Discussion of "Note on a Nonlinearity Parameter of Surface Waves" by S. Beji

James T. Kirby

Center for Applied Coastal Research
Ocean Engineering Laboratory
University of Delaware
Newark, DE 19716, U.S.A.

Discussion of "Note on a nonlinearity parameter of surface waves" by S. Beji

James T. Kirby
Center for Applied Coastal Research, University of Delaware
Newark, DE. 19716, kirby@coastal.udel.edu

Abstract

A recently proposed rescaling of the governing equations for inviscid, irrotational, free surface hydrodynamics is reconsidered. A redefinition of the proposed scaling parameters leads to a nondimensionalized problem which contains both the usual intermediate depth scaling and the shallow water scaling as asymptotic limits, thus eliminating the need for a separate development of scaling arguments for the two regimes as given in most standard references.

1 Introduction

Typical scalings of the water wave problem revolve around the use of three representative length scales given by water depth h, wave amplitude a and inverse wavenumber k^{-1}, and introduce three nondimensional parameters representing ratios of these scales, only two of which can be independent. In the following, we denote these ratios by

$$\mu = kh \quad \epsilon = ka \quad \delta = a/h \tag{1}$$

Of these, μ represents the ratio of water depth to wavelength and typically is used to distinguish between short wave ($\mu = O(1)$) and long wave ($\mu \ll 1$) regimes. The remaining two parameters characterize the importance of nonlinearity in the problem, with ϵ arising naturally in the short wave limit and δ arising naturally in the long wave limit, as discussed below.

Recently, Beji (1995) has suggested a scaling of the problem which introduces a so-called wave Froude number, which we denote in the following by

$$\alpha = \frac{ga}{c^2} \tag{2}$$

where g is gravitational acceleration and c denotes wave phase speed as dictated by linear theory,

$$c^2 = \frac{g}{k} \tanh kh = gh \frac{\tanh \mu}{\mu} \tag{3}$$

1

The resulting scaling identifies α as the principle parameter denoting the importance of nonlinearity within the governing equations. However, the formulation retains wave steepness ϵ as a distinct nonlinear parameter and characterizes one linear term by the ratio of the two nonlinear parameters ϵ and α.

In the following, we review the conventional scaling analysis of the short and long wave regimes for subsequent reference. We then review Beji's suggested rescaling and consider its asymptotic relation to the scaled equations obtained with conventional methods. A new scaling is then suggested and is shown to contain both short and long wave regimes as proper asymptotic limits. Finally, the relationship of Beji's wave Froude number to previous work is discussed.

2   A general framework and conventional scalings

Since the primary difference between short and long wave scaling resides in the different assumptions about vertical length scales, we proceed here by leaving the vertical length scale L_v unspecified during the initial stages of the analysis. Denoting dimensional variables by primes, we introduce dimensionless variables according to

$$x, y = kx', ky'; \quad z = z'/L_v; \quad t = kct'; \quad \eta = \eta'/a \tag{4}$$

We also leave the explicit expression for the scale of the velocity potential ϕ' somewhat open; instead, we note that its scale should be given by the product of a velocity scale, a horizontal length scale and a parameter characterising wave amplitude. We thus write

$$\phi = \phi'/\phi_0; \quad \phi_0 = ck^{-1}\alpha \tag{5}$$

where α is an unspecified parameter characterising amplitude. The resulting nondimensional Laplace equation and bottom boundary condition are given by

$$\phi_{xx} + \phi_{yy} + (kL_v)^{-2}\phi_{zz} = 0; \quad -(h/L_v) \leq z \leq (a/L_v)\eta \tag{6}$$

$$\phi_z = 0; \quad z = -(h/L_v) \tag{7}$$

The dynamic boundary condition is given by

$$\eta + \left[\frac{kc\phi_0}{ga}\right]\phi_t + \left[\frac{k^2\phi_0^2}{2ga}\right]\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{(kL_v)^2}(\phi_z)^2\right) = 0; \quad z = (a/L_v)\eta \tag{8}$$

Forcing a leading order balance between pressure gradient and local acceleration in the Bernoulli equation would cause the first square-bracketed coefficient in (8) to be unity, which leads to the choice $\alpha = ga/c^2$, as indicated in (2). Beji's parameter is thus a natural choice for a nonlinear parameter in the absence of any further specification of an asymptotic

form for c^2. Applying this choice then leads to a revised dimensionless dynamic condition given by

$$\eta + \phi_t + \frac{\alpha}{2}\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{(kL_v)^2}(\phi_z)^2\right) = 0; \qquad z = (a/L_v)\eta \qquad (9)$$

