70}$$

and L_1 and L_2 are some complicated coefficients. The terms in sin 3θ can be accommodated by solutions η_3 and u_3 which are proportional to cos 3θ, but the sin θ terms resonate with the operators on the left. There is a secular solution which is proportional to θ sin θ and is unbounded in θ. To eliminate the resonant term, we first combine (3.56) and (3.57) by eliminating η_3'. Thus, the right-hand side of the resulting equation becomes

$$k\,(RHS \text{ of } (3.56)) + \omega_o\,(RHS \text{ of } (3.57)) = \Bigg\{ - \omega_2(k + \omega_o U)$$

$$+ \frac{1}{2}k^2(D + BU) + \frac{1}{2}\,k\omega_o UD - \beta\mu^2 \Bigg[\Bigg(\frac{k^4h^2}{2} \frac{5\alpha - 2}{1 - \mu^2\alpha k^2 h^2}$$

$$+ 2k^2 h\omega_o^2 + 2\alpha k^4 h^2 \Bigg) D + \frac{k^2}{2}\,(\alpha k^2 h^2 + \omega_o^2 h) BU - \frac{3}{8}k^4 hU$$

$$+ \frac{3k^3\omega_o}{8(1 - \mu^2\alpha\, k^2h^2)} - \frac{k^5 h}{2\omega_o(1 - \mu^2\alpha k^2h^2)^2} \Bigg] \Bigg\} \sin\theta + (kL_1 + \omega_o L_2) \sin 3\theta$$

We now make the coefficient of sin θ be zero so that the resonant term vanishes. The final result for ω_2 is

$$\omega_2 = \frac{k}{2}\,D + \frac{k^2}{2\omega_o} \frac{1}{2 - \mu^2\alpha k^2 h^2}\,B - \beta\mu^2 \frac{k^2h^2}{2 - \mu^2\alpha k^2h^2}$$

$$\Bigg\{ 2k \left[\frac{5}{4}\alpha - 1 + \left(\alpha + \frac{\omega_o^2}{k^2 h}\right) (1 - \mu^2\alpha k^2 h^2) \right] D$$

$$+ \frac{\omega_o}{2h}\left(1 + \alpha\frac{k^2h}{\omega_o^2}\right) B + \frac{3}{8}\,\frac{\omega_o}{h^2}\left(1 - \frac{k^2 h}{\omega_o^2}\right) - \frac{k^2}{2\omega_o h(1 - \mu^2\alpha k^2h^2)} \Bigg\} \tag{3.71}$$

where B and D are defined in (3.66) and (3.67). ω_2 represents the dependence of the dispersion relation on amplitude. If the Boussinesq approximation is adopted (i.e. $\beta = 0$), ω_2 can be simplified to be

$$\omega_2 = \left\{ \frac{1}{8} \frac{k^3}{\omega_o} \frac{1}{\mu^2 k^3 h^3} (3 - 2\mu^2 \alpha k^2 h^2)(3 - 5\mu^2 \alpha k^2 h^2) \right.$$

$$\left. - \frac{k^4}{8\omega_o^3} \frac{1}{1 - \mu^2 \alpha k^2 h^2} \right\} \frac{1}{(1 - \mu^2 \alpha k^2 h^2)(2 - \mu^2 \alpha k^2 h^2)} \tag{3.72}$$

Returning to dimensional variables, we can express the dispersion relation, (3.51) as

$$\frac{\omega}{\sqrt{ghk}} = \left[1 - \left(\alpha + \frac{1}{3}\right) k^2 h^2\right]^{1/2} (1 - \alpha k^2 h^2)^{-1/2}$$

$$+ \left(\frac{a}{h}\right)^2 \left\{ \frac{1}{8k^2h^2} (3 - \alpha k^2 h^2)(3 - 5\alpha k^2 h^2) - \frac{1}{8}\left[1 - \left(\alpha + \frac{1}{3}\right) k^2 h^2\right]^{-1} \right\}$$

$$\left[1 - \left(\alpha + \frac{1}{3}\right) k^2 h^2\right]^{-1/2} (1 - \alpha k^2 h^2)^{-1/2} (2 - \alpha k^2 h^2)^{-1} \tag{3.73}$$

For the special case $\alpha = 0$, i.e., the Boussinesq equations written in terms of the free surface velocity, the dispersion relation can be expressed as

$$\omega = k\sqrt{h}\left(1 - \frac{\mu^2}{3} k^2 h^2\right)^{1/2} + \varepsilon^2 \left(\frac{9}{16} \frac{k^3}{\omega_o} \frac{1}{\mu^2 k^3 h^3} - \frac{1}{16} \frac{k^4}{\omega_o^3}\right) \tag{3.74}$$

in the dimensionless form, and

$$\frac{\omega}{\sqrt{ghk}} = \left(1 - \frac{1}{3}k^2h^2\right)^{\frac{1}{2}} + \left(\frac{a}{h}\right)^2 \left\{ \frac{9}{16k^2h^2} \right.$$

$$\left. - \frac{1}{16}\left(1 - \frac{1}{3}k^2h^2\right)^{-1} \right\} \left(1 - \frac{1}{3}k^2h^2\right)^{-\frac{1}{2}} \tag{3.75}$$

in the dimensional form.

4. Extension of Shallow-Water Equations to Deep Water

From the analysis shown in the previous section, we know that the Boussinesq equations are unstable subject to high frequency disturbance if $0 > \alpha > -1/3$ (see equation (3.60)). Even if the Boussinesq equations are stable for all kh (for example, $\alpha = -1/3$), the accuracy of these equations in the intermediate and the deep water depth is in question. A simple way to evaluate the accuracy of various forms of the nonlinear shallow water equations is to compare the dispersion relation derived from the these equations to that from Stokes' wave theory in the intermediate and deep water.

4.1 *Comparison Between Shallow-Water Equations and Stokes' Wave Theory*

For a uniform Stokes' wave train in a constant depth, the dispersion relation can be written as (Whitham 1974):

$$\frac{\omega^2}{gk \tanh kh} = 1 + \left(\frac{9 \tanh^4 kh \, - \, 10 \tanh^2 kh \, + \, 9}{8 \tanh^4 kh} \right) k^2 a^2 + \dots \quad (4.1)$$

where ka denotes the wave slope and is considered as a small parameter. The dispersion relation can be further approximated as

$$\frac{\omega}{\sqrt{gk \tanh kh}} = 1 + \frac{1}{2} \left(\frac{9 \tanh^4 kh \, - \, 10 \tanh^2 kh \, + \, 9}{8 \tanh^4 kh} \right) k^2 a^2 \quad (4.2)$$

The first term on the right-hand side of the above equation represents the linear wave dispersion relation, while the second term denotes the amplitude effect on the dispersion. In the shallow water limit, $kh << 1$, the dispersion relation becomes

$$\frac{\omega}{\sqrt{ghk}} = 1 - \frac{1}{6} k^2 h^2 + \left(\frac{a}{h} \right)^2 \frac{9}{16 \, k^2 h^2} + \dots \quad (4.3)$$

Once again, the shallow water limit of the Stokes' wave theory is valid only if $0(a/h) << 0(kh)^2$ (Whitham 1974). Hence, the Stokes' theory works well in the shallow water only for extremely small amplitude waves.

Because the Boussinesq equations are derived from the assumption that $0(a/h) = 0(kh)^2$ which covers the special case when the nonlinearity is weak. Therefore, it is not surprising to observe that the dispersion relation for the conventional Boussinesq equations, (3.75), after the higher-order terms in kh are dropped, is the same as (4.3).

However, we are more interested in knowing if the dispersion relation derived from the nonlinear shallow-water equations can be matched with that from the Stokes' wave theory in the intermediate and the deep water. Denoting ω and ω^s as the frequencies associated with the nonlinear shallow-water equations and the Stokes' theory, respectively, we can define the ratios between these frequencies at two separate orders:

$$\frac{\omega_o}{\omega_o^s} = \sqrt{\frac{kh}{\tanh kh}} \left[\frac{1-(\alpha+\frac{1}{3})k^2h^2}{1-\alpha k^2h^2}\right]^{1/2} \tag{4.4}$$

$$\frac{\omega_2}{\omega_2^s} = \omega_2^* \sqrt{\frac{kh}{\tanh kh}} \left(\frac{1}{k^2h^2}\right) \frac{2\tanh^4 kh}{9\tanh^4 kh - 10\tanh^2 kh + 9} \tag{4.5}$$

where

$$\omega_2^* = \frac{1}{2}D^* + \frac{1}{2}B^* \left[\frac{1-\alpha k^2h^2}{1-(\alpha+\frac{1}{3})k^2h^2}\right] \frac{1}{2-\alpha k^2h^2}$$

$$-\frac{\beta k^2h^2}{2-\alpha k^2h^2} \left\{ 2\left[\frac{5\alpha-2}{4} + \left(\alpha + \frac{1-(\alpha+\frac{1}{3})k^2h^2}{1-\alpha k^2h^2}\right)(1-\alpha k^2h^2)\right] D^* \right.$$

$$+\frac{1}{2}\left[1+\alpha\frac{1-\alpha k^2h^2}{1-(\alpha+\frac{1}{3})k^2h^2}\right] B^* + \frac{3}{8}\left[1 - \frac{1-\alpha k^2h^2}{1-(\alpha+\frac{1}{3})k^2h^2}\right]$$

$$\left. -\frac{1}{2}\left[\frac{1-\alpha k^2h^2}{1-(\alpha+\frac{1}{3})k^2h^2}\right]^{\frac{1}{2}} \frac{1}{1-\alpha k^2h^2} \right\} \tag{4.6}$$

$$D^* = \frac{1}{4k^2h^2} \frac{[1-(\alpha+\frac{1}{3})k^2h^2]^{\frac{1}{2}}}{(1-\alpha k^2h^2)^{\frac{3}{2}}} \left\{ 3 - 2\alpha k^2h^2 \right.$$

$$\left. -\beta\left[(4\alpha+3)k^2h^2 + 2\left(4\alpha^2+\alpha+\frac{1}{3}\right)k^4h^4\right] \right\} \tag{4.7}$$

$$B^* = \left[\frac{1 - (\alpha + \frac{1}{3})k^2h^2}{1 - \alpha k^2h^2}\right]^{\frac{1}{2}} D^* - \frac{1}{4[1 - (\alpha + \frac{1}{3})k^2h^2]}$$

$$\frac{1}{1 - \alpha k^2h^2}\left\{1 - \beta k^2h^2[(2\alpha + 3) - 2(\alpha + \frac{1}{3})k^2h^2]\right\} \qquad (4.8)$$

When $\beta = 0$, the leading order frequency ratio remains the same as (4.4). But the second order frequency ratio can be simplified to be

$$\omega_2^* = \left\{\frac{(3 - \alpha k^2h^2)(3 - 5\alpha k^2h^2)}{k^2h^2} - \frac{1}{1 - (\alpha + \frac{1}{3})k^2h^2}\right\}$$

$$\left[1 - (\alpha + \frac{1}{3})k^2h^2\right]^{-1/2} (1 - \alpha k^2h^2)^{-1/2} (2 - \alpha k^2h^2)^{-1} \qquad (4.9)$$

These two frequency ratios are calculated for different kh, α and β values. In Figure 1, we show the leading order frequency ratio, (4.4), for the conventional Boussinesq equations with $\alpha = 0$. The agreement degenerates quickly for $kh \approx 1.5$. On the other hand, the agreement becomes much better if the velocity along the bottom ($\alpha = -0.5$) is used in the Boussinesq equations; roughly 20% differences occur in the deep water. As far as the conventional Boussinesq equations are concerned, the best agreement occurs when the mean velocity is used, i.e. $\alpha = -1/3$. Since any value between 0 and -0.5 can be used for α without reducing the accuracy in Boussinesq equations, one can find the optimal α value such that the differences in phase velocities and group velocities derived from ω_o and ω_o^s, respectively, are minimized over the entire range of water depth. Chen and Liu (1993) reported that the optimal value is $\alpha = -0.3855$. Using a different approach, Madsen *et al.* (1991) derived a set of Boussinesq-type equations whose linear dispersion relation has the same form as that given in (3.59). Madsen *et al.* suggested that the optimal value for α should be $-8/21 = -0.381$. The approach used by Madsen *et al.* will be discussed in section 4.3.

The second order frequency ratio is also calculated for the conventional Boussinesq equations ($\beta = 0$) and the nonlinear shallow water equations ($\beta = 1$). Figure 2 shows that the conventional Boussinesq equations underestimate the second order frequency in the intermediate and the deep water. The second order frequency for the Boussinesq equation is insensitive to the α value when it is less than $-1/3$. Moreover, the second order frequency is practically zero for $kh > 1.5$. On the other hand, the nonlinear shallow water equations give slightly better estimations of the second order frequency (Figure 3). Using the optimal α value determined from the first order frequency comparison, we show that the second order frequency for the nonlinear shallow water equation is about twenty five per cent of that from the Stokes' wave theory.

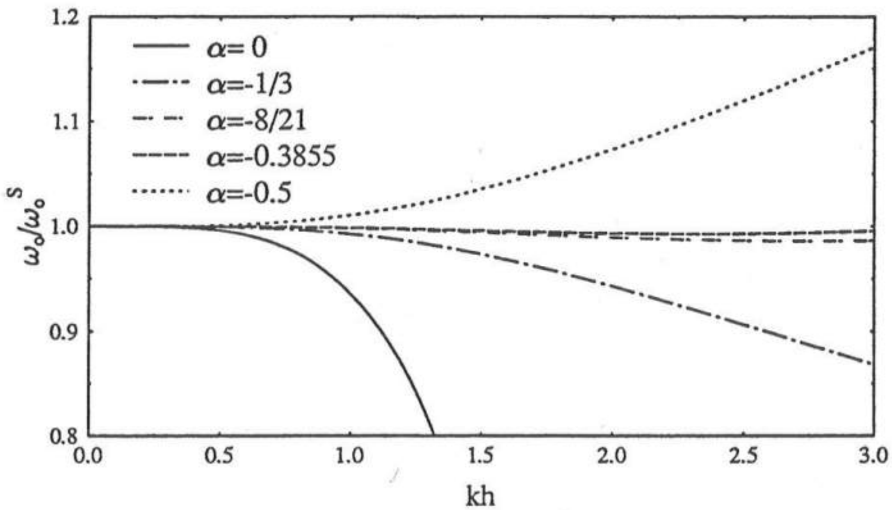

Figure 1. First order frequency ratios between the Stokes wave theory and Boussinesq-type equations.

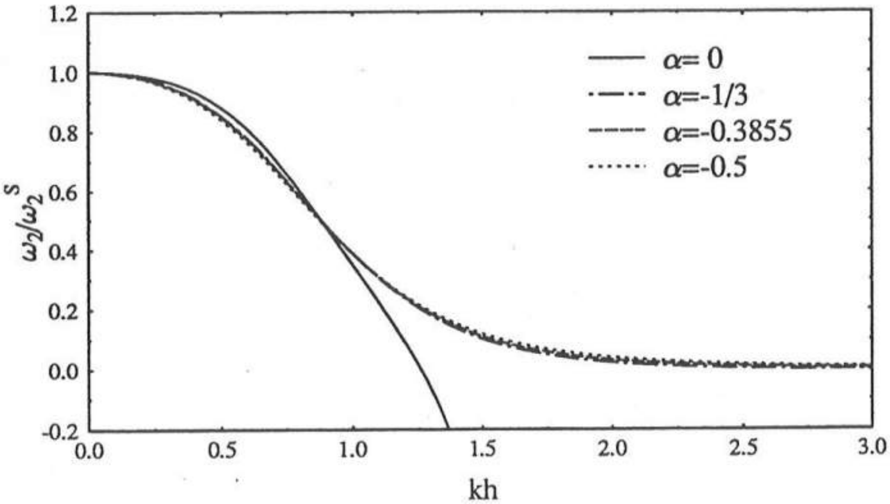

Figure 2. Second order frequency ratios between the Stokes wave theory and conventional Boussinesq equations (β= 0).

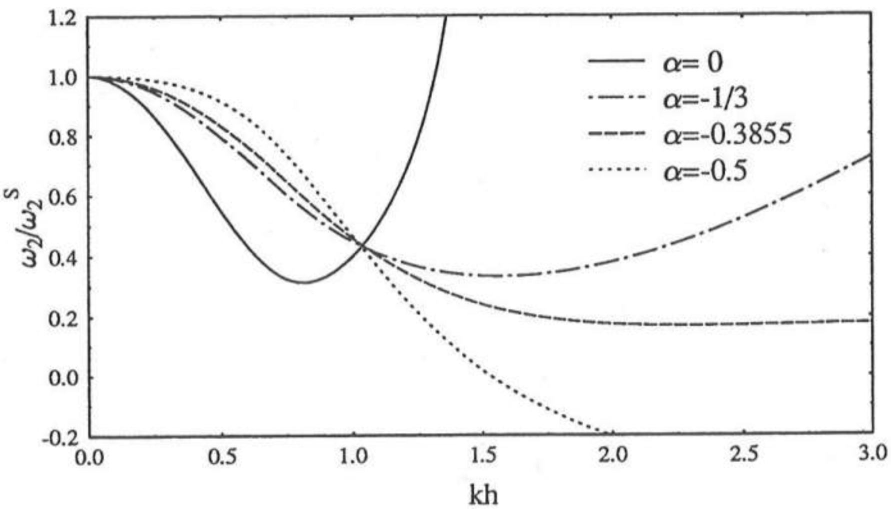

Figure 3. Second order frequency ratios between the Stokes wave theory and the nonlinear shallow water equations (β= 1).

4.2 *Hamiltonian Approach*

As shown in section 3.1, the Boussinesq equations derived from the Hamiltonian given in (3.24) becomes unstable when μkh is greater than $\sqrt{3}$. The instability occurs when the Hamiltonian becomes negative. The remedy is to reconstruct the Hamiltonian into a quadratic form, which is always positive and definite. First, the Hamiltonian, (3.24), is rewritten as

$$\mathcal{H} = \frac{1}{2}\int\int_{\Omega} \left[\nabla\Phi_o \cdot (dR\nabla\Phi_o) + \eta^2\right] dxdy + 0(\mu^4, \varepsilon\mu^2, \varepsilon^2) \quad (4.10)$$

where R is a two by two symmetric tensor operator defined by

$$R = \begin{bmatrix} 1+\mu^2 a & \mu^2 b \\ \mu^2 b & 1+\mu^2 c \end{bmatrix} \quad (4.11)$$

$$a = \frac{h}{2}\frac{\partial^2}{\partial x^2}h - \frac{h^2}{6}\frac{\partial^2}{\partial x^2} \quad (4.12a)$$

$$b = \frac{h}{2}\frac{\partial^2}{\partial x\partial y}h - \frac{h^2}{6}\frac{\partial^2}{\partial x\partial y} \quad (4.12b)$$

$$c = \frac{h}{2}\frac{\partial^2}{\partial y^2}h - \frac{h^2}{6}\frac{\partial^2}{\partial y^2} \quad (4.12c)$$

and $d = \varepsilon\eta + h$ is the total depth. Following the approach suggested by Broer *et al.* (1987) and Mooiman (1991), we seek for a positive definite Hamiltonian in a quadratic form:

$$\mathcal{H} = \frac{1}{2}\int\int_{\Omega} \left[d(F\,\nabla\Phi_o)^2 + \eta^2\right] dxdy + 0(\varepsilon^2, \varepsilon\mu^2, \mu^4) \quad (4.13)$$

where F is another tensor operator to be found from the following relationship

$$\nabla\Phi_o \cdot (R\nabla\Phi_o) = (F\nabla\Phi_o)\cdot(F\nabla\Phi_o) + 0(\mu^4)$$

Using (4.11) and (4.12), we obtain

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{4.14}$$

$$F_{11} = \frac{1}{\sqrt{2}} \left[1 + \frac{\mu^2}{2} (a + b) \right] \tag{4.15a}$$

$$F_{12} = \frac{1}{\sqrt{2}} \left[1 + \frac{\mu^2}{2} (b + c) \right] \tag{4.15b}$$

$$F_{21} = -\frac{1}{\sqrt{2}} \left[1 + \frac{\mu^2}{2} (a - b) \right] \tag{4.15c}$$

$$F_{22} = \frac{1}{\sqrt{2}} \left[1 - \frac{\mu^2}{2} (b - c) \right] \tag{4.15d}$$

where a, b, and c are operators defined in (4.12). The Hamiltonian defined in (4.13) is always positive and has the same accuracy as those given in (4.10) and (3.24). However, the Hamiltonian, (4.13), becomes unbounded for short waves, $\mu^2 \to \infty$. We must further approximate the operator F in the following manner

$$F_{11}^* = \frac{1}{\sqrt{2}[1 - \frac{\mu^2}{2}(a + b)]} = F_{11} + 0(\mu^4) \tag{4.16a}$$

$$F_{12}^* = \frac{1}{\sqrt{2}[1 - \frac{\mu^2}{2}(b + c)]} = F_{12} + 0(\mu^4) \tag{4.16b}$$

$$F_{21}^* = -\frac{1}{\sqrt{2}[1 - \frac{\mu^2}{2}(a - b)]} = F_{21} + 0(\mu^4) \tag{4.16c}$$

$$F_{22}^* = \frac{1}{\sqrt{2}[1 - \frac{\mu^2}{2}(b - c)]} = F_{22} + 0(\mu^4) \tag{4.16d}$$

The canonical equations given the final form of the stable Boussinesq-type equations:

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \left[F^T (d F \nabla \Phi_o) \right]$$

$$= -\frac{\partial}{\partial x}[F^*_{11}\, d(F^*_{11}u_o + F^*_{12}v_o) + F^*_{21}\, d(F^*_{21}u_o + F^*_{22}v_o)]$$

$$-\frac{\partial}{\partial y}[F^*_{12}\, d(F^*_{11}u_o + F^*_{12}v_o) + F^*_{22}\, d(F^*_{21}u_o + F^*_{22}v_o)] \qquad (4.17)$$

for continuity equation and

$$\frac{\partial \Phi_o}{\partial t} = -\frac{\varepsilon}{2}(F\nabla\Phi_o)^2 - \eta \qquad (4.18)$$

for momentum equations. Taking the gradient of the momentum equation, we obtain

$$\frac{\partial u_o}{\partial t} = -\varepsilon\left[(F^*_{11}\, u_o + F^*_{12}v_o)\frac{\partial}{\partial x}(F^*_{11}u_o + F^*_{12}v_o) + (F^*_{21}u_o + F^*_{22}v_o)\frac{\partial}{\partial x}(F^*_{21}u_o + F^*_{22}v_o)\right] - \frac{\partial \eta}{\partial x} \qquad (4.19)$$

$$\frac{\partial v_o}{\partial t} = -\varepsilon\left[(F^*_{11}u_o + F^*_{12}v_o)\frac{\partial}{\partial y}(F^*_{11}u_o + F^*_{12}v_o) + (F^*_{21}u_o + F^*_{22}v_o)\frac{\partial}{\partial y}(F^*_{21}u_o + F^*_{22}v_o)\right] - \frac{\partial \eta}{\partial y} \qquad (4.20)$$

Equations (4.17), (4.19) and (4.20) represent the modified Boussinesq equations which are stable for short waves.

Because the differential operators appear in the denominators on the right-hand side of (4.17), (4.19) and (4.20), numerical schemes for solving these equations must be designed with special care (Mooiman 1991). To demonstrate the improved characteristics of the modified Boussinesq equations, we examine the one-dimensional waves ($v_o = 0$, $\partial/\partial y = 0$) over a constant depth. From (4.17) and (4.19), we obtain

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}[F^*_{11}(dF^*_{11}u_o) + F^*_{21}(dF^*_{21}u_o)] \qquad (4.21)$$

$$\frac{\partial u_o}{\partial t} = -\varepsilon\left[F^*_{11}\, u_o\frac{\partial}{\partial x}(F^*_{11}u_o) + F^*_{21}\, u_o\frac{\partial}{\partial x}(F^*_{21}u_o)\right] - \frac{\partial \eta}{\partial x} \qquad (4.22)$$

where

$$F_{11}^* = -F_{21}^* = \frac{1}{\sqrt{2}} \frac{1}{1 - \frac{\mu^2 h^2}{6} \frac{\partial^2}{\partial x^2}} \tag{4.23}$$

The linearized version of the above equations becomes

$$\left(1 - \frac{\mu^2 h^2}{6} \frac{\partial^2}{\partial x^2}\right)^2 \frac{\partial \eta}{\partial t} = -h \frac{\partial u_o}{\partial x} \tag{4.24}$$

$$\frac{\partial u_o}{\partial t} = -\frac{\partial \eta}{\partial x} \tag{4.25}$$

For a small amplitude periodic wave given in (3.58) the dispersion relation can be determined from (4.24) and (4.25) as

$$c^2 = \frac{\omega_o^2}{k^2} = h \left(\frac{1}{1 + \frac{1}{6}\mu^2 k^2 h^2}\right)^2 \tag{4.26}$$

It is quite obvious that the quantity on the right-hand side is always positive. Therefore, the system is stable for all kh values. The ratio of the frequency calculated from (4.26) to that from Stokes' wave can be expressed as

$$\frac{\omega_o}{\omega_o^s} = \sqrt{\frac{kh}{\tanh kh}} \left(\frac{1}{1 + \frac{1}{6}k^2 h^2}\right) \tag{4.27}$$

Following the same procedure presented in section 3.3, we derive the second order frequency ratio defined in (4.5) with the following parameters:

$$\omega_2^* = \frac{D^*}{4}\left[2\left(1 + \frac{2k^2h^2}{3}\right)^{-2} - \left(1 + \frac{k^2h^2}{6}\right)^{-2}\right]$$

$$+ \frac{1}{4}\left(1 + \frac{k^2h^2}{6}\right)^{-1}\left[B^*\left(1 + \frac{k^2h^2}{6}\right)^{-1} + D^*\left(1 + \frac{2k^2h^2}{3}\right)^{-1}\right] \tag{4.28}$$

$$D^* = \frac{1}{4k^2h^2}\left(1 + \frac{15}{36}k^2h^2\right)^{-1}\left(1 + \frac{k^2h^2}{6}\right)(3 + k^2h^2)\left(1 + \frac{2k^2h^2}{3}\right) \tag{4.29}$$

$$B^* = \left(1 + \frac{k^2h^2}{6}\right)\left(1 + \frac{2k^2h^2}{3}\right)^{-1}\left[\frac{1}{2} + \left(1 + \frac{2k^2h^2}{3}\right)^{-1} D^*\right] \tag{4.30}$$

In Figure 4 the first order frequency ratio, (4.27), is plotted for different kh values. As a reference the frequency ratio derived from the original Hamiltonian ($\alpha = 0$) is also plotted. The improvement made by the modified Hamiltonian is rather significant. However, the behavior of the shallow water equation with the optimal α value is still slightly better than that of the modified Boussinesq equations in the deep water limit (see Figure 1). The characteristics of the second order frequency for the modified Boussinesq equations are more or less the same as those of the original Boussinesq equations with $\alpha < -1/3$ (see Figures 2 and 5).

4.3 *Other Approaches*

Madsen *et al.* (1991) took a different approach and derived a set of Boussinesq-type equations. They rewrote the conventional Boussinesq equations in terms of the depth-averaged velocity, (3.43) and (3.45), in the following conservative form:

$$\frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \tag{4.31}$$

$$\frac{\partial P}{\partial t} + \varepsilon\frac{\partial}{\partial x}\left(\frac{P^2}{d}\right) + \varepsilon\frac{\partial}{\partial y}\left(\frac{PQ}{d}\right) + d\frac{\partial \eta}{\partial x} - \frac{\mu^2}{3}h^2\left(\frac{\partial^3 P}{\partial x^2 \partial t} + \frac{\partial^3 Q}{\partial x \partial y \partial t}\right) = 0 \tag{4.32}$$

$$\frac{\partial Q}{\partial t} + \varepsilon\frac{\partial}{\partial x}\left(\frac{PQ}{d}\right) + \varepsilon\frac{\partial}{\partial y}\left(\frac{Q^2}{d}\right) + d\frac{\partial \eta}{\partial y} - \frac{\mu^2}{3}h^2\left(\frac{\partial^3 P}{\partial x \partial y \partial t} + \frac{\partial^3 Q}{\partial y^2 \partial t}\right) = 0 \tag{4.33}$$

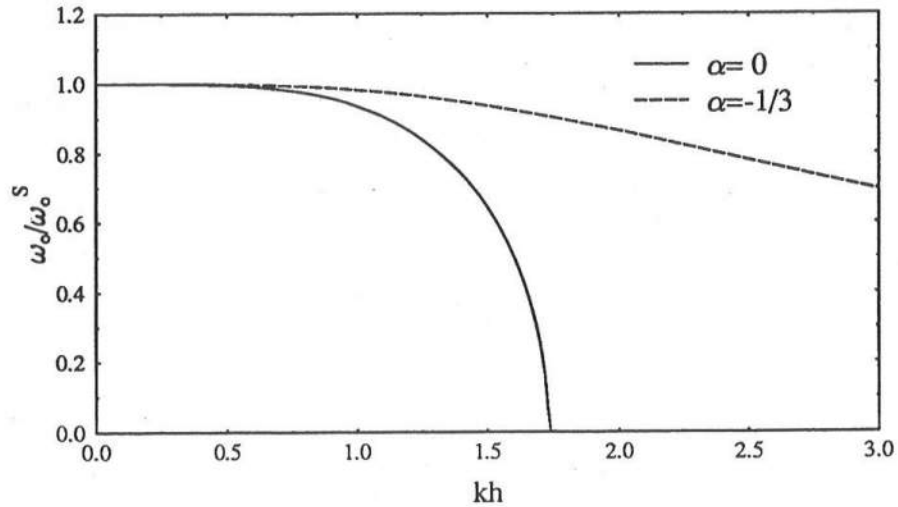

Figure 4. First order frequency ratios between the Stokes wave theory and the Boussinesq equation derived from the modified Hamiltonian.

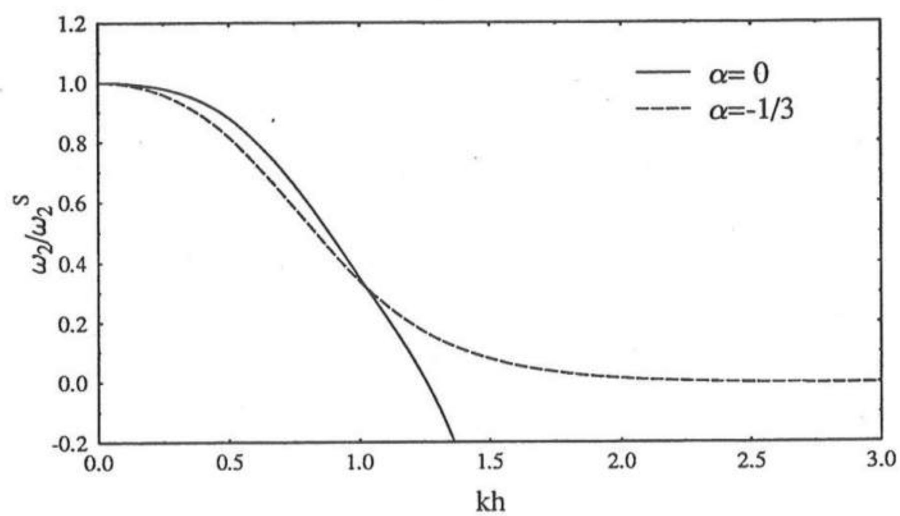

Figure 5. Second order frequency ratios between the Stokes wave theory and the modified Boussinesq equation.

where $d = \varepsilon\eta + h$ is the total depth and $P = \bar{u}(\varepsilon\eta + h)$ and $Q = \bar{v}(\varepsilon\eta + h)$ are the volume flux component in the $x-$ and $y-$direction, respectively. From the leading order terms in the momentum equations we obtain

$$\frac{\partial^3 P}{\partial x^2 \partial t} + \frac{\partial^3 Q}{\partial x \partial y \partial t} + h\left(\frac{\partial^3 \eta}{\partial x^3} + \frac{\partial^3 \eta}{\partial x \partial y^2}\right) = 0(\varepsilon, \mu^2) \tag{4.34}$$

$$\frac{\partial^3 Q}{\partial y^2 \partial t} + \frac{\partial^3 P}{\partial x \partial y \partial t} + h\left(\frac{\partial^3 \eta}{\partial y^3} + \frac{\partial^3 \eta}{\partial x^2 \partial y}\right) = 0(\varepsilon, \mu^2) \tag{4.35}$$

Madsen *et al.* (1991) argued that because the above quantities are in the same order of magnitude as the truncation errors in the Boussinesq equations, one can add a portion of these quantities into the Boussinesq equations without affecting the accuracy of the resulting equations. Hence, they multiplied (4.34) and (4.35) by $-\mu^2 B h^2$ and added them to the momentum equations (4.32) and (4.33), respectively. The resulting model equation becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \tag{4.36}$$

$$\frac{\partial P}{\partial t} + \varepsilon\frac{\partial}{\partial x}\left(\frac{P^2}{d}\right) + \varepsilon\frac{\partial}{\partial y}\left(\frac{PQ}{d}\right) + d\frac{\partial \eta}{\partial x}$$

$$-\mu^2\left(B + \frac{1}{3}\right)h^2\left(\frac{\partial^3 P}{\partial x^2 \partial t} + \frac{\partial^3 Q}{\partial x \partial y \partial t}\right) - \mu^2 B h^3\left(\frac{\partial^3 \eta}{\partial x^3} + \frac{\partial^3 \eta}{\partial x \partial y^2}\right) = 0 \tag{4.37}$$

$$\frac{\partial Q}{\partial t} + \varepsilon\frac{\partial}{\partial x}\left(\frac{PQ}{d}\right) + \varepsilon\frac{\partial}{\partial y}\left(\frac{Q^2}{d}\right) + d\frac{\partial \eta}{\partial y}$$

$$-\mu^2\left(B + \frac{1}{3}\right)h^2\left(\frac{\partial^3 Q}{\partial y^2 \partial t} + \frac{\partial^3 P}{\partial x \partial y \partial t}\right) - \mu^2 B h^3\left(\frac{\partial^3 \eta}{\partial y^3} + \frac{\partial^3 \eta}{\partial x^2 \partial y}\right) = 0 \tag{4.38}$$

where B is an empirical coefficient. The linear dispersion relation for the above equations can be written as

$$\frac{\omega_o}{\omega_o^s} = \sqrt{\frac{kh}{\tanh kh}} \left[\frac{1 + Bk^2h^2}{1 + (B + \frac{1}{3})k^2h^2}\right]^{1/2} \tag{4.39}$$

Comparing (4.39) and (4.4), we find $B = -(\alpha + 1/3)$. Madsen *et al.* suggested that choosing the value $B = 1/21$ (or $\alpha = -8/21 = -0.381$) leads to phase velocity errors less than 3 % for the entire range $0 < h/\lambda_o < 0.75$ and to group velocity errors of less than 6% for the range $0 < h/\lambda_o < 0.55$, where λ_o is the wave length in deep water. The frequency ratio, (4.39), is plotted in Figure 1 and is very close to the curve created by using the optimal α value, -0.3855.

5. Concluding Remarks

Nonlinear shallow water equations are derived based on the assumption that the frequency parameter, μ, is small, while the nonlinearity parameter, ϵ, is an order one quantity. The Boussinesq equations become a subset of the nonlinear shallow water equations, i.e., $0(\varepsilon) \approx 0(\mu^2)$. It is shown that the Boussinesq equations can take on several different forms with the same order magnitude of accuracy. However, some of these equations are unstable in the range of short waves, which not only limits the applications of these Boussinesq-type equations to very shallow water, but also makes these equations vulnerable to any numerical disturbances. Several different methods for extending the shallow water equations to deep water are discussed.

Based on analyses for the constant depth, the linear dispersion characteristics of the Boussinesq-type equations can be matched with those of the Stokes' waves in deep water by either using an appropriate velocity variable in the governing equations, or adding some of the higher derivative terms in the equations. A modified Hamiltonian has also been derived, which is always finite and positive. The associated modified Boussinesq equations also show significant improvements in the linear dispersion characteristics. However, all of these Boussinesq-type equations have very poor nonlinear dispersion characteristics in deep water. The nonlinear shallow water equations seem to give a better match with the Stokes' wave theory in deep water. Nevertheless, the second order frequency (amplitude dispersion) is still significantly under-estimated.

The future research efforts should continue on identifying the most suitable model equation describing wave propagation from deep water to shallow water. The nonlinearity needs to be properly included. Moreover, the issue concerning the vertical structure of velocity field in the deep water should also be addressed.

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