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Abstract

In this paper, we investigate effects of a weak and slowly varying random topography on interfacial long wave propagation near the critical depth level, where the cubic nonlinearity is comparable to the quadratic nonlinearity. The evolution equation is derived from the Euler equations for two fluid layers. This equation is completely integrable and can be transformed into the modified KdV equation. Hirota's method is used to find two-soliton and soliton-shock-like solutions. For a steady wave propagating over a random topography with zero-mean Gaussian distribution, all its moments satisfy the same diffusion equation. The randomness of the topography causes an averaged solitary wave to deform into a spreading Gaussian wavepacket with its height decreasing and width increasing at the same rate determined by the correlation function of the topography. The front of an averaged shock wave also increases at the same rate. Asymptotic behaviours of an averaged two-solitary wave and solitary-shock wave are also discussed and the results are generalized to an averaged N-solitary wave and N-solitary-shock wave.

1 Introduction

Nonlinear wave propagations in randomly inhomogeneous media or under random excitation have attracted much attention in the last decade, because of their wide range of applications. However, due to nonlinearity, it is in general impossible to obtain closed equations for the mean field or its higher-order moments directly from a nonlinear stochastic wave equation. Tremendous computational demand also prevents one from performing direct numerical analysis of nonlinear stochastic wave equations, except for very simple cases. Thus, hypotheses or approximate methods have to be used to close the equations for moments of the wave field. This renders the study

of exactly solvable nonlinear stochastic wave equations very important, because not only equations of this type are analytically tractable but also their solutions can be used to verify the applicability and accuracy of various hypotheses and approximate methods adopted to derive the closed equations.

It is well known that the Korteweg-de Vires (KdV) equation is the prototype equation describing the evolution of nonlinear waves propagating in a dispersive medium, whereas the Burgers equation is the simplest equation governing the propagation of nonlinear waves with dissipation. Both equations are exactly solvable (by the inverse scattering transform and Hopf–Cole transformation, respectively) and admit steady-wave solutions such as solitary wave and shock wave [1]. It is of great interest to study how these steady waves deform when they propagate in randomly inhomogeneous media or under stochastic excitation.

Although there have been several studies of the KdV equation and Burgers equation under external random forcing (e.g. see [2]–[8]), the study of the KdV equation and Burgers equation with random coefficients, which appear as the governing equations for the propagation of solitary waves and shock waves in random media or over randomly varying boundaries, has just started recently. Wadati [9] studied the deformation of a soliton propagating in nonlinear lattices with random mass distribution. Under the Gaussian white noise assumption, he showed that an averaged soliton deforms into a spreading Gaussian wavepacket with decreasing height and increasing width proportional to the square root of the distance of propagation. Applying Wadati's analysis to shallow-water waves, Ono [10] investigated the effect of a weak and gentle random topography on the propagation of a surface solitary wave. Assuming that the topography is a Gaussian white noise process, he confirmed that the soliton diffusion phenomenon found by Wadati also exists in shallow-water waves. Ono also examined the effect of randomness on the propagation of a Burgers shock wave by

studying wave propagation in a duct with varying cross-section and found that the front of an averaged Burgers shock wave also diffuses in proportional to the square root of the distance of propagation under the assumption that the small variation of the cross-section is a Gaussian white noise process.

In parametric regime where the cubic nonlinearity and the quadratic nonlinearity are comparable, the governing equation for the propagation of interfacial long waves in two-layer fluids or internal long waves in a continuously stratified fluid over a flat bottom is a combined KdV and modified KdV equation with negative cubic nonlinearity. This evolution equation admits a family of solitary-wave solutions and a shock-like solution [11, 12, 13]. However, the effects of a random topography on the propagation of these internal solitary waves and shock wave have not been studied. Moreover, it is not clear if a random topography has different effect on cubic nonlinear waves than on quadratic nonlinear waves. Therefore, it is the objective of this paper to investigate the effect of a random topography on the propagation of interfacial waves and internal waves, in which the cubic nonlinearity is significant. Since the governing equations for the propagation of weakly nonlinear, long internal waves and corresponding interfacial waves in the same setting are equivalent under a rescaling (in fact, internal waves include interfacial waves as a limited case), for simplicity, we shall study the random effect on the propagation of interfacial waves only in this paper. However, the conclusions drawn on interfacial waves are also applied to internal waves.

Helfrich, Melville & Miles [13] have derived the evolution equation for interfacial waves propagating over a finite but slowly varying topography. However, their equation cannot pass the Painlevé PDE test (see [14]), i.e. the equation is not completely integrable. Consequently, when it is used to study the effect of a random topography on wave propagation, the resulting stochastic equation is analytically intractable. To

make the problem analytically tractable, the topography is assumed to be weak and slowly varying in this paper. When the resulting evolution equation is used to study the effect of randomness, the topography is further assumed to be a stationary Gaussian process with an arbitrary correlation function, which includes the Gaussian white noise used in Wadati and Ono's papers [9, 10] as a special case. In addition to averaged steady waves such as solitary wave and shock-like wave, we also investigate the evolution of an averaged two-solitary wave and an averaged solitary-shock wave, and extend the results to an averaged N-solitary wave and an averaged N-solitary-shock wave.

2 Derivation of the evolution equation

We consider interfacial waves propagating along the interface of two fluid layers. Cartesian coordinates are employed with the $\tilde{x}$-axis along the still interface and $\tilde{z}$-axis pointing vertically upwards. The disturbed interface is denoted by $\tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t})$. The densities of the upper and lower layers are $\tilde{\rho}^+$ and $\tilde{\rho}^-$ ($\tilde{\rho}^- > \tilde{\rho}^+$), respectively (hereafter superscripts $+$ and $-$ are used to identify quantities in the upper and lower layers, respectively). The upper and lower layers are bounded by $\tilde{z} = \tilde{H}^+$ (the rigid-lid assumption is adopted to approximate the free surface) and $\tilde{z} = \tilde{H}^-(\tilde{x})$, respectively.

The fluids in both layers are assumed to be inviscid and incompressible. The dimensionless governing equations and boundary conditions for flows in the upper and lower layers are

$$\frac{\partial u^\pm}{\partial x} + \frac{\partial w^\pm}{\partial z} = 0, \tag{2.1a}$$

$$\frac{\partial u^\pm}{\partial t} + \epsilon \left(u^\pm \frac{\partial u^\pm}{\partial x} + w^\pm \frac{\partial u^\pm}{\partial z} \right) = -\frac{\partial p^\pm}{\partial x}, \tag{2.1b}$$

$$\mu^2 \left[\frac{\partial w^\pm}{\partial t} + \epsilon \left(u^\pm \frac{\partial w^\pm}{\partial x} + w^\pm \frac{\partial w^\pm}{\partial z} \right) \right] = -\frac{\partial p^\pm}{\partial z}, \tag{2.1c}$$

$$w^{\pm} = \frac{\partial \eta}{\partial t} + \epsilon u^{\pm} \frac{\partial \eta}{\partial x} \qquad \text{on} \quad z = \epsilon\eta, \tag{2.1d}$$

$$\rho^{+} p^{+} - \rho^{-} p^{-} + \eta = 0 \qquad \text{on} \quad z = \epsilon\eta, \tag{2.1e}$$

$$w^{+} = 0 \qquad \text{on} \quad z = H^{+}, \tag{2.1f}$$

$$w^{-} = -u^{-} \frac{\mathrm{d}H^{-}}{\mathrm{d}x} \qquad \text{on} \quad z = -H^{-}(x), \tag{2.1g}$$

where signs are vertically ordered; $u^{\pm}$ and $w^{\pm}$ are the dimensionless velocity components in the horizontal and vertical direction respectively, and $p^{\pm}$ is the dimensionless hydrodynamic pressure. The dimensionless variables are related to the dimensional variables (denoted by a tilde) by:

$$\left.\begin{aligned} \tilde{x} = l_0 x, \qquad \tilde{z} = h_0 z, \qquad \tilde{t} = \frac{l_0}{c_0} t, \qquad \tilde{H}^{\pm} = h_0 H^{\pm}, \\ \tilde{\eta} = a_0 \eta, \qquad \tilde{u}^{\pm} = \frac{a_0 c_0}{h_0} u^{\pm}, \qquad \tilde{w}^{\pm} = \frac{a_0 c_0}{l_0} w^{\pm}, \\ \tilde{\rho}^{\pm} = \rho_0 \rho^{\pm}, \qquad c_0^2 = (\rho^{-} - \rho^{+}) g h_0, \qquad \tilde{p}^{\pm} = \frac{a_0}{h_0} c_0^2 \rho_0 \rho^{\pm} p^{\pm}, \end{aligned}\right\} \tag{2.2}$$

where g is the gravitational acceleration; l_0 and h_0 are the characteristic wavelength and depth, respectively; ρ_0 is the characteristic density; a_0 and c_0 are the characteristic amplitude and phase velocity of linear long waves, respectively. The parameters ϵ and μ^2 appearing in (2.1) are defined as

$$\epsilon = a_0/h_0, \qquad \mu^2 = (h_0/l_0)^2, \tag{2.3}$$

which measure nonlinearity and dispersion respectively.