The corresponding kinematic surface boundary condition becomes

$$\eta_t + \alpha\left(\phi_x\eta_x + \phi_y\eta_y\right) - \frac{\alpha}{(kL_v)^2(a/L_v)}\phi_z = 0; \qquad z = (a/L_v)\eta \qquad (10)$$

2.1 Short wave scaling

Conventional results for short and long wave scaling follow from a choice of the vertical length scale L_v and an asymptotic form for c^2. For short waves, we assume

$$L_v = k^{-1}; \qquad c^2 = gk^{-1}; \qquad \alpha = \epsilon \qquad (11)$$

and obtain the usual dimensionless problem

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0; \qquad -\mu \le z \le \epsilon\eta \qquad (12)$$

$$\phi_z = 0; \qquad z = -\mu \qquad (13)$$

$$\eta + \phi_t + \frac{\epsilon}{2}\left((\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right) = 0; \qquad z = \epsilon\eta \qquad (14)$$

$$\eta_t + \epsilon\left(\phi_x\eta_x + \phi_y\eta_y\right) - \phi_z = 0; \qquad z = \epsilon\eta \qquad (15)$$

The problem is characterized by a single wave amplitude parameter ϵ. The dispersion parameter μ appears but only as a scale for the total water depth.

2.2 Long wave scaling

For the case of long waves, we adopt a fixed vertical scale for the wave and invoke shallow water asymptotes, and get

$$L_v = h; \qquad c^2 = gh; \qquad \alpha = \delta \qquad (16)$$

The revised dimensionless problem is then given by

$$\phi_{xx} + \phi_{yy} + \frac{1}{\mu^2}\phi_{zz} = 0; \qquad -1 \le z \le \delta\eta \qquad (17)$$

$$\phi_z = 0; \qquad z = -1 \qquad (18)$$

$$\eta + \phi_t + \frac{\delta}{2}\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{\mu^2}(\phi_z)^2\right) = 0; \qquad z = \delta\eta \qquad (19)$$

$$\eta_t + \delta\left(\phi_x\eta_x + \phi_y\eta_y\right) - \frac{1}{\mu^2}\phi_z = 0; \qquad z = \delta\eta \qquad (20)$$

and the dynamical significance of the dispersion parameter μ^2 in the establishment of a distinction between horizontal and vertical motions is apparent.

3

3 Beji's scaling

Beji (1995) suggests a different approach which revolves around the retention of the parameter α. Beji then chooses a fixed vertical length scale k^{-1}. The general set of equations (6)-(10) then reduces to

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0; \quad -\mu \leq z \leq \epsilon\eta \tag{21}$$

$$\phi_z = 0; \quad z = -\mu \tag{22}$$

$$\eta + \phi_t + \frac{\alpha}{2}\left((\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right) = 0; \quad z = \epsilon\eta \tag{23}$$

$$\eta_t + \alpha\left(\phi_x\eta_x + \phi_y\eta_y\right) - \frac{\alpha}{\epsilon}\phi_z = 0; \quad z = \epsilon\eta \tag{24}$$

as given in Beji (1995). The resulting set of equations has two unusual features; the retention of two wave amplitude parameters, and the characterization of a linear term by the ratio of the two wave amplitude parameters. The first result is merely inconvenient, making the system somewhat more complex than any corresponding conventional form of the equations. The second result must be regarded as a defect of the proposed scaling, since there is no reason that the linear balance between vertical motion and surface movement should be influenced by any scale describing the wave amplitude. Beji recognized that this ratio constitutes a revision of the shallow water dispersion parameter, but the correct approach to writing it in such a way is not contained within Beji's proposed scaling.

Another defect of the proposed scaling lies in its inability to reproduce the shallow water theory. This fault results from the apparent uniform choice of k^{-1} as a vertical length scale. Taking the asymptotic limit $c^2 \to gk^{-1}$ in (21)-(24) reproduces the short-wave analysis presented above. However, taking the long wave limit $c^2 \to gh$ produces the defective system

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0; \quad -\mu \leq z \leq \epsilon\eta \tag{25}$$

$$\phi_z = 0; \quad z = -\mu \tag{26}$$

$$\eta + \phi_t + