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Research Report No. CACR-96-01

April, 1996

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This paper was presented at the *Coastal Dynamics '95 Conference* in Gdansk, Poland, September 4–8, 1995. The final manuscript was apparently not forwarded to the Proceedings editor for inclusion in the Proceedings volume, and so it being made available here.

J.T. Kirby, 4/4/96

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Abstract

Based on time-domain Boussinesq equations, a comprehensive model is constructed to simulate wave shoaling, wave breaking and wave runup in coastal regions. To illustrate the accuracy of various forms of Boussinesq models for large effect of nonlinearity, solitary wave solutions corresponding to several height to depth ratios are obtained numerically. Comparisons of permanent solutions are made between Boussinesq models and other existing closed-form expressions. Additional energy dissipation terms are included in the Boussinesq equations to account for the effects of wave breaking and bottom friction, which play an important role for wave transformation in surf zone and swash zone areas. Model results model are compared to data from a laboratory study of random wave breaking over a constant slope (Mase and Kirby, 1992). Good agreement between numerical results and experimental data is found.

Introduction

Accurate prediction of wave transformation from deep to shallow water is important to the understanding of coastal processes. As waves propagate from the deep ocean towards coastal regions, the effect of bottom topography causes wave height and wave shape to change accordingly. A combination of wave refraction-diffraction, reflection and nonlinear interaction takes place. The increase of wave height and the steepening of wave crest eventually lead to wave breaking, resulting in the generations of large scale turbulent motion and nearshore circulation, as well as the movement of sediment along beaches and coastlines.

With the inclusions of lowest order effects of nonlinearity and frequency dispersion, Boussinesq equations provide a sound and increasingly well-tested

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basis for the simulation of wave propagation in coastal regions. By using depth-averaged velocity as a dependent variable, Peregrine (1967) derived the Boussinesq equations for variable water depth. Numerical models based on Peregrine's equations or equivalent formulations have been shown to give predictions which compare quite well with field data (Elgar and Guza, 1985) and laboratory data (Goring, 1978; Liu, Yoon and Kirby, 1985; Rygg, 1988), when applied within their range of validity.

In recent years, efforts have been made to extend the validity range of standard Boussinesq equations to intermediate-depth areas. Madsen *et al.* (1991) improved the linearized model by introducing expressions in the equations which are formally equivalent to zero within the accuracy of the model, thus obtaining a rearrangement of higher-order terms in the momentum equations. Nwogu (1993) used the velocity at a certain depth as a dependent variable and pursued a consistent derivation of the governing equations using this non-standard dependent variable. The resulting dispersion relations of Madsen *et al.* (1991) and Nwogu(1993) are formally equivalent and are much closer to the exact solution in intermediate water depths than are the standard Boussinesq equations.

By using the velocity at a certain water depth as a dependent variable (Nwogu, 1993) and making no assumption for small nonlinearity, Wei *et al.* (1995) derived fully nonlinear Boussinesq equations which further extend the range of validity for Boussinesq models. In addition to obtaining the same dispersion relation in intermediate water depths as those of Madsen *et al.* (1991) and Nwogu (1993), the fully nonlinear Boussinesq equations provide an improved theory for second and third order nonlinear interactions, and can be applied to simulate wave propagation prior to wave breaking where nonlinearity is expected to be large. Numerical examples given by Wei *et al.* (1995) have shown that the fully nonlinear Boussinesq model predict wave heights, wave shapes and wave celerities much more accurately compared to those from weakly nonlinear extended Boussinesq equations.

To extend Boussinesq models to surf zone and swash regions where the effects of wave breaking and bottom friction are important, additional energy dissipation terms are included to the model equations. Observing the similarity between broken wave propagation and hydraulic jumps, Heitner and Housner (1970) proposed an eddy viscosity model to dissipate energy for breaking waves. Energy loss is limited to the front face of waves where the change of wave properties exceeds a certain criteria. Zelt (1991) implemented the eddy viscosity formula in a Lagrangian Boussinesq model to simulate solitary wave breaking and runup. Good agreements between numerical results and experimental data were obtained.

In this study, we are constructing a coastal processes model based on the extended Boussinesq equations. With the improved dispersion relation in intermediate water depth, the model can be applied in relatively deeper water.

Energy dissipation terms similar to those used by Heitner and Housner (1970) and by Zelt (1991) are added to the Boussinesq model to simulate wave propagation in surf zone and swash zone areas. The same high-order predictor-corrector finite difference scheme as that in Wei *et al.* (1995) is used to discretize the model equations and to obtain corresponding solutions.

The Boussinesq equations derived by Nwogu (1993) and by Wei *et al.* (1995) are similar, with extra higher order nonlinear dispersive terms including in the latter set of equations. To illustrate the importance of these terms, we apply the two models to obtain solitary wave solutions for several cases of large nonlinearity. Results with and without high order nonlinear dispersive terms are compared with existing closed-form solutions. To demonstrate the effect of energy dissipation terms included in the model equations, we apply to laboratory study of random wave breaking over a slope (Mase and Kirby, 1992). Comparisons of numerical results and experimental data for wave heights, wave phase and third moment wave statistics will be provided.

Model Description

The fully nonlinear Boussinesq equations of Wei *et al.* (1995) are derived in a similar way as that of Nwogu (1993) by using the velocity at a certain water depth as a dependent variable. In the derivation, however, no assumption was made about the relative size of nonlinear effects. The resulting equations therefore include additional high order nonlinear dispersive terms. The fully nonlinear Boussinesq equations in dimensionless form are

$$\eta_t + \nabla\cdot\Big\{ (h+\delta\eta)\Big[\mathbf{u}_\alpha + \mu^2\left(\frac{1}{2}z_\alpha^2 - \frac{1}{6}(h^2 - h\delta\eta + (\delta\eta)^2)\right)\nabla(\nabla\cdot\mathbf{u}_\alpha) + \mu^2\left(z_\alpha + \frac{1}{2}(h-\delta\eta)\right)\nabla(\nabla\cdot(h\mathbf{u}_\alpha))\Big]\Big\} = 0 \quad (1)$$

$$\mathbf{u}_{\alpha t} + \delta(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + \nabla\eta + \mu^2\left\{\frac{1}{2}z_\alpha^2\nabla(\nabla\cdot\mathbf{u}_{\alpha t}) + z_\alpha\nabla[\nabla\cdot(h\mathbf{u}_{\alpha t})]\right\}$$
$$+ \delta\mu^2\nabla\Big\{ (z_\alpha - \delta\eta)(\mathbf{u}_\alpha\cdot\nabla)[\nabla\cdot(h\mathbf{u}_\alpha)] - \eta\left[\frac{1}{2}\delta\eta\nabla\cdot\mathbf{u}_{\alpha t} + \nabla\cdot(h\mathbf{u}_{\alpha t})\right]$$
$$+ \frac{1}{2}\left[z_\alpha^2 - (\delta\eta)^2\right](\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot\mathbf{u}_\alpha) + \frac{1}{2}[\nabla\cdot(h\mathbf{u}_\alpha) + \delta\eta\nabla\cdot\mathbf{u}_\alpha]^2\Big\} = 0 \quad (2)$$

where η is the surface elevation, h the water depth, $\mathbf{u}_\alpha$ the horizontal velocity vector at the water depth of $z = z_\alpha h$, ∇ the horizontal gradient operator, subscript t the partial derivative in time, δ the ratio of typical wave height to typical water depth, μ^2 the product square of typical wavenumber and typical water depth. The two dimensionless parameters δ and μ^2 represent the effects of nonlinearity and frequency dispersion, respectively. For the case of small nonlinearity where terms with order of $\delta\mu^2$ and higher may be neglected, the above equations reduce to Nwogu's extended Boussinesq equations. As will be shown in the next section, however, it is these high order nonlinear dispersive terms that provide accurate predictions for wave heights and wave shapes of solitary wave solutions.

In addition to the necessary corrections for waves with strong nonlinearity, the fully nonlinear Boussinesq equations also provide a physically correct condition for mass balance at the shoreline. As shown in the continuity equation (1), all the mass flux terms have a common factor $h+\delta\eta$. It is thus clear that the mass flux at the shoreline will go to zero when the total water depth $h+\delta\eta$ becomes zero. This result is expected on physical grounds and appears in the nonlinear shallow water equations and in Boussinesq equations where the depth-averaged velocity is the dependent variable. However, this condition is not automatically satisfied by Nwogu's or other weakly nonlinear Boussinesq model based on a velocity other than the depth-averaged value, making the application of these models problematic at the shoreline.

Following the approach by Heitner and Housner (1970) and by Zelt (1991), the overall effect of wave breaking is simulated by adding the following energy dissipation term into the right hand side of equation (2)

$$F_{break} = [(\nu_b u_x)_x + (\nu_b u_y)_y, (\nu_b v_x)_x + (\nu_b v_y)_y] \tag{3}$$

where u and v are the x and y component of $\mathbf{u}_\alpha$. The notation ν_b is the eddy viscosity defined as

$$\nu_b = -B\alpha^2 h^2 \nabla \cdot \mathbf{u}_\alpha \tag{4}$$

where B is a coefficient related to the local property of the waves and the corresponding critical value for wave breaking to take place. The coefficient α is the mixing length parameter, whose value is determined empirically. In the computation which will be shown below, we use it as a constant, *i.e.* $\alpha = 2$. Zelt (1991) defined the critical value of velocity gradient as $u_x^* = -0.3\sqrt{g/h}$, and coefficient B is given by

$$B = \begin{cases} 1 & \text{if} \quad \nabla\cdot\boldsymbol{u} \le 2u_x^* \\ \left(\frac{\nabla\cdot\boldsymbol{u}}{u_x^*} - 1\right) & \text{if} \quad 2u_x^* < \nabla\cdot\boldsymbol{u} \le u_x^* \\ 0 & \text{if} \quad \nabla\cdot\boldsymbol{u} > u_x^* \end{cases} \tag{5}$$

In swash zone areas where water moves up and down along the beach face, the effect of bottom friction is no longer small. However, exact expressions for friction force are difficult to obtain and parameterization formula has to be used instead. Here, we use Chevy's formula to compute bottom friction, *i.e.*, adding the following expression to the right hand side of equation (2)

$$F_{bottom} = -\frac{\mathbf{u}_\alpha|\mathbf{u}_\alpha|}{C_f^2(h+\delta\eta)} \tag{6}$$

where C_f is the Chevy's dimensional coefficient which is a function of bottom roughness and the magnitude of velocity $\mathbf{u}_\alpha$. For simplicity, a constant coefficient is used in the model. To overcome the difficulty of defining physical variables in the physically dry grids in a Eulerian system, a minimum thickness of water is maintained for those dry grids. This thin layer of water remains almost motionless due to the balance between gravitational force and bottom friction.

A high-order numerical scheme is utilized to obtain solutions to the model equations described above. The scheme is similar to that of Wei *et al.* (1995), with the inclusion of extra energy dissipation terms in the governing equations. Standard five-point and three-point finite difference schemes are used to discretize the first-order spatial derivative terms and other higher order terms, respectively. For time-stepping, we use the fourth-order Adams-Bashforth- Moulton predictor-corrector scheme. Detailed formulations can be found in Wei *et al.* (1995).

Results and Comparisons

To demonstrate the importance of high order nonlinear dispersion terms in the fully nonlinear Boussinesq equations, we apply Boussinesq models with and without those terms to obtain solitary wave solutions for large effects of nonlinearity. These permanent wave solutions are then compared with other existing solutions given by Tanaka (1986) and by Seabra-Santos *et al.* (1987). The approximate analytical solution of solitary waves derived by Wei and Kirby (1995) is not valid for this case due to the assumption of small nonlinearity used in the derivation. As far as we know, there is no closed form solutions for solitary wave for the extended Boussinesq equations. Therefore, numerical experiments are performed here to obtain solitary waves corresponding to Boussinesq equations.

Using the approximate expressions of Wei and Kirby (1995) as inputs, we ran both models over a long distance with constant depth. At the beginning of these computations, wave heights and wave shapes changed constantly, and small oscillatory tails developed behind the main waves as they propagated forward. After running the models for a long time, however, the changes of form became negligible and wave shapes stabilized, indicating that a numerical permanent-form solitary wave solution corresponding to each of the Boussinesq models was obtained. Due to the discrepancy between initial and stabilized wave forms, several runs of the models were required to obtain a solitary wave with desired height.

Closed-form solutions of permanent waves corresponding to different wave equations have been investigated in the past. Problems for wave propagation in inviscid and impressible fluid are governed by the Laplace equation inside the fluid and associate surface and bottom boundary conditions. By assuming small effect of dispersion and/or nonlinearity, approximate forms of equations for wave propagation are derived to reduce the dependence of vertical space. Shallow water equations, Boussinesq equations and the equations derived by Serre (1953) all belong to approximate equations. Permanent solutions corresponding to the original wave problem were obtained by Tanaka (1986), who used conformal mapping to transform the curved surface elevation into a straight line. Though numerical iteration is required to solve the closed- form expression for given wave height, Tanaka's solutions are the most accurate to compare with. For long waves propagating over constant depth, Serre (1953) derived a set of equations which include more high order nonlinear dispersion terms than standard Boussinesq equations, but not as many as the fully nonlinear Boussinesq equations of Wei *et al.* (1995). As shown in the paper by Seabra-Santos *et al.* (1987), Serre's equations has closed-form solitary wave solutions.

Figure 1 shows the comparisons between numerical solutions of solitary waves by extended Boussinesq models and the closed-form solutions of Tanaka (1986), and the solutions of Serre's equations. The results from the fully nonlinear Boussinesq model match very well Tanaka's solution for all three height to depth ratios. The permanent solutions corresponding to the extended Boussinesq equations without high order nonlinear dispersive terms and to Serre's equations, on the other hand, predicted either narrower or wider solitary wave shapes. The results imply that all the high order nonlinear dispersive terms are important for simulating the propagation of waves with strong nonlinearity. Wei *et al.* (1995) applied the Boussinesq models to study solitary wave shoaling over different slopes and undular bore propagation over constant depth. Results of wave height, wave shape and wave celerity from the fully nonlinear Boussinesq models are much better than those from weakly nonlinear forms of Boussinesq equations, especially near wave breaking regions where the effect of nonlinearity is no long small.

By including energy dissipation terms into the extended Boussinesq equations, we could further extend the models to surf zone and swash zone areas, where the effects of wave breaking and bottom friction are substantial. Mase and Kirby (1992) conducted a laboratory study of random wave breaking over an impermeable beach with constant slope. Two sets of random wave with peak frequencies $0.75Hz$ (test one) and $1.0Hz$ (test two) were generated by a piston wavemaker. Random waves propagated over a constant depth of 47.0 cm and then over a constant slope of 1/20. Time series of surface elevation were collected at 12 gage locations along the slope, which will serve as a verification for the comprehensive numerical model. The corresponding water depths for the wave gauges are: 47 cm, 35 cm, 30 cm, 25 cm, 20 cm, 17.5 cm, 15.0 cm, 12.5 cm, 10.0 cm, 7.5 cm, 5.0 cm, 2.5 cm. Wave breaking was observed to start at the water depth of 17.5 cm.

The typical kh value (k is the wavenumber corresponding to the peak frequency and h the water depth) for test two is close to 2, making it impossible to employ the standard Boussinesq model due to the dispersion errors. The

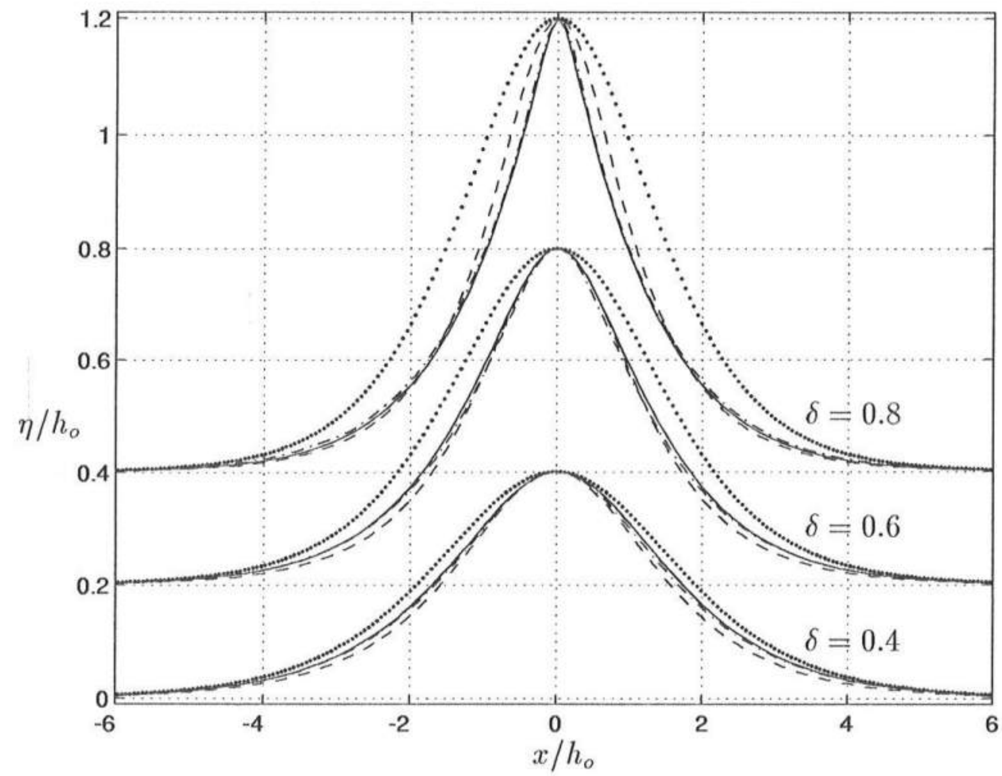

Figure 1: Comparison of solitary wave shapes for $\delta = 0.4, 0.6, 0.8$. Tanaka's solution (——); fully nonlinear Boussinesq model (-.-.-.); Boussinesq model without high-order nonlinear dispersive terms (- - -); Closed-form solutions of Serre's equations (.....)

extended Boussinesq models, on the other hand, could be applied here to simulate wave transformation due to improved dispersion relation. In the numerical computation, experimental data from gage 1 were used as input to the model. Figure 2 shows the comparison for surface elevation in the shallow regions where wave breaking becomes dominant, for a duration of 30 seconds. Except for small discrepancy in wave phase, the model predicts the change of wave form and the decrease of wave height due to the effects of wave shoaling and breaking. The results indicate that the model is capable of simulating spectral waves whose kh values vary over a large range.

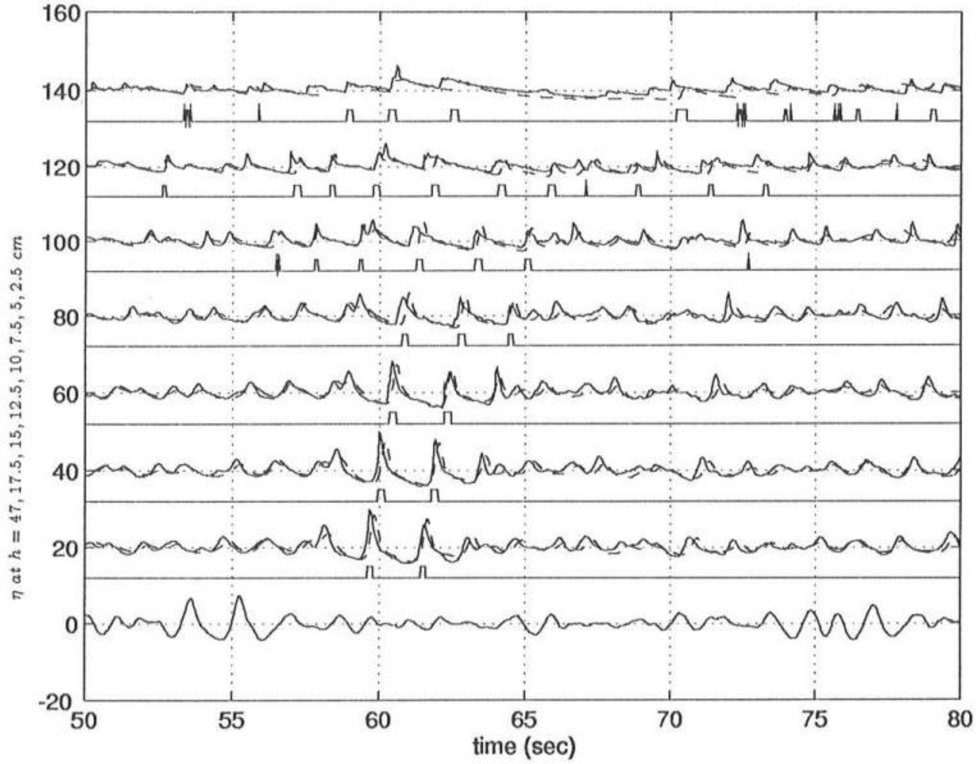

Figure 2: Comparison of surface elevation at different water depths. Solid lines is experimental data (Mase and Kirby, 1992). Dashed line is numerical result. Wave breaking indicated by blips below the water surface traces.

To further compare the wave properties between the model and data, we ran the model for the entire data set. Similar to those in Figure 2, time series comparisons are consistently good. Wave statistics were then computed from the resulting time series. Figure 3 shows the comparisons of wave skewness and asymmetry for all gage locations. Except for the last gage at depth of 2.5 cm where skewness from numerical model underpredicted the data, good agreement were found between data and model. Since the rate of sediment transport is closely related to the third moment of wave statistics, the model has potential coastal engineering application for estimating sand movement along beaches and coastlines.

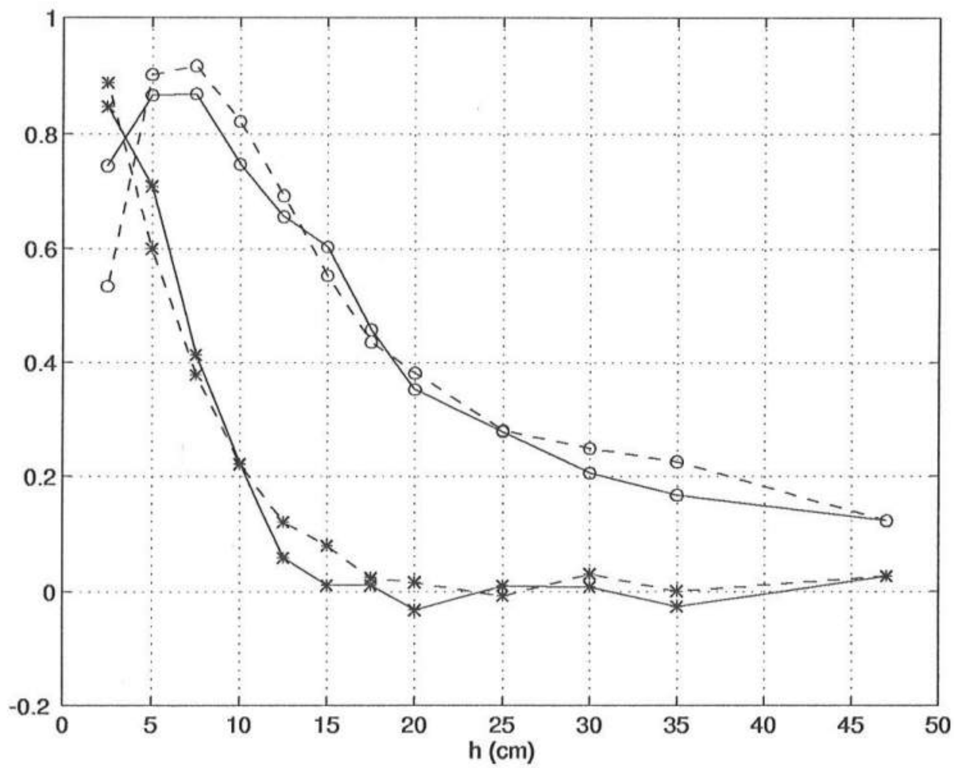

Figure 3: Comparison of skewness (o) and asymmetry (*) at different water depths. Solid lines is experimental data (Mase and Kirby, 1992). Dashed line is numerical result.

Conclusion

The fully nonlinear Boussinesq equations have been shown to be capable of predicting permanent wave solutions more accurately than the extended Boussinesq and Serre's equations, indicating that the high order nonlinear dispersion terms in the equations are important for simulating wave transformation with strong nonlinear interaction. By including energy dissipation terms in the model equations, the effects of wave breaking and bottom friction can be simulated. Comparison of numerical results with laboratory data for random wave breaking over a slope are quite good. Agreements have been found not only for wave heights and wave shapes, but also for skewness and asymmetry. This results is significant since sediment transport calculation depends on the third moment of wave statistics. Future work will be to extend the model to 2-D cases and to obtain wave-induced currents by time averaging method.

Acknowledgment

This research was supported by the Army Research Office through University Research Initiative grant DAAL 03-92-G-0116.

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