We are interested in weakly nonlinear and dispersive waves, in which nonlinear and dispersion effects balance each other, leading to the formation of permanent waves, such as solitary waves. Under the Boussinesq assumption, i.e. $O(\epsilon) = O(\mu^2) \ll 1$, the nonlinear term in the evolution equation (e.g. see equation (2.14) in Chen & Liu's paper [15]) is quadratic and its coefficient is proportional to $D_{-2} = \rho^{-}/(h^{-})^2 - \rho^{+}/(h^{+})^2$, which vanishes at the critical depth level defined as $h^{+}/h^{-} = (\rho^{+}/\rho^{-})^{1/2}$. When $D_{-2} = O(\epsilon)$, the coefficient of the nonlinear term is so small that the balance between

the nonlinearity and dispersion becomes impossible under the Boussinesq assumption. However, in the parametric regime where $O(\epsilon) = O(\mu)$, the balance between these two effects is still possible. In this situation, the cubic nonlinearity becomes comparable to the quadratic nonlinearity and must be taken into consideration.

We shall derive the evolution equation for weakly nonlinear and dispersive interfacial waves propagating near the critical depth level over a weak and gentle topography. More specifically, we assume that

$$D_{-2} = \rho^-/(h^-)^2 - \rho^+/(h^+)^2 = O(\epsilon), \quad \epsilon = \alpha\mu, \quad \text{with } \mu \ll 1, \tag{2.4}$$

where $\alpha = O(1)$ is a constant, and H^- can be expressed as

$$H^- = h^- + \mu^2 B(\mu^2 x), \tag{2.5}$$

where h^- is a constant and $B = O(1)$ is a function of slow variable $X = \mu^2 x$. μ is used as the basic perturbation parameter.

To derive a single evolution equation for the interfacial displacement, we introduce the following transformation:

$$\xi = x/C - t, \qquad X = \mu^2 x, \qquad Z = z, \tag{2.6a}$$

where C is given by

$$C = \left(\rho^+/h^+ + \rho^-/h^-\right)^{-1/2} \tag{2.6b}$$

and $h^+ \equiv H^+ \equiv$ constant is used for convenience. Note that C is the leading order of the local linear-long-wave speed and $\xi = O(1)$ is the characteristic coordinate moving at the speed of C to the right.

In terms of the new coordinates (ξ, X, Z), equations and boundary conditions (2.1) can be rewritten as

$$\frac{1}{C}\frac{\partial u^\pm}{\partial \xi} + \mu^2 \frac{\partial u^\pm}{\partial X} + \frac{\partial w^\pm}{\partial Z} = 0, \tag{2.7a}$$

$$\frac{\partial u^{\pm}}{\partial \xi} - \alpha\mu\left[u^{\pm}\left(\frac{1}{C}\frac{\partial u^{\pm}}{\partial \xi} + \mu^2\frac{\partial u^{\pm}}{\partial X}\right) + w^{\pm}\frac{\partial u^{\pm}}{\partial Z}\right] = \frac{1}{C}\frac{\partial p^{\pm}}{\partial \xi} + \mu^2\frac{\partial p^{\pm}}{\partial X}, \tag{2.7b}$$

$$\mu^2\left\{\frac{\partial w^{\pm}}{\partial \xi} - \alpha\mu\left[u^{\pm}\left(\frac{1}{C}\frac{\partial w^{\pm}}{\partial \xi} + \mu^2\frac{\partial w^{\pm}}{\partial X}\right) + w^{\pm}\frac{\partial w^{\pm}}{\partial Z}\right]\right\} = \frac{\partial p^{\pm}}{\partial Z}, \tag{2.7c}$$

$$w^{\pm} = -\frac{\partial \eta}{\partial \xi} + \alpha\mu u^{\pm}\left(\frac{1}{C}\frac{\partial \eta}{\partial \xi} + \mu^2\frac{\partial \eta}{\partial X}\right) \quad \text{on} \quad Z = \epsilon\eta, \tag{2.7d}$$

$$\rho^{+}p^{+} - \rho^{-}p^{-} + \eta = 0 \quad \text{on} \quad Z = \epsilon\eta, \tag{2.7e}$$

$$w^{+} = 0 \quad \text{on} \quad Z = h^{+}, \tag{2.7f}$$

$$w^{-} = -\mu^4 u^{-}\frac{\mathrm{d}B}{\mathrm{d}X} \quad \text{on} \quad Z = -h^{-} - \mu^2 B, \tag{2.7g}$$

where (2.5) has been used.

A solution to the governing equations and boundary conditions (2.7) is sought in the following series forms:

$$G(\xi, X, Z; \mu) = G_0(\xi, X, Z) + \mu G_1(\xi, X, Z) + \mu^2 G_2(\xi, X, Z) + O(\mu^3), \tag{2.8a}$$

$$\eta(\xi, X; \mu) = \eta_0(\xi, X) + \mu\eta_1(\xi, X) + \mu^2\eta_2(\xi, X) + O(\mu^3), \tag{2.8b}$$

where $G = \{u^{\pm}, w^{\pm}, p^{\pm}\}$. Substituting (2.8) into (2.7) and expanding the interfacial boundary conditions (2.7d) and (2.7e) at $Z = 0$ and the bottom boundary condition (2.7g) at $Z = -h^{-}$, we obtain a sequence of initial-boundary-value problems by collecting coefficients of μ^n. Carrying out the perturbation analysis to the second order ($n = 2$), we find out that the compatability condition of the second-order problem requires η_0 and η_1 to satisfy the following evolution equation (for details see [14]):

$$\frac{\partial \eta_0}{\partial X} + \frac{3\alpha C}{4\mu}D_{-2}\frac{\partial \eta_0^2}{\partial \xi} - \alpha^2 C D_{-3}\frac{\partial \eta_0^3}{\partial \xi} + \frac{D_1}{6C}\frac{\partial^3 \eta_0}{\partial \xi^3} + \frac{\rho^{-}BC}{2(h^{-})^2}\frac{\partial \eta_0}{\partial \xi} + \frac{3\alpha C}{2}D_{-2}\frac{\partial \eta_0\eta_1}{\partial \xi} = 0, \tag{2.9}$$

where D_1, D_{-2} and D_{-3} are defined as

$$D_n(X) = \rho^{-}(h^{-})^n + (-1)^{n-1}\rho^{+}(h^{+})^n \quad (n = 1, -2, -3). \tag{2.10}$$

Because $D_{-2} = O(\epsilon)$, the last term involving η_1 in (2.9) can be dropped and we obtain the evolution equation, including both quadratic and cubic nonlinearity, for waves propagating over a weak and gentle topography near the critical depth level:

$$\frac{\partial \eta_0}{\partial X} + \frac{3\epsilon}{4\mu^2} C D_{-2} \frac{\partial \eta_0^2}{\partial \xi} - \frac{\epsilon^2}{\mu^2} C D_{-3} \frac{\partial \eta_0^3}{\partial \xi} + \frac{D_1}{6C} \frac{\partial^3 \eta_0}{\partial \xi^3} + \frac{\rho^- BC}{2(h^-)^2} \frac{\partial \eta_0}{\partial \xi} = 0. \tag{2.11}$$

Note that if $O(\epsilon) = O(\mu^2)$ and $D_{-2} = O(1)$, the cubic nonlinear term in (2.11) becomes higher order and can be neglected; the resulting equation is the evolution equation for weakly nonlinear and dispersive interfacial waves propagating over a weak and slowly varying topography far away from the critical layer (cf. (3.27) in [15]). We remark that both (2.11) and equation (2.7) in Helfrich *et al.*'s paper [13] recover the same equation when the bottom is flat.

3 Analytical solutions

It is straightforward to show that (2.11) passes the Painlevé PDE test without requiring any constraints on its coefficients (see [14, 15]). Therefore, (2.11) is a completely integrable equation, i.e. it can be solved by the inverse scattering transform. Moreover, (2.11) can be transformed into a constant-coefficient equation

$$\frac{\partial u}{\partial \tau} + 6\frac{\partial u^2}{\partial \theta} - 2\frac{\partial u^3}{\partial \theta} + \frac{\partial^3 u}{\partial \theta^3} = 0, \tag{3.1}$$

through the following transformation:

$$\theta = c_1 \left[\xi - c_2 W(X)\right], \qquad \tau = c_1 c_3 X, \qquad \eta_0 = c_4 u, \tag{3.2a}$$

$$W(X) = \int_0^X B(X')\mathrm{d}X', \tag{3.2b}$$

where $c_i (i = 1, 2, 3, 4)$ are constants given by

$$c_1 = \frac{C|D_{-2}|}{4\mu}\sqrt{\frac{3}{D_1 D_{-3}}}, \quad c_2 = \frac{\rho^- C}{2(h^-)^2}, \quad c_3 = \frac{C D_{-2}^2}{32\mu^2 D_{-3}}, \quad c_4 = \frac{D_{-2}}{4\epsilon D_{-3}}. \tag{3.2c}$$

Equation (3.1) is a combined KdV and modified KdV equation with negative cubic nonlinearity. Similar equations with different coefficients were obtained by Kakutani & Yamasaki and Miles [11, 12]. They showed that this kind of equations admits a family of solitary-wave solutions and a shock-like solution. We could obtain these solutions by transforming (3.1) into either one of the equations they studied. However, we prefer to seek for all possible bounded traveling-wave solutions of (3.1) first and then find the solitary-wave and shock-wave solutions.

Substituting $u(\phi) = u(\theta - \omega\tau)$ (ω is a constant) into (3.1) and integrating the resulting equation once, we have

$$u'' = 2u^3 - 6u^2 + \omega u + A, \tag{3.3}$$

where $' = \mathrm{d}/\mathrm{d}\phi$ and A is the constant of integration. Equation (3.3) can be recast into

$$\chi' = \vartheta, \qquad \vartheta' = 2\chi^3 - P\chi + Q, \tag{3.4}$$

with

$$\chi = u - 1, \quad P = 6 - \omega, \quad Q = -4 + \omega + A. \tag{3.5}$$

Carrying out analysis on the phase plane (χ, ϑ), we first look for periodic solutions (close orbits) which can be expressed in terms of elliptic functions. Then, by letting the modulus go to 1, we obtain solitary-wave solutions and shock-wave solutions, which correspond to homoclinic and heteroclinic trajectories, respectively. By further imposing $u \to 0$ as $\phi \to +\infty$, we obtain a family of solitary-wave solutions and a shock-wave solution, which are given as follows.

The family of solitary-wave solutions is given by

$$u(\theta, \tau; \lambda) = 2\lambda^2 \left[1 + \sqrt{1 - \lambda^2}\cosh 2\psi\right]^{-1}, \quad \psi = \lambda\left(\theta - 4\lambda^2\tau\right) + \psi^0, \tag{3.6}$$

where $\lambda \in (0, 1)$ is the family parameter and ψ^0 is a phase constant. Note that contrary to the Boussinesq solitary wave, when λ is close to 1, the width of the

solitary wave given by (3.6) actually increases as its height increases, which has an upper bound 2 as $\lambda \to 1^-$ (see figure 1). The shock-wave solution is given by

$$u(\theta, \tau) = 1 - \tanh(\theta - 4\tau + \psi^0). \tag{3.7}$$

Since (3.1) is invariant under transformation $u \to 2 - u$, from (3.7), we have $u = 1 + \tanh(\theta - 4\tau + \psi^0)$, which is the shock-wave solution satisfying $u \to 0$ as $\theta \to -\infty$. For definiteness, we assume that $u \to 0$ as $\theta \to +\infty$. Note that the shock wave travels faster than any solitary wave given by (3.6).

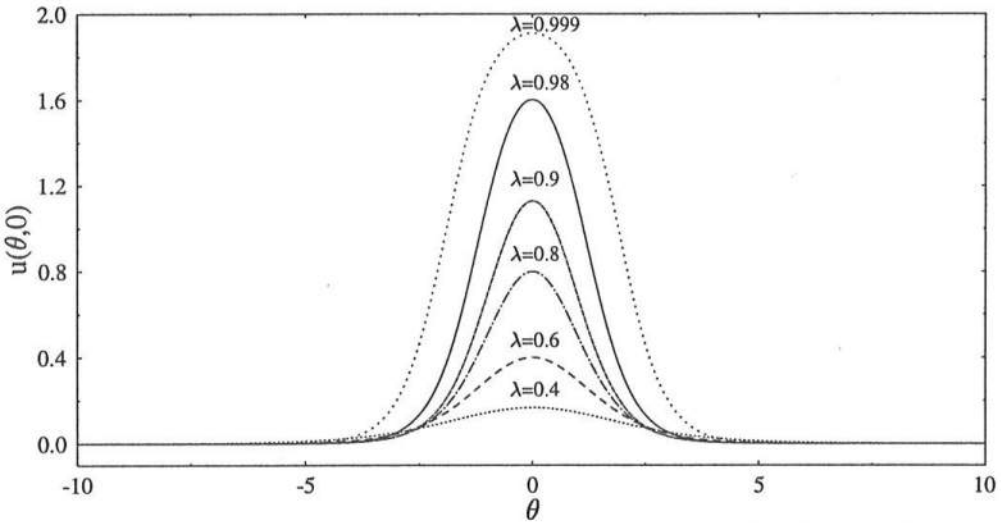

Figure 1: Different profiles of the solitary wave given by (3.6) for different λ values at $\tau = 0$.

The two-soliton solution was not given in Kakutani & Yamasaki and Miles' papers. Here we use Hirota's method, a more straightforward approach than the inverse scattering transform, to find the two-soliton solution of (3.1). Carrying out rather tedious but straightforward manipulation, we find that $u(\theta, \tau)$ given by the following expression:

$$u(\tau, \theta; \lambda_1, \lambda_2) = \frac{\partial}{\partial \theta} \ln \frac{f_-}{f_+}, \tag{3.8a}$$

$$f_\pm(\tau, \theta) = 1 + (1 \pm \lambda_1)E_1 + (1 \pm \lambda_2)E_2 + \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}(1 \pm \lambda_1)(1 \pm \lambda_2)E_1E_2, \tag{3.8b}$$

$$E_i = \exp(-2\psi_i), \quad \psi_i = \lambda_i \left(\theta - 4\lambda_i^2 \tau\right) + \psi_i^0, \quad (i = 1, 2), \tag{3.8c}$$

where $\lambda_1, \lambda_2 \in (0, 1] (\lambda_1 \neq \lambda_2$; without loss of generality, we assume $\lambda_1 > \lambda_2$) are parameters and ψ_1^0, ψ_2^0 are phase constants, is a solution of (3.1). Expression (3.8) can be further simplified; however, the result depends on whether $\lambda_1 = 1$.

(a) When $0 < \lambda_2 < \lambda_1 < 1$, expression (3.8) can be simplified as

$$\begin{aligned} u(\theta, \tau) = (\lambda_1^2 - \lambda_2^2) \Big\{ & [\lambda_1 \coth(\psi_1 + \delta_1) - \lambda_2 \tanh(\psi_2 + \delta_2)]^{-1} \\ & - [\lambda_1 \coth(\psi_1 - \delta_1) - \lambda_2 \tanh(\psi_2 - \delta_2)]^{-1} \Big\}, \end{aligned} \tag{3.9}$$

where

$$\delta_i = \tfrac{1}{4} \ln[(1 + \lambda_i)/(1 - \lambda_i)], \qquad (i = 1, 2). \tag{3.10}$$

Expanding (3.9) as $\tau \to \pm\infty$, keeping ψ_1 and ψ_2 fixed respectively, we obtain

$$u \sim \frac{2\lambda_1^2}{1 + \lambda_{c1} \cosh 2(\psi_1 + \Delta)} + \frac{2\lambda_2^2}{1 + \lambda_{c2} \cosh 2(\psi_2 - \Delta)}, \quad \text{as } \tau \to -\infty, \tag{3.11a}$$

$$u \sim \frac{2\lambda_1^2}{1 + \lambda_{c1} \cosh 2(\psi_1 - \Delta)} + \frac{2\lambda_2^2}{1 + \lambda_{c2} \cosh 2(\psi_2 + \Delta)}, \quad \text{as } \tau \to +\infty, \tag{3.11b}$$

with

$$\lambda_{ci} = \sqrt{1 - \lambda_i^2}, \qquad (i = 1, 2), \tag{3.11c}$$

$$\Delta = \tfrac{1}{2} \ln[(\lambda_1 + \lambda_2)/(\lambda_1 - \lambda_2)], \tag{3.11d}$$

which represents the superposition of two solitary waves at infinity. Therefore, (3.9) with $\lambda_1, \lambda_2 \in (0, 1)$ is the two-soliton solution of (3.1). From the asymptotic expression (3.11), one can see that after collision, two solitary waves undergo phase shifts but do not change their amplitudes and speeds; the taller solitary wave moves forward by an amount $\theta = 2\Delta/\lambda_1$, whereas the short one moves backward by an amount $\theta = 2\Delta/\lambda_2$.

(b) When $\lambda_1 = 1$ and $0 < \lambda_2 < 1$, expression (3.8) can be simplify as

$$u(\theta, \tau) = 1 + \lambda_2 \tanh \psi_2 - \frac{1 - \lambda_2^2}{\coth \psi_1 - \lambda_2 \tanh \psi_2}. \tag{3.12}$$

The asymptotic expansion of this solution as $\tau \to \pm\infty$ is given by

$$u \sim 1 - \tanh(\psi_1 + 2\delta_2) + \frac{2\lambda_2^2}{1 + \lambda_{c2}\cosh 2(\psi_2 - \delta_2)}, \quad \text{as } \tau \to -\infty, \tag{3.13a}$$

$$u \sim 1 - \tanh(\psi_1 - 2\delta_2) - \frac{2\lambda_2^2}{1 + \lambda_{c2}\cosh 2(\psi_2 + \delta_2)}, \quad \text{as } \tau \to +\infty, \tag{3.13b}$$

which represents the superposition of a shock wave and a solitary wave. Therefore, (3.12) is the solitary-shock-wave solution describing the interaction between a shock wave and a solitary wave (see figure 2). From (3.13) and figure 2, one can see that after the shock wave surpasses the solitary wave, the solitary wave reverses its polarity and gains a backward shift $2\delta_2/\lambda_2$, whereas the shock wave obtains a forward shift $4\delta_2$. Accidentally, the interaction between a shock wave and a solitary wave can also be viewed as the interaction among three shock waves, because the solitary wave can be rewritten as the sum of two shock waves:

$$u = \frac{2\lambda_2^2}{1 + \lambda_{c2}\cosh 2\psi_2} = \lambda_2 \left\{ [1 - \tanh(\psi_2 - \delta_2)] - [1 - \tanh(\psi_2 + \delta_2)] \right\}. \tag{3.14}$$

This may explain the double phase shift in the shock wave in (3.13).

On substitution of transformation (3.2) into (3.6) and (3.7), we obtain a family of interfacial solitary waves, which are concave when $D_{-2} < 0$ and convex when $D_{-2} > 0$, and an interfacial non-dissipative jump ($D_{-2} < 0$) or bore ($D_{-2} > 0$) propagating over an arbitrary weak and gentle topography near the critical depth level. Note that the mass and energy of a non-dissipative bore or jump is infinite. Thus, the bore or jump solution cannot be uniformly valid over the entire domain $-\infty < \xi < \infty$; however, it may represents a useful approximation in some limited interval (see [12, 16]). When $D_{-2} = 0$, we have only trivial solution $\eta_0 = 0$ which vanishes at infinity. The corresponding two-soliton solution and bore-soliton (or jump-soliton) solutions to equation (2.11) can also be obtained by substituting (3.2) into (3.9) and (3.12), respectively.

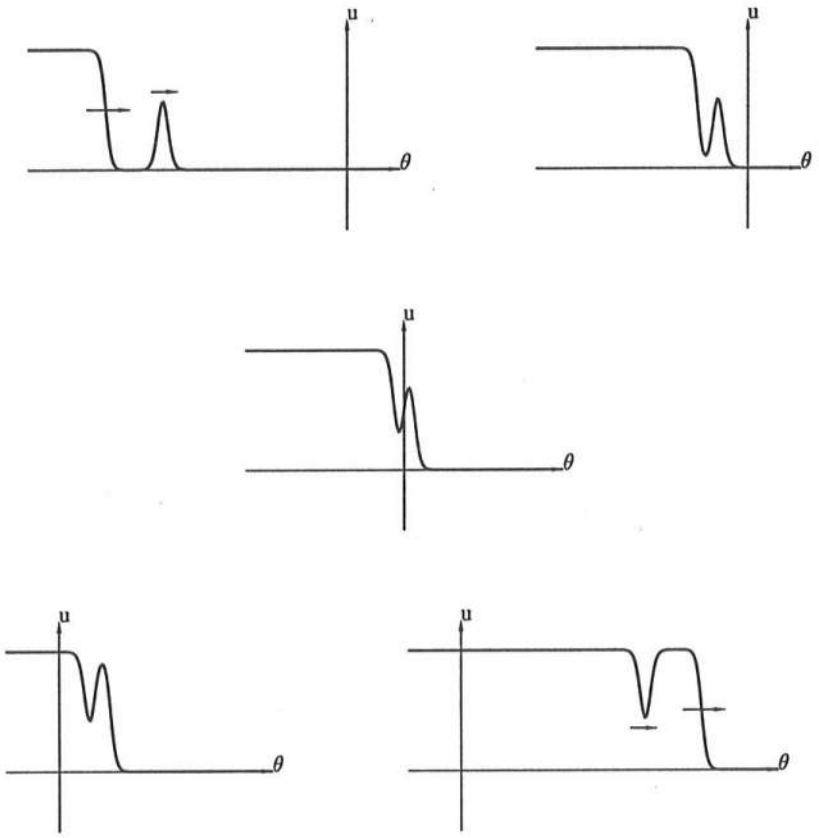

Figure 2: Interaction between a shock wave and a solitary wave in equation (3.1).

From transformation (3.2), one can see that the existence of a weak and gentle topography only affects the phase velocity of a solitary wave or a shock wave in the moving coordinates (ξ, X, Z), which is given by

$$V = \frac{\mathrm{d}\xi}{\mathrm{d}X} = \frac{\rho^- BC}{2(h^-)^2} + \lambda^2 \frac{CD_{-2}^2}{8\mu^2 D_{-3}}, \qquad (0 < \lambda \leq 1). \tag{3.15}$$

According to this expression, the effect of the topography cannot only change the magnitude of the phase velocity but also change its direction. Therefore, in the moving frame, under different circumstances, a solitary wave (or a shock wave) can remain still or bounce forward and backward; two solitary waves (or a shock wave and a solitary wave) can collide against each other; a small solitary wave can catch up with a large one (or a shock wave) after both have reversed their directions. These phenomena cannot exist if the bottom is flat.

4 Over random topography

In the previous section, we have showed that a weak and gentle deterministic topography affects only the phase velocity of a solitary wave or a shock wave, but not their shapes. In this section, we shall investigate the effect of a weak and gentle random topography on interfacial wave propagation.

We assume that $B(X)$ is a stationary Gaussian stochastic process with zero mean $\langle B(X) \rangle = 0$ and with an arbitrary correlation function:

$$R_B(Y) = \langle B(X)B(X+Y) \rangle, \tag{4.1}$$

where the angle brackets $\langle\ \rangle$ denotes the ensemble average. Then, $W(X)$ given by (3.2b) is also a Gaussian stochastic process with zero mean and variance given by

$$\sigma_W^2(X) = \langle W^2(X) \rangle = 2\int_0^X (X-Y)R_B(Y)\mathrm{d}Y. \tag{4.2}$$

4.1 Steady waves

Let $u_1(\theta-\omega\tau)$ be a steady-wave solution of (3.1). Then, according to the transformation (3.2), a steady incident interfacial wave propagating over a random topography can be written as

$$\eta_0(\xi,X)=c_4u_1(c_1(\xi-c_2W-\omega c_3X))=U_1(\xi-c_2W-\omega c_3X), \tag{4.3}$$

which can be regarded as a nonlinear transformation of the Gaussian process $W(X)$. By definition, the mth (m is an integer) moment of the interfacial displacement is given by

$$\begin{aligned}\langle\eta_0^m(\xi,X)\rangle&=\int_{-\infty}^{+\infty}U_1^m(\xi-c_2w-\omega c_3X)\frac{1}{\sqrt{2\pi\sigma_W^2}}\exp\left(-\frac{w^2}{2\sigma_W^2}\right)\mathrm{d}w\\&=\int_{-\infty}^{+\infty}\frac{U_1^m(s)}{\sqrt{2\pi c_2^2\sigma_W^2}}\exp\left[-\frac{(s-p)^2}{2c_2^2\sigma_W^2}\right]\mathrm{d}s,\end{aligned} \tag{4.4}$$

where

$$p=\xi-\omega c_3X. \tag{4.5}$$

It is easy to verify that $\langle\eta_0^m(p,X)\rangle$ given by (4.4) satisfies the following diffusion equation

$$\frac{\partial\langle\eta_0^m\rangle}{\partial X}=\frac{c_2^2}{2}(\sigma_W^2)'\frac{\partial^2\langle\eta_0^m\rangle}{\partial p^2}, \tag{4.6}$$

with the initial condition given by

$$\langle\eta_0^m(p,0)\rangle=U_1^m(p)=\eta_0^m(\xi,0)|_{\xi=p}, \tag{4.7}$$

where $'$ denotes the first derivative of a function with respect to its argument. Note that the variable diffusion coefficient in equation (4.6) depends on the correlation function $R_B(Y)$ (see (4.2)). Therefore, the randomness of the topography causes all the moments of a steady incident wave to diffuse and their shapes vary as the wave propagates downstream. The fact that all the moments of the interfacial displacement

for a steady wave satisfy the same linear diffusion equation (although the original equation is a nonlinear evolution equation with random coefficients) hinges on two important factors: $W(X)$ linearly depends on $B(X)$ and $B(X)$ is assumed to be a Gaussian process. It has nothing to do with the specific form of the nonlinear function $U_1(s)$, which is used only to provide the initial condition at $X = 0$ for the diffusion equation (see (4.7)).

In terms of the moving coordinates (ξ, X, Z), we find that the first three moments satisfy the following equation

$$\frac{\partial\langle\eta_0\rangle}{\partial X} + \frac{3\epsilon}{4\mu^2} CD_{-2}\frac{\partial\langle\eta_0^2\rangle}{\partial\xi} - \frac{\epsilon^2}{\mu^2} CD_{-3}\frac{\partial\langle\eta_0^3\rangle}{\partial\xi} + \frac{D_1}{6C}\frac{\partial^3\langle\eta_0\rangle}{\partial\xi^3} = \frac{c_2^2}{2}(\sigma_W^2)'\frac{\partial^2\langle\eta_0\rangle}{\partial\xi^2}, \qquad (4.8)$$

where

$$-\omega c_3 U_1' + \frac{3\epsilon}{4\mu^2} CD_{-2}(U_1^2)' - \frac{\epsilon^2}{\mu^2} CD_{-3}(U_1^3)' + \frac{D_1}{6C}U_1''' = 0$$

has been used. From (4.8), it is evident that the randomness gives rise to the dissipation term appearing on the right-hand side of the equation. Comparing the exact mean-field equation (4.8) with the approximate mean-field equation derived from an approximate method (such as the mean field method [17]) can shed lights on the applicability and accuracy of the approximate method.

From (4.4), we also find

$$\int_{-\infty}^{+\infty}\langle\eta_0^m(\xi, X)\rangle \mathrm{d}\xi = \int_{-\infty}^{+\infty} U^m(s)\mathrm{d}s = \int_{-\infty}^{+\infty}\eta_0^m(\xi, 0)\mathrm{d}\xi, \qquad (4.9)$$

assuming that the integrals exist. Therefore, for a solitary wave propagating over a random topography, although $\langle\eta_0^m\rangle$ will spread out as X increases, its integral over the entire domain, $\int_{-\infty}^{+\infty}\langle\eta_0^m\rangle\mathrm{d}\xi$, is a constant. Furthermore, the invariant $\int_{-\infty}^{+\infty}\langle\eta_0^m\rangle\mathrm{d}\xi$ is independent of the correlation function $R_B(X)$ and its value can be determined by the incident solitary wave.

The asymptotic behavior of $\langle\eta_0^m(\xi, X)\rangle$ given by (4.4) as $X \to +\infty$ depends on

the initial condition, i.e. the incident wave. We now discuss the asymptotic behaviour of the mean value of a solitary wave and a shock wave as $X \to +\infty$.

Through the Fourier transform, the solitary-wave solution (3.6) can be rewritten as

$$u_1(\theta,\tau) = \int_{-\infty}^{+\infty} \sin\left(\frac{bk}{2\lambda}\right) \operatorname{csch}\left(\frac{\pi k}{2\lambda}\right) \exp[-ik(\theta - 4\lambda^2\tau)]\mathrm{d}k, \tag{4.10}$$

where $b = \cosh^{-1}(1/\sqrt{1-\lambda^2})$. Using the following identities (see [9])

$$\langle \exp[iaW(X)]\rangle = \exp\left[-\tfrac{1}{2}a^2\langle W^2(X)\rangle\right] = \exp(-a^2\sigma_W^2/2), \tag{4.11}$$

where a is an arbitrary constant, we obtain

$$\begin{aligned}
&\langle \eta_0(\xi,X)\rangle = c_4 \langle u_1\left(c_1[\xi - c_2W(X)], c_1c_3X\right)\rangle \\
&= c_4 \int_{-\infty}^{+\infty} \sin\left(\frac{bk}{2\lambda}\right) \operatorname{csch}\left(\frac{\pi k}{2\lambda}\right) \exp\left[-ikc_1(\xi - 4\lambda^2c_3X)\right] \langle \exp[ikc_1c_2W(X)]\rangle \mathrm{d}k \\
&= c_4 \int_{-\infty}^{+\infty} \sin\left(\frac{bk}{2\lambda}\right) \operatorname{csch}\left(\frac{\pi k}{2\lambda}\right) \exp\left[-ikc_1(\xi - 4\lambda^2c_3X) - k^2c_1^2c_2^2\sigma_W^2/2\right] \mathrm{d}k.
\end{aligned} \tag{4.12}$$

