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August 11, 1994

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Abstract

The Hamiltonian for two-dimensional long waves over a slowly varying depth is derived. The vertical variation of the velocity field is obtained by using a perturbation method in terms of velocity potential. Employing the canonical theorem, the conventional Boussinesq equations are recovered. The Hamiltonian becomes negative when the wavelength becomes short. A modified Hamiltonian is constructed so that it remains positive and finite for short waves. The corresponding Boussinesq-type equations are then given.

[1] Assistant Professor, Department of Civil Engineering, Hanyang University, Ansan-Si, Korea 425-791

[2] Professor, School of Civil and Environmental Engineering, Cornell University, Ithaca, NY

1 Introduction

The development of accurate wave theories is essential for engineering applications in offshore and coastal environment. Outside of the surf zone, Boussinesq-type equations have been commonly used to describe weakly nonlinear and weakly dispersive waves in shallow water. The Boussinesq-type equations are derived based on the assumption that the nonlinearity represented by the amplitude to depth ratio, $\varepsilon = a/h$, is in the same order of magnitude as the frequency dispersion denoted by the depth to wave length ratio, $\mu^2 = (h/\lambda)^2$.

There are various methods for deriving the Boussinesq-type equations. One can integrate the continuity equation and the momentum equation over the entire water depth and apply the nonlinear free surface boundary conditions (e.g., [16], [20]). An alternative method is based on the Hamiltonian theory of surface waves (e.g., [3], [13], [14], [17], [18], [21]). The Hamiltonian is the total energy of the flow motion; the free surface displacement and the velocity potential on the free surface are canonical variables in the Hamilton's sense. One of the important tasks in the Hamiltonian approach is to determine the vertical distribution of the velocity field.

Broer and his associates have worked extensively on the derivation of Boussinesq-type long wave equations based on the Hamiltonian principle (e.g., [2], [3], [4], [5]). Collectively, they have obtained Hamiltonians for one-dimensional long waves over a constant or a very slowly varying depth. Benjamin [1] gave the Hamiltonian for two-dimensional long waves over a constant water depth. The Hamiltonian for two-dimensional long waves over a slowly

varying topography is still unknown.

Broer and his associates further demonstrate that Hamiltonians become negative in the range of short waves and the corresponding Boussinesq-type equations are unstable. One might argue that since these short waves are outside the range of validity of the Boussinesq regime, it should not cause any concern. However, in practical applications, the Boussinesq-type equations are solved numerically and numerical errors contain disturbances with the wavelength which is twice of the numerical mesh size. These numerically generated waves are short waves as far as the Boussinesq-type system is concerned and could cause instability. Broer and his associates have suggested several ways to modify the Hamiltonians (by adding some higher order terms) such that the system becomes stable even in the range of short waves (also in [12], [19]). Specific examples have been given to cases of constant depth.

In this paper we would like to achieve two goals. First, the Hamiltonian for two-dimensional long waves over a slowly varying depth will be derived. Using the canonical theorem, the conventional Boussinesq equations are recovered. The second objective of the paper is to modify the Hamiltonian such that it remains positive and finite for short waves. The corresponding modified Boussinesq-type equations are then obtained.

2 The Governing Equations and the Canonical Theorem

In this section we first summarize the governing equation and boundary conditions for irrotational gravity waves propagating in a layer of inviscid fluid. Denoting $\vec{x}'' = (x', y')$ as the horizontal coordinates, and z' the vertical coordinate, the flow domain is bounded by a free surface, $z' = \eta'(\vec{x}'', t')$, and a solid bottom, $z' = -h'(\vec{x}'')$. Introducing the characteristic wavelength, $(k')^{-1}$ as the horizontal length scale,, the characteristic depth, h'_o, as the vertical length scale, and $(k'\sqrt{gh'_o})^{-1}$ as the time scale, the following dimensionless variables can be defined:

$$\vec{x} = k'\vec{x}'' \quad , \quad z = z'/h'_o \quad , \quad h = h'/h'_o$$

$$\eta = \eta'/a'_o \quad , \quad \Phi = \frac{k'\sqrt{gh'_o}}{ga'_o}\Phi' \quad , \quad t = k'\sqrt{gh'_o}\, t' \tag{2.1}$$

in which a'_o denotes the characteristic amplitude of wave motions, g the gravitational acceleration, and Φ the dimensionless velocity potential. The dimensionless continuity equation and boundary conditions are in the following form:

$$\beta\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0, -h < z < \varepsilon\eta \tag{2.2}$$

4

$$\beta \nabla h \bullet \nabla \Phi + \frac{\partial \Phi}{\partial z} = 0, z = -h \tag{2.3}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \varepsilon [(\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial \Phi}{\partial y})^2 + \frac{1}{\beta} (\frac{\partial \Phi}{\partial z})^2] + \eta = 0, z = \varepsilon \eta \tag{2.4}$$

$$\frac{\partial \eta}{\partial t} + \varepsilon \nabla \eta \bullet \nabla \Phi = \frac{1}{\beta} \frac{\partial \Phi}{\partial z}, \; z = \varepsilon \eta \tag{2.5}$$

where

$$\beta = (k'h'_o)^2 \tag{2.6}$$

and

$$\varepsilon = \frac{a'_o}{h'_o} \tag{2.7}$$

are parameters denoting frequency dispersion and nonlinearity. In (2.3) and (2.5) $\nabla = (\partial/\partial x, \partial/\partial y)$ represents the gradient vector on the horizontal plane.

The total energy of the fluid can be written in the dimensionless form:

$$H = \frac{1}{2} \int \int_\Omega \int_{-h}^{\varepsilon \eta} [(\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial \Phi}{\partial y})^2 + \frac{1}{\beta} (\frac{\partial \Phi}{\partial z})^2] \, dz \, dxdy + \frac{1}{2} \int \int_\Omega \eta^2 \, dxdy \tag{2.8}$$

The total energy (or the Hamiltonian) has been normalized by a factor $\rho g a_o l'^2 / k l'^2$. The canonical theorem states that the free surface boundary conditions, (2.4) and (2.5), are equivalent to the following canonical equation [2].

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \phi}, \; \frac{\partial \phi}{\partial t} = -\frac{\delta H}{\delta \eta} \tag{2.9}$$

5

in which ϕ is the potential function on the free surface, i.e.,

$$\phi(\vec{x}, t) = \Phi(\vec{x}, \eta(\vec{x}, t), t) \qquad (2.10)$$

To evaluate the canonical equations the velocity potential Φ in the Hamiltonian, (2.8), must be written in terms of the surface potential value, ϕ. In the following sections, approximated Hamiltonians will be found based on the Boussinesq assumptions, i.e., $0(\varepsilon) = 0(\beta) \ll 1$.

3   An Approximated Hamiltonian

The total energy (Hamiltonian), (2.8), can be written as the sum of kinetic energy, E_k, and the potential energy, E_p, i.e.,

$$H = E_k + E_p \qquad (3.1)$$

where

$$E_k = E_{ko} + E_{k\eta} \qquad (3.2a)$$

with

$$E_{ko} = \frac{1}{2} \int \int_{\Omega} \int_{-h}^{0} [(\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial \Phi}{\partial y})^2 + \frac{1}{\beta}(\frac{\partial \Phi}{\partial z})^2] \, dz \, dxdy \qquad (3.2b)$$

6

$$E_{k\eta} = \frac{1}{2} \int \int_\Omega \int_0^{\varepsilon\eta} [(\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2 + \frac{1}{\beta}(\frac{\partial\Phi}{\partial z})^2] \, dz \, dxdy \qquad (3.2c)$$

$$E_p = \frac{1}{2} \int \int_\Omega \eta^2 \, dxdy \qquad (3.2d)$$

Applying Green's theorem to the right-hand side of (3.2b) ([7], [9]), we obtain

$$E_{ko} = \frac{1}{2} \int \int_\Omega [\frac{1}{\beta}(\Phi \frac{\partial\Phi}{\partial z})]_{z=0} \, dxdy \qquad (3.3a)$$

where the integrand is evaluated on the still-water level, z=0. Using the Taylor's series expansion, we approximate (3.2c) as

$$E_{k\eta} = \frac{1}{2} \int \int_\Omega \varepsilon\eta \, [(\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2 + \frac{1}{\beta}(\frac{\partial\Phi}{\partial z})^2]_{z=0} \, dxdy + 0(\varepsilon^2) \qquad (3.3b)$$

We need to replace Φ and its derivatives in (3.3) by the potential value at z=0.

Because $\Phi(\vec{x}, z, t)$ is an analytical function, we can express it in a power series form:

$$\Phi(\vec{x}, z, t) = \sum_{n=0}^{\infty} (z + h)^n \, \phi^{(n)}(\vec{x}, t) \qquad (3.4)$$

Note that (z+h) presents the vertical distance measured from the bottom; $\phi^{(0)}$ denotes the potential value at the bottom. The solution form given in (3.4) was first introduced by Lin and Clark [10] in studying the shallow-water wave theory. Liu and Earickson [11] have applied the similar approach to study the generation and propagation of tsunamis with a moving seafloor. Substituting (3.4) into the continuity equation, (2.2), and the bottom boundary condition, (2.3), we obtain the following recursive relations among $\phi^{(n)}$,

7

$$\phi^{(1)} = -\frac{\beta \nabla h \bullet \nabla \phi^{(0)}}{1 + \beta |\nabla h|^2} \tag{3.5a}$$

$$\phi^{(n+2)} = -\frac{\beta[\nabla^2 \phi^{(n)} + 2(n+1)\nabla h \bullet \nabla \phi^{(n+1)} + (n+1)\nabla^2 h \phi^{(n+1)}]}{(n+1)(n+2)(1 + \beta|\nabla h|^2)} \tag{3.5b}$$

where n = 0, 1, 2, ... Using the recursive relations, we can express $\phi^{(n)}$ in terms of $\phi^{(0)}$, which is the velocity potential on the bottom, z = -h. For completeness, the approximate formulas for $\phi^{(n)}$ (n = 1, 2, 3, and 4) are listed in Appendix A.

The velocity potential at the still water surface, z = 0, can be readily obtained from (3.4) as

$$\phi = \Phi(\vec{x}, z = 0, t) = \sum_{n=0}^{\infty} h^n \phi^{(n)}(\vec{x}, t)$$

Substituting the formulas for $\phi^{(n)}$ given in the appendix, we find

$$\phi = Q \phi^{(0)} + 0(\beta^2) \tag{3.6}$$

where Q is a pseudo-differential operator defined as

$$Q = 1 - \beta h \nabla h \bullet \nabla - \frac{\beta}{2} h^2 \nabla^2 + 0(\beta^2) \tag{3.7}$$

We can rewrite equation (3.6) as

$$\phi^{(0)} = Q^{-1}\phi + 0(\beta^2) \tag{3.8}$$

8

with

$$Q^{-1} = 1 + \beta h \, \nabla h \bullet \nabla + \frac{\beta}{2} \, h^2 \nabla^2 + 0(\beta^2) \tag{3.9}$$

To evaluate the integrand in (3.3a), we first express $\partial \Phi / \partial z$ at $z = 0$ in the following form:

$$\frac{\partial \Phi}{\partial z} \Big|_{z=0} = \sum_{n=0}^{\infty} (n+1) \, h^n \, \phi^{(n+1)} = S\phi^{(0)} + 0(\beta^3) \tag{3.10}$$

where

$$S = -\beta(1 - \beta|\nabla h|^2) \, \nabla h \bullet \nabla - \beta h(1 - \beta|\nabla h|^2)\nabla^2 + 2\beta^2 h \nabla h \bullet \nabla(\nabla h \bullet \nabla)$$

$$+ \beta^2 h \, \nabla^2 h \, (\nabla h \bullet \nabla) + \beta^2 h^2 [\frac{1}{2}\nabla^2(\nabla h \bullet \nabla) + \nabla h \bullet \nabla(\nabla^2) + \frac{1}{2} \, \nabla^2 h \, \nabla^2]$$

$$+ \frac{1}{6} \, \beta^2 h^3 \nabla^2 \nabla^2 \tag{3.11}$$

Substituting (3.8) into (3.10), we obtain

$$\frac{\partial \Phi}{\partial z} \Big|_{z=0} = SQ^{-1}\phi + 0(\beta^3) \tag{3.12}$$

Substitutions of (3.12) and (3.4) into (3.3a) yield, after some lengthy manipulations,

9

$$E_{ko} = \frac{1}{2\beta} \int \int_{\Omega} \phi \, SQ^{-1} \phi \, dxdy = \int_{\Omega} \int \left[h(\nabla\phi)^2 + \frac{\beta h^2}{2} \nabla\phi \bullet \nabla (\nabla h \bullet \nabla\phi) \right.$$

$$\left. + \frac{\beta h^2}{2} \nabla^2\phi \, (\nabla h \bullet \nabla\phi) + \frac{\beta h^3}{3} \nabla^2 (\nabla\phi) \bullet \nabla\phi \right] dxdy \, + \, 0(\beta^2)$$

$$= \frac{1}{2} \int \int_{\Omega} \nabla\phi \bullet (hR\nabla\phi) \, dxdy \, + \, 0(\beta^2) \tag{3.13}$$

in which hR is a symmetric (self-adjoint) tensor operator, where R is defined by

$$R = 1 + \beta h \left[\frac{1}{2} \nabla (\nabla h\bullet) + \frac{1}{2} \nabla h \, \nabla \bullet + \frac{h}{3} \nabla^2 \right] \tag{3.14}$$

or

$$R = \begin{bmatrix} 1 + \beta a & \beta b \\ \\ \beta b & 1 + \beta c \end{bmatrix} \tag{3.15}$$

where

$$a = \frac{h}{2} \frac{\partial^2}{\partial x^2} h - \frac{h^2}{6} \frac{\partial^2}{\partial x^2} \tag{3.16a}$$

$$b = \frac{h}{2} \frac{\partial^2}{\partial x \partial y} h - \frac{h^2}{6} \frac{\partial^2}{\partial x \partial y} \tag{3.16b}$$

$$c = \frac{h}{2} \frac{\partial^2}{\partial y^2} h - \frac{h^2}{6} \frac{\partial^2}{\partial y^2} \tag{3.16c}$$

Mooiman [15] presented an operator R by the straightforward extension of one-dimensional result of Katopodes and Dingemans ([7], [8]) as:

$$R_m = \begin{bmatrix} 1 + \beta a & 0 \\ 0 & 1 + \beta a \end{bmatrix} \tag{3.17}$$

where

$$a = \frac{h}{2} \left(\frac{\partial^2}{\partial x^2} h + \frac{\partial^2}{\partial y^2} h \right) - \frac{h^2}{6} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{3.18}$$

Mooiman's choice of R_m leads to an erroneous Hamiltonian. This wil be discussed later.

Equation (3.3b) can be approximated as

$$E_{k\eta} = \frac{1}{2} \int \int_{\Omega} [\varepsilon \eta \, \nabla \phi \bullet \nabla \phi] \, dx dy + 0(\varepsilon^2, \, \varepsilon \beta) \tag{3.19}$$

because

$$\frac{\partial \Phi}{\partial x}|_{z=0} = \frac{\partial \phi}{\partial x} + 0(\beta), \quad \frac{\partial \Phi}{\partial y}|_{z=0} = \frac{\partial \phi}{\partial y} + 0(\beta), \quad \frac{\partial \Phi}{\partial z}|_{z=0} = 0(\beta) \tag{3.20}$$

Finally, an approximate Hamiltonian for two-dimensional waves over a slowly varying depth can be written as

$$H = \frac{1}{2} \int \int_{\Omega} [\nabla \phi \bullet (h \, R \, \nabla \phi) + \varepsilon \eta \, \nabla \phi \bullet \nabla \phi + \eta^2] \, dx dy + 0(\varepsilon^2, \, \varepsilon \beta, \, \beta^2) \tag{3.21}$$

The Hamiltonian given in (3.21) can be simplified to several special cases. For the case of a constant depth, h = 1, the operator R with (3.16) becomes

11

$$R = \begin{bmatrix} 1 + \frac{\beta}{3}\nabla^2 & 0 \\ 0 & 1 + \frac{\beta}{3}\nabla^2 \end{bmatrix} \tag{3.22}$$

and the Hamiltonian (3.21) becomes

$$H = \frac{1}{2}\int\int_\Omega [(1 + \varepsilon\eta)(\nabla\phi \bullet \nabla\phi) - \frac{1}{3}\beta(\nabla^2\phi)^2 + \eta^2]\,dxdy + 0(\varepsilon^2) \tag{3.23}$$

which is the same as that given in Benjamin [1]. For the case of unidirectional waves over a slowly varying depth, the Hamiltonian, (3.21), can be simplified to be

$$H = \frac{1}{2}\int [\frac{\partial\phi}{\partial x}hR\frac{\partial\Phi}{\partial x} + \varepsilon\eta(\frac{\partial\Phi}{\partial x})^2 + \eta^2]\,dx + 0(\varepsilon^2, \varepsilon\beta, \beta^2) \tag{3.24}$$

with

$$R = 1 + \frac{\beta h}{2}\frac{\partial^2}{\partial x^2}h - \frac{\beta h^2}{6}\frac{\partial^2}{\partial x^2} \tag{3.25}$$

which has been obtained by Katopodes and Dingemans ([7] and [8]).

The conventional Boussinesq equations in two-dimension over variable depth can be obtained using (3.21), (3.16) and (2.9) as:

$$\frac{\partial\eta}{\partial t} = -\nabla \bullet (hR\nabla\phi) - \nabla \bullet (\varepsilon\eta\nabla\phi) = -\nabla \bullet [(h + \varepsilon\eta)\vec{u}]$$

$$-\frac{1}{2}\beta\nabla \bullet [h^2\nabla(\nabla \bullet (h\vec{u})) - \frac{h^3}{3}\nabla(\nabla \bullet \vec{u})] \tag{3.26}$$

for continuity equation, and

12

$$\frac{\partial \phi}{\partial t} = -\frac{\varepsilon}{2} (\nabla \phi)^2 - \eta \tag{3.27}$$

for momentum equation.

Taking the gradient of the momentum equation yields,

$$\frac{\partial \vec{u}}{\partial t} = -\varepsilon \vec{u} \bullet \nabla \vec{u} - \nabla \eta \tag{3.28}$$

where $\vec{u}$ is the velocity vector defined by $\nabla \phi$. The irrotationality has been employed in deriving (3.22). This set of Boussinesq equations, (3.26) and (3.28), can also be obtained directly using (2.4), (2.5) and (3.4) [6]. Mooiman's [15] operator R_m given by (3.17), however, does not give these classical Boussinesq equations.

4 Stable Boussinesq Equations

Broer ([2], [3] and [4]) and his associates [5] demonstrated that for the case of a constant depth Boussinesq equations derived from the Hamiltonian, (3.23), are unstable for short waves. These short waves could be introduced either numerically or physically into the system. The instability occurs because the Hamiltonian, (3.23), becomes negative for short waves. Many alternate approximations for the Hamiltonian have been suggested. Here, we follow the approach suggested by Broer et al. [5] and Mooiman [15] and seek for a positive definite Hamiltonian in a quadratic form.

First, the Hamiltonian (3.21) is modified as

$$H = \frac{1}{2} \int \int_{\Omega} [\nabla\phi \bullet (DR\nabla\phi) + \eta^2] \, dxdy + 0(\varepsilon^2, \, \varepsilon\beta, \, \beta^2) \tag{4.1}$$

where

$$R = 1 + 0(\beta), \quad \varepsilon\eta R = \varepsilon\eta + 0(\varepsilon\beta) \tag{4.2}$$

is used and D($=$h$+\varepsilon\eta$) is the total depth. The Hamiltonians, (3.21) and (4.1), have the same accuracy in terms of small parameters ε and β.

We now rewrite (4.1) in a quadratic form as

$$H = \frac{1}{2} \int \int_{\Omega} [D(F\nabla\phi)^2 + \eta^2] \, dxdy + 0(\varepsilon^2, \, \varepsilon\beta, \, \beta^2) \tag{4.3}$$

where F is another tensor operator to be found from the following relationship:

$$\nabla\phi \bullet (R\nabla\phi) = (F\nabla\phi) \bullet (F\nabla\phi) + 0(\beta)^2 \tag{4.4}$$

Using (3.14) and (3.15), we find (see Appendix B)

$$F = \begin{bmatrix} F_{11} & F_{22} \\ F_{21} & F_{12} \end{bmatrix} \tag{4.5}$$

$$F_{11} = \frac{1}{\sqrt{2}} [1 + \frac{\beta}{2}(a + b)] \tag{4.6a}$$

$$F_{12} = \frac{1}{\sqrt{2}} [1 + \frac{\beta}{2}(b + c)] \tag{4.6b}$$

14

$$F_{21} = -\frac{1}{\sqrt{2}} \left[1 + \frac{\beta}{2} (a - b) \right] \qquad (4.6c)$$

$$F_{22} = \frac{1}{\sqrt{2}} \left[1 - \frac{\beta}{2} (b - c) \right] \qquad (4.6d)$$

where a, b, and c are operators defined in (3.16). The Hamiltonian given in (4.3) is always positive and has the same accuracy as those given in (3.21) and (4.1). However, (4.3) becomes unbounded for short waves. We must further approximate the operator F in the following manner

$$F_{11}^* = \frac{1}{\sqrt{2}} \frac{1}{1 - \frac{\beta}{2} (a + b)} \qquad (4.7a)$$

$$F_{12}^* = \frac{1}{\sqrt{2}} \frac{1}{1 - \frac{\beta}{2} (b + c)} \qquad (4.7b)$$

$$F_{21}^* = -\frac{1}{\sqrt{2}} \frac{1}{1 - \frac{\beta}{2} (a - b)} \qquad (4.7c)$$

$$F_{22}^* = \frac{1}{\sqrt{2}} \frac{1}{1 + \frac{\beta}{2} (b - c)} \qquad (4.7d)$$

The canonical equations (2.9) gives the final form of stable Boussinesq-type equations:

$$\frac{\partial \eta}{\partial t} = -\nabla \bullet [F^T (DF \nabla \phi)]$$

$$= -\frac{\partial}{\partial x} [F_{11}^* (D(F_{11}^* u + F_{12}^* v)) + F_{21}^* (D(F_{21}^* u + F_{22}^* v))]$$

15

$$-\frac{\partial}{\partial y}\left[F_{12}^* \left(D(F_{11}^* u + F_{12}^* v)\right) + F_{22}^*(D(F_{21}^* u + F_{22}^* v))\right] \tag{4.8}$$

for continuity equation and

$$\frac{\partial \phi}{\partial t} = -\frac{\varepsilon}{2}\left(F\nabla\phi\right)^2 - \eta \tag{4.9}$$

for momentum equation.

Taking the gradient of the momentum equation, we obtain

$$\frac{\partial u}{\partial t} = -\varepsilon\left[(F_{11}^* u + F_{12}^* v)\frac{\partial}{\partial x}(F_{11}^* u + F_{12}^* v) + (F_{21}^* u + F_{22}^* v)\frac{\partial}{\partial x}\right. \tag{4.10}$$
$$\left. (F_{21}^* u + F_{22}^* v)\right] - \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\varepsilon\left[(F_{11}^* u + F_{12}^* v)\frac{\partial}{\partial y}(F_{11}^* u + F_{12}^* v) + (F_{21}^* u + F_{22}^* v)\frac{\partial}{\partial y}\right. \tag{4.11}$$
$$\left. (F_{21}^* u + F_{22}^* v)\right] - \frac{\partial \eta}{\partial y}$$

Equations (4.8), (4.10) and (4.11) constitute the modified Boussinesq equations which are stable subject to short wave disturbances.

For later uses, we present the modified Boussinesq equations for one-dimensional waves (v = 0 and $\partial/\partial y = 0$) over a constant depth. From (4.8) and (4.10), we get

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\left[F_{11}^* \left(D F_{11}^* u\right) + F_{21}^* \left(DF_{21}^* u\right)\right] \tag{4.12}$$

$$\frac{\partial u}{\partial t} = -\varepsilon\left[F_{11}^* u \frac{\partial}{\partial x}\left(F_{11}^* u\right) + F_{21}^* u \frac{\partial}{\partial x}(F_{21}^* u)\right] - \frac{\partial \eta}{\partial x} \tag{4.13}$$

16

where

$$F_{11}^* = -F_{21}^* = \frac{1}{\sqrt{2}} \frac{1}{1 - \frac{\beta h^2}{6} \frac{\partial^2}{\partial x^2}}$$ (4.14)

5 Phase velocities and linear instability

An effective way to demonstrate that the modified Boussinesq equations have a better stability characteristics and wider range of applicabilities in the short waves than the conventional Boussinesq equations is to examine the phase velocities corresponding to each equation. For simplicity, we consider the linear version of the equations for one horizontal dimension and constant depth. From the conventional Boussinesq equations (3.22) and (3.24), we get

$$\frac{\partial \eta}{\partial t} = -h \frac{\partial u}{\partial x} - \frac{\beta h^3}{3} \frac{\partial^3 u}{\partial x^3}$$ (5.1)

$$\frac{\partial u}{\partial t} = -\frac{\partial \eta}{\partial x}$$ (5.2)

On the other hand, from the modified Boussinesq equations (4.12) and (4.13), we get

$$(1 - \frac{\beta h^2}{6} \frac{\partial^2}{\partial x^2})^2 \frac{\partial \eta}{\partial t} = -h \frac{\partial u}{\partial x}$$ (5.3)

$$\frac{\partial u}{\partial t} = -\frac{\partial \eta}{\partial x}$$ (5.4)

17

Consider a small amplitude periodic disturbance with frequency ω and wave number k,

$$\eta = A \, exp[i \, (kx - \omega t)] \tag{5.5}$$

$$u = U \, exp[i \, (kx - \omega t)] \tag{5.6}$$

Substituting (5.5) and (5.6) into (5.1) and (5.2), we get the following well-known dispersion relation

$$C_{CB}^2 = \frac{\omega^2}{k^2} = h \left[1 - \frac{1}{3} \beta (kh)^2 \right] \tag{5.7}$$

for the conventional Boussinesq equation. Following the same procedure, we find the dispersion relation based on the modified Boussinesq equation, (5.3) and (5.4) as:

$$C_{MB}^2 = \frac{\omega^2}{k^2} = h \left[\frac{1}{1 + \frac{1}{6} \beta (kh)^2} \right]^2 \tag{5.8}$$

In terms of dimensional variables, these dispersion relations read

$$C_{CB}^2 = \frac{\omega^2}{k^2} = gh \left[1 - \frac{1}{3} (kh)^2 \right] \tag{5.9}$$

$$C_{MB}^2 = \frac{\omega^2}{k^2} = gh \left[\frac{1}{1 + \frac{1}{6} (kh)^2} \right]^2 \tag{5.10}$$

We note that (5.9) becomes negative when kh $> \sqrt{3}$. It implies that the phase velocity or the frequency becomes imaginary for the conventional Boussinesq equations. Therefore, the

18

disturbance grows exponentially in time. On the other hand, the dispersion relation for the modified Boussinesq equation is always positive. Therefore, the solution is always stable.

To compare further the range of applicability for the conventional Boussinesq equations and the modified Boussinesq equations in terms of the relative water depth, we calculate the phase velocities, normalized by the phase velocity based on the linear Airy theory,

$$C_{Airy} = \sqrt{gh} \sqrt{\frac{tanhkh}{kh}} \tag{5.11}$$

for different relative depth. The results are shown in Table 1. The relative depth is defined as the ratio of the depth, h, to the deep water wavelength $\lambda_o = 2\pi g/\omega^2$. The deep water depth limit corresponds to $h/\lambda_o \approx 0.5$. The phase velocity associated with the modified Boussinesq equation is roughly within 10% of that associated with Airy theory up to kh $\approx$ 0.45. The conventional Boussinesq equations degenerate quickly around kh = 0.2.

6   Concluding Remarks

Two major tasks have been accomplished in this paper. First, a Hamiltonian for two-dimensional long waves over a slowly varying depth is derived. Using the canonical theorem the conventional Boussinesq equations are recovered. Because the Hamiltonian becomes negative infinity when the wavelength becomes short, the conventional Boussinesq equations become unstable in the short wave range (or deep water). Therefore, the conventional Boussinesq equations are susceptible to short waves generated by numerical errors. To improve the

19

Table 1: Comparison of phase velocities derived from the Conventional Boussinesq equations, the Modified Boussinesq equations and the Airy Theory

k/λ_o	kh	C_{CB}/C_{Airy}	C_{MB}/C_{Airy}
0.01	0.253	1.001	1.000
0.02	0.362	0.998	1.000
0.04	0.523	0.994	0.998
0.06	0.655	0.987	0.996
0.08	0.774	0.976	0.993
0.10	0.886	0.960	0.988
0.15	1.152	0.886	0.972
0.20	1.414	0.728	0.946
0.25	1.683	0.317	0.912
0.30	1.961	—	0.870
0.35	2.249	—	0.823
0.40	2.545	—	0.772
0.45	2.847	—	0.720
0.50	3.141	—	0.671

characteristics of the Boussinesq equations, the second task is to derive a modified Hamiltonian. The corresponding modified Boussinesq equations contain many higher derivatives. For the one-dimensional constant depth case, the phase velocity derived from the linearized modified Boussinesq equations is shown to be a real and finite number. Therefore, the modified Boussinesq equations are stable for a wide range of water depth. For the general two-dimensional problems, an accurate numerical scheme needs to be developed for these equations.

Acknowledgement

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Appendix A

From the recursive relations, (3.5), the following approximate formulas can be derived:

$$\phi^{(1)} = -\beta \left(1 - \beta|\nabla h|^2\right) \nabla h \bullet \nabla \phi^{(0)} + 0(\beta^3) \tag{A.1}$$

$$\phi^{(2)} = -\frac{\beta}{2} \left(1 - \beta|\nabla h|^2\right) \nabla^2 \phi^{(0)} + \beta^2 \nabla h \bullet \nabla \left(\nabla h \bullet \nabla \phi^{(0)}\right)$$

$$+ \frac{\beta^2}{2} \nabla^2 h \left(\nabla h \bullet \nabla \phi^{(0)}\right) + 0(\beta^3) \tag{A.2}$$

$$\phi^{(3)} = \beta^2 \left[\frac{1}{6} \nabla^2 \left(\nabla h \bullet \nabla \phi^{(0)}\right) + \frac{1}{3} \nabla h \bullet \nabla \left(\nabla^2 \phi^{(0)}\right)\right.$$

$$\left. + \frac{1}{6} \nabla^2 h \, \nabla^2 \phi^{(0)}\right] + 0(\beta^3) \tag{A.3}$$

$$\phi^{(4)} = \frac{\beta^2}{24} \nabla^2 \left(\nabla^2 \phi^{(0)}\right) + 0(\beta^3) \tag{A.4}$$

Appendix B

Using the expression given in (3.14) for the operator R, we can rewrite the first integrand of (4.1) as

$$\nabla \phi \bullet (R \, \nabla \phi) = [u \, v] \begin{bmatrix} 1 + \beta a & 1 + \beta b \\ -1 + \beta b & 1 + \beta c \end{bmatrix} \left\{ {u \atop v} \right\}$$

$$= u(1 + \beta a)u + u(1 + \beta b)v + v(-1 + \beta b)u + v(1 + \beta c)v \qquad (B.1)$$

where u and v are the x- and y- component of velocity vector. Note that the operator R has been slightly modified for convenience. We assume that the operator F has the following form:

$$F = \begin{bmatrix} A + \beta I & B + \beta J \\ C + \beta K & D + \beta L \end{bmatrix} \qquad (B.2)$$

in which A, B, C, and D are constants and I, J, K, and L are differential operators. We can determine these unknown quantities by satisfying (4.4). The right-hand side of (4.4) can be expressed approximately as

$$(F\nabla\phi) \bullet (F\nabla\phi) = u[A^2 + C^2 + 2\beta \, (AI + CK)]u + 2u[AB + \beta(AJ + CL)]v$$

$$+ 2v[DC + \beta \, (DK + BI)]u + v[B^2 + D^2 + 2\beta(BJ + DL)]v + 0(\beta^2) \qquad (B.3)$$

Equating (B.1) and (B.3), we obtain the following set of equations

$$A^2 + C^2 = 1,$$

26

$$AB = \frac{1}{2}$$

$$CD = -\frac{1}{2}$$

$$B^2 + D^2 = 1,$$

$$AI + CK = \frac{a}{2}$$

$$AJ + CL = \frac{b}{2}$$

$$DK + BI = \frac{b}{2}$$

$$DL + BJ = \frac{c}{2}$$

which leads to the solution

$$A = B = D = -C = \frac{1}{\sqrt{2}} \qquad (B.4a)$$

$$I = \frac{1}{2\sqrt{2}}(a+b), \quad J = \frac{1}{2\sqrt{2}}(b+c) \qquad (B.4b)$$

$$K = -\frac{1}{2\sqrt{2}}(a-b), \quad L = -\frac{1}{2\sqrt{2}}(b-c) \qquad (B.4c)$$

The operator F can be written as

$$F = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 + \frac{\beta}{2}(a+b) & 1 + \frac{\beta}{2}(b+c) \\ -1 - \frac{\beta}{2}(a-b) & 1 - \frac{\beta}{2}(b-c) \end{bmatrix} \qquad (B.5)$$