Under the assumption that $\sigma_W(X) \to +\infty$ as $X \to +\infty$, the asymptotic expansion of the above integral as $X \to +\infty$ is given by

$$\langle \eta_0(\xi,X)\rangle \sim \sqrt{\frac{2}{\pi}} \frac{bc_4}{c_1c_2\sigma_W} \exp\left[-\frac{(\xi - 4\lambda^2c_3X)^2}{2c_2^2\sigma_W^2}\right] = \langle \eta_0(\xi,X)\rangle^a. \tag{4.13}$$

One can check that

$$\int_{-\infty}^{+\infty} \langle \eta_0(\xi,X)\rangle^a \mathrm{d}\xi = \int_{-\infty}^{+\infty} \langle \eta_0(\xi,X)\rangle \mathrm{d}\xi = \int_{-\infty}^{+\infty} \eta_0(\xi,0)\mathrm{d}\xi = \frac{2bc_4}{c_1}. \tag{4.14}$$

Therefore, far away in the downstream (X becomes sufficiently large), an averaged solitary wave evolves into a spreading Gaussian wavepacket, whose height decreases as $\sigma_W^{-1}(X)$ and whose width increases as $\sigma_W(X)$ around $\xi = 4\lambda^2c_3X$. Note that the velocity of the Gaussian packet is equal to the velocity of the incident solitary wave.

For a shock wave propagating over the same random topography, the mean value of the shock wave is given by

$$\langle \eta_0(\xi,X)\rangle = c_4 \left\{1 - \frac{1}{\sqrt{2\pi\sigma_W^2}} \int_{-\infty}^{+\infty} \tanh\left[c_1(\xi - c_2w - 4c_3X)\right] \exp\left(-\frac{w^2}{2\sigma_W^2}\right) \mathrm{d}w\right\}$$

$$= c_4 \left\{ 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \tanh \left[c_1(\xi - \sqrt{2}\, c_2 \sigma_W s - 4c_3 X) \right] \exp(-s^2) \mathrm{d}s \right\} \quad (4.15)$$

For large X, the asymptotic expansion of (4.15) is

$$\begin{aligned} \langle \eta_0(\xi, X) \rangle &\sim c_4 \left\{ 1 - \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{\frac{\xi - 4c_3 X}{\sqrt{2}\, c_2 \sigma_W}} \exp(-s^2) \mathrm{d}s - \int_{\frac{\xi - 4c_3 X}{\sqrt{2}\, c_2 \sigma_W}}^{+\infty} \exp(-s^2) \mathrm{d}s \right] \right\} \\ &= c_4 \mathrm{erfc} \left(\frac{\xi - 4c_3 X}{\sqrt{2}\, c_2 \sigma_W} \right), \quad \text{as } X \to +\infty, \end{aligned} \quad (4.16)$$

where erfc is the complementary error function defined as

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2) \mathrm{d}s. \quad (4.17)$$

Thus, the randomness of the topography causes the front of an averaged shock wave to increase proportionally to σ_W, coincident with its effect on the width of an averaged solitary wave. This phenomenon may be called the diffusion of a shock wave.

We remark that when $B(X)$ is Gaussian white noise with strength D, i.e. $R_B(Y) = 2D\delta(Y)$, $\sigma_W^2 = 2DX$ and the asymptotic behaviours of an averaged solitary wave and an averaged shock wave we just obtained are the same as those given by Ono [10] for an averaged surface solitary wave and an averaged Burgers shock wave, despite the fact that these waves appear in different physical situations.

4.2 Unsteady waves

The mean values of a two-solitary wave (3.9) and a solitary-shock wave (3.12) are given by

$$\langle \eta_0(\xi, X) \rangle = c_4 \int_{-\infty}^{+\infty} u_2 \left(c_1(\xi - c_2 w), c_1 c_3 X \right) \frac{1}{\sqrt{2\pi\sigma_W^2}} \exp\left(-\frac{w^2}{2\sigma_W^2} \right) \mathrm{d}w, \quad (4.18)$$

where $u_2(\theta, \tau)$ is given by (3.9) and (3.12), respectively. Recall that as $\tau \to +\infty$, (3.9) becomes the sum of two single solitary waves, whereas (3.12) becomes the sum of a single solitary wave and a shock wave (see (3.11b) and (3.13b)). Therefore, the

asymptotic expansion of an averaged two-solitary wave can be expressed as the sum of two averaged single solitary waves as $X \to +\infty$, whereas the asymptotic expansion of an averaged solitary-shock wave is the sum of an averaged single solitary wave and an averaged shock wave as $X \to +\infty$. Since the asymptotic expressions of an averaged single solitary wave and an averaged shock wave have been given by (4.13) and (4.16), one can readily obtain the asymptotic form of an averaged two-solitary wave:

$$\langle \eta_0(\xi, X) \rangle \sim \sqrt{\frac{2}{\pi}} \frac{c_4}{c_1 c_2 \sigma_W} \sum_{i=1}^{2} b_i \exp\left[-\frac{(\xi - 4\lambda_i^2 c_3 X + \delta_i^+)^2}{2c_2^2 \sigma_W^2} \right], \quad \text{as } X \to +\infty, \quad (4.19)$$

where

$$b_i = \cosh^{-1}(1/\sqrt{1-\lambda_i^2}), \quad \delta_1^+ = (\psi_1^0 - \Delta)/(\lambda_1 c_1), \quad \delta_2^+ = (\psi_2^0 + \Delta)/(\lambda_2 c_1), \quad (4.20)$$

and the asymptotic form of an averaged solitary-shock wave:

$$\begin{aligned} \langle \eta_0(\xi, X) \rangle \sim\ & c_4 \mathrm{erfc}\left(\frac{\xi - 4c_3 X + \delta_1^+}{\sqrt{2}\, c_2 \sigma_W} \right) \\ & - \sqrt{\frac{2}{\pi}} \frac{b_2 c_4}{c_1 c_2 \sigma_W} \exp\left[-\frac{(\xi - 4\lambda_2^2 c_3 X + \delta_2^+)^2}{2c_2^2 \sigma_W^2} \right], \text{ as } X \to +\infty, \quad (4.21) \end{aligned}$$

where

$$\delta_1^+ = (\psi_1^0 - 2\delta_2)/c_1, \quad \delta_2^+ = (\psi_2^0 + \delta_2)/(\lambda_2 c_1). \quad (4.22)$$

To see how a two-solitary wave evolves into two Gaussian packets (according to (4.19)) and a solitary-shock wave evolves into a Gaussian packet and a diffusing shock wave (according to (4.21)), we numerically evaluate the integral (4.18). The numerical results presented below are obtained under the assumption that $B(X)$ is the Gaussian white noise with strength $D = c_3/(c_1 c_2^2)$. Figures 3 and 4 show the evolution of an averaged two-solitary wave and an averaged solitary-shock wave, respectively. The phase constants have been chosen as $\psi_1^0 = \psi_2^0 = 0$ for the two-solitary wave and the solitary-shock wave so that the corresponding incident wave is symmetric with respect

to $\xi = 0$ and anti-symmetric with respect to $\eta_0/c_4 = 1$, respectively (see figures 3(a) and 4(a)). From figure 3, one can see that as the two-solitary wave propagates downstream, the averaged wave indeed evolves into two Gaussian packets with each own height decreasing and width increasing proportionally to $X^{1/2}$. Figure 4 shows that as X increases, the averaged shock wave evolves into a diffusing shock wave with its front increasing as $X^{1/2}$, followed by a spreading Gaussian packet, which decays as $X^{1/2}$ and will disappear as $X \to +\infty$. To verify the asymptotic expansions (4.19) and (4.21), we compare the exact averaged two-solitary wave and solitary-shock wave given by (4.18) with the asymptotic expression given by (4.19) at $X = 50$ (see figure 5) and the asymptotic expression given by (4.21) at $X = 25$ (see figure 6), respectively. In both cases the agreement is excellent. In fact, the agreement is so good that we have to magnify the ordinate to see the difference between the exact mean value and its asymptotic prediction in figure 5; otherwise, they are indistinguishable as in figure 6.

Using the fact that N solitary waves will separate from one and other as $X \to +\infty$ (see [18]), we can extend the conclusions drawn on an averaged two-solitary wave and an averaged solitary-shock wave to an averaged N-solitary wave and an averaged N-solitary-shock wave, respectively. We remark that if $\sigma_W \sim X^{1+\beta}$ with $\beta > 0$ as $X \to +\infty$ for some Gaussian process $B(X)$, the final state of an averaged N-solitary wave is one large Gaussian packet instead of N Gaussian packets propagating at various speeds. The reason is that each constituent Gaussian packet is mutually separated by the distance of order X, whereas the width of each packet is proportional to $X^{1+\beta}$, which exceeds the separation distances. The overlapping among each packet tends to form one large Gaussian packet (cf. [3]). Therefore, the statistical characteristic of the topography essentially affects the evolution of the mean value of an N-solitary wave. If there is a shock wave component in the incident wave, the final state of the

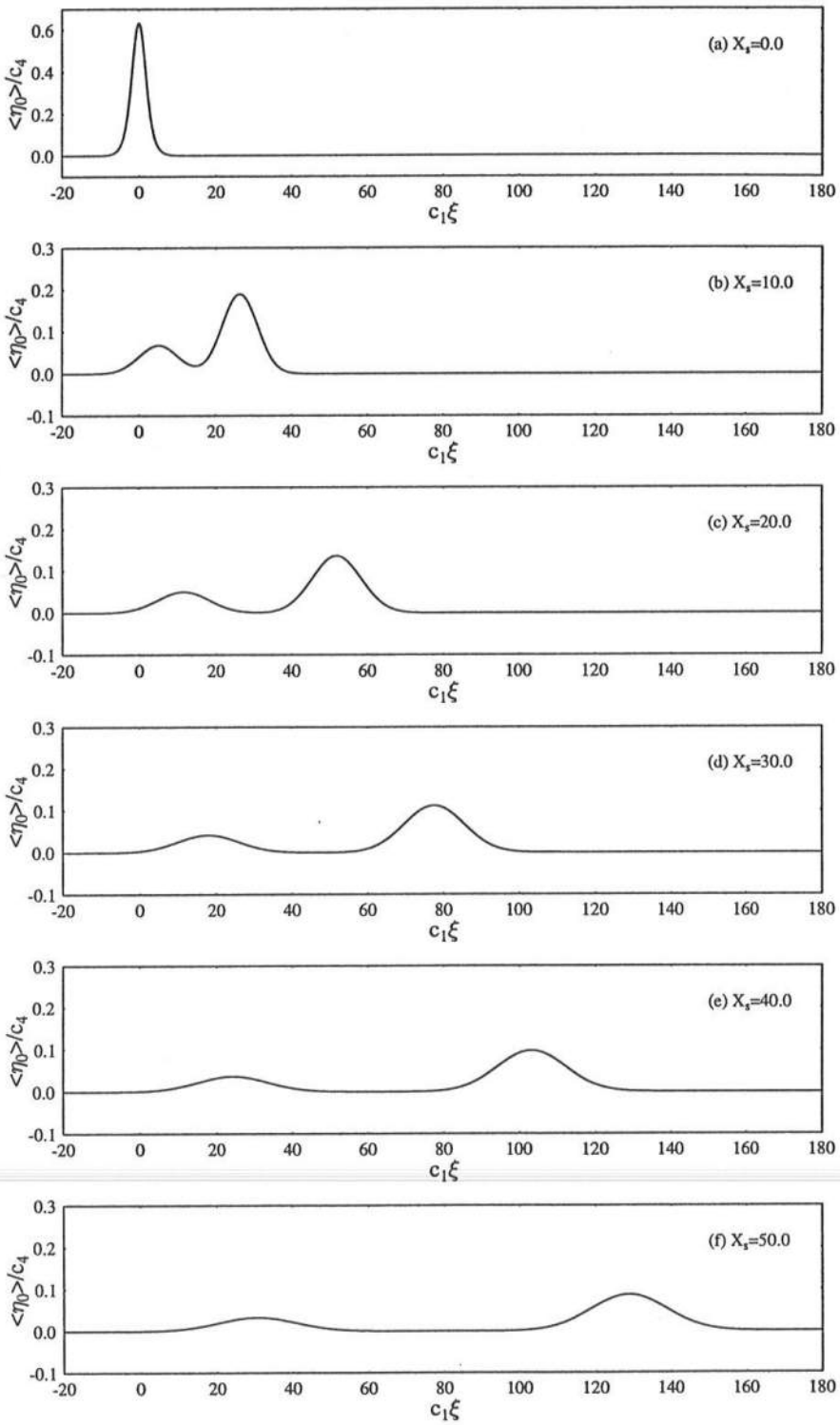

Figure 3: Evolution of an averaged two-solitary wave ($X_s = c_1 c_2 X$ and $\lambda_1 = 2\lambda_2 = 0.8$).

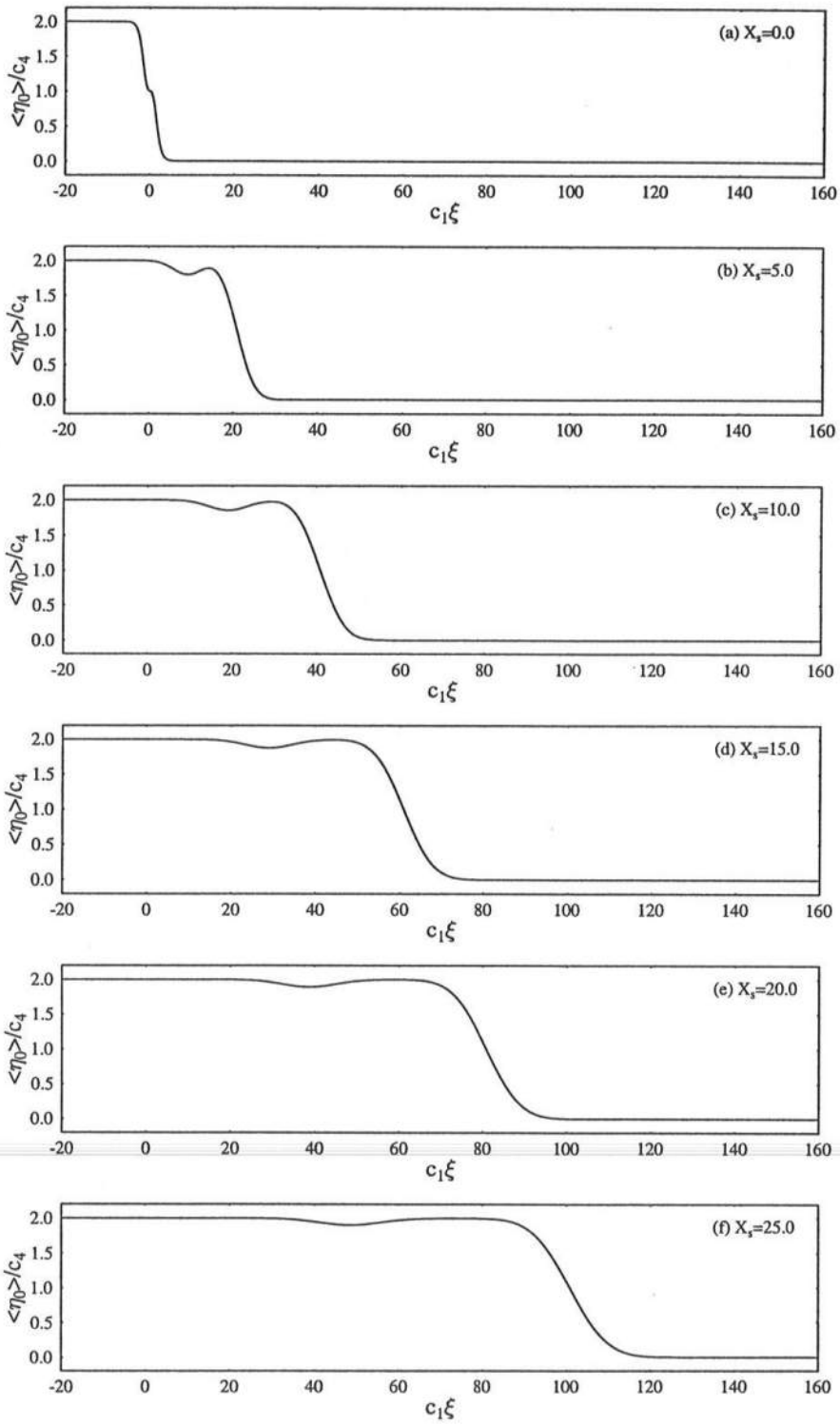

Figure 4: Evolution of an averaged solitary-shock wave ($X_s = c_1 c_2 X$ and $\lambda_2 = 0.7$).

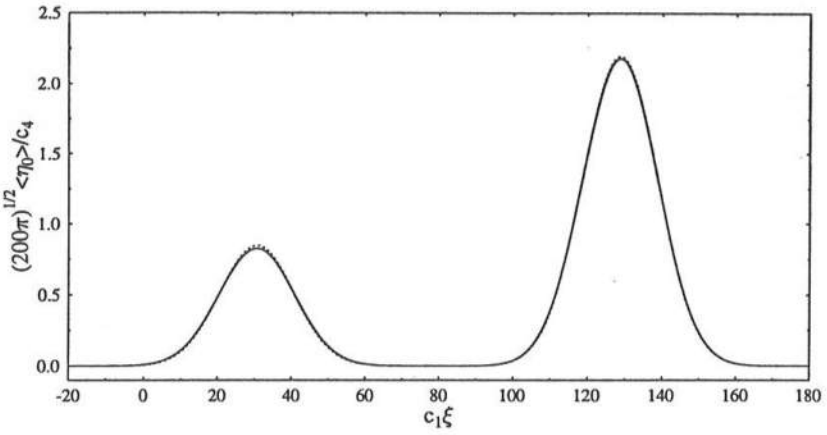

Figure 5: Comparison between the exact averaged two-solitary wave (——) and its asymptotic form ($\cdots$) given by (4.19) at $X = 50$.

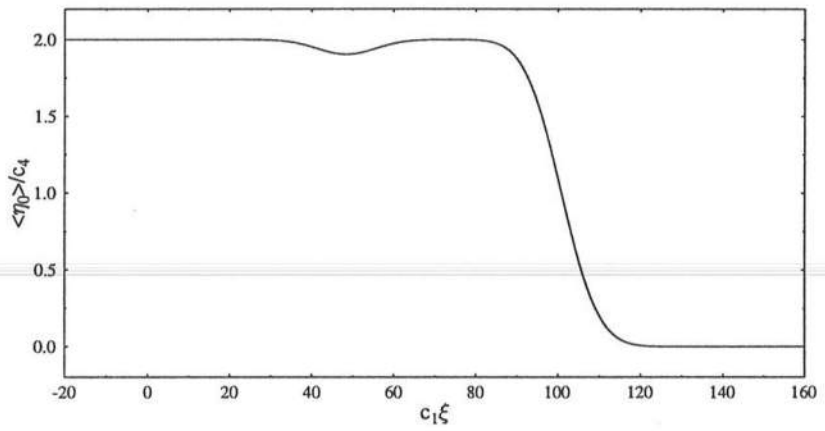

Figure 6: Comparison between the exact averaged solitary-shock wave (——) and its asymptotic form ($\cdots$) given by (4.21) at $X = 25$.

averaged wave field is one diffusing shock wave whose front increases as σ_W, because the Gaussian packets, traveling far behind, decay rapidly as X becomes sufficiently large.

5 Concluding remarks

We have studied the effect of a weak and gentle topography on interfacial long wave propagation near the critical depth level, where the cubic nonlinearity is comparable to the quadratic nonlinearity. The evolution equation, derived from the Euler equations, is completely integrable and can be transformed to a deterministic combined KdV and modified KdV equation. This allows us to investigate the effect of a random topography on wave propagation analytically.

While a weak and slowly varying deterministic topography only affects the phase velocity of a steady incident wave but not its shape, the effect of a random topography with Gaussian characteristic will cause an averaged steady wave to diffuse. An averaged solitary wave deforms into a spreading Gaussian wavepacket, whose height decreases and whose width increases proportionally to σ_W, which can be determined from the correlation function of the topography (see (4.2)); the front of an averaged shock wave also increases as σ_W. For Gaussian white noise, $\sigma_W \sim X^{1/2}$ and the asymptotic behaviours of an averaged interfacial solitary wave and shock wave are the same as those of an averaged surface solitary wave and Burgers shock wave (see [10]) respectively, despite the fact that these waves appear in different physical situations. The spreading rate σ_W substantially affects the asymptotic behaviour of an averaged N-solitary wave. If $\sigma \sim X^{1+\beta}$ with $\beta > 0$, then an averaged N-solitary wave evolves into one wide Gaussian packet far away downstream. On the other hand, if $\sigma \sim X^{1+\beta}$ with $\beta < 0$, we have N spreading Gaussian packets moving away from

one and other instead. If there is a shock wave component in the incident wave, the final state of the averaged wave field is a diffusing shock wave. Finally, we remark that these conclusions are also applied to those internal long waves in which the cubic nonlinearity and quadratic nonlinearity are comparable propagating over a weak and gentle random topography in a continuously stratified fluid.

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References

[1] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York (1974).

[2] M. Wadati, "Stochastic Korteweg-de Vires equation", *J. Phys. Soc. Japan 52*, 2642–2648 (1983).

[3] M. Wadati and Y. Akutsu, "Stochastic Korteweg-de Vires equation with and without damping", *J. Phys. Soc. Japan 53*, 3342-3350 (1984).

[4] H. Konno, "On stochastic Burgers equation", *J. Phys. Soc. Japan 54*, 4475-4478 (1985).

[5] A. Orlowski and K. Sobczyk, "Solitons and shock waves under random external noise", *Rep. Math. Phys. 27*, 59–71 (1989).

[6] R.L. Herman, "The stochastic, damped KdV equation", *J. Phys. A: Math. Gen. 23*, 1063–1084 (1990).

[7] V.M. Loginov, "Influence of flicker noise on a Korteweg-de Vires soliton", *Sov. Phys. Tech. Phys. 36*, 480–481 (1991).

[8] A. Orlowski, "Randomly driven Korteweg-de Vires-Burgers equation", *Physical Review E 49*, 2465–2467 (1994).

[9] M. Wadati, "Deformation of solitons in random media", *J. Phys. Soc. Japan 59*, 4201–4203 (1990).

[10] H. Ono, "Nonlinear wave propagation under weak inhomogeneity", *J. Phys. Soc. Japan 60*, 3692–3697 (1991).

[11] T. Kakutani and N. Yamasaki, "Solitary waves on a two-layer fluid", *J. Phys. Soc. Japan 45*, 674–679 (1978).

[12] J.W. Miles, "On internal solitary waves", *Tellus 31*, 456–462 (1979).

[13] K.R. Helfrich, W.K. Melville and J.W. Miles, "On interfacial solitary waves over slowly varying topography", *J. Fluid Mech. 149*, 305–317 (1984).

[14] Y. Chen, *Modelling of Surface Water Wave and Interfacial Wave Propagation*, Doctoral dissertation, Cornell University, Ithaca, New York (1995).

[15] Y. Chen and P. L.-F. Liu, "The unified Kadomtsev–Petviashvili equation for interfacial waves", *J. Fluid Mech. 288*, 383–408 (1995).

[16] W.K. Melville and K.R. Helfrich, "Transcritical two-layer flow over topography", *J. Fluid Mech. 178*, 31–52 (1987).

[17] M.S. Howe, "On wave scattering by random inhomogeneities, with application to the theory of weak bores", *J. Fluid Mech. 45*, 785–804 (1971).

[18] J.W. Miles, "On internal solitary waves. II", *Tellus 33*, 397–401 (1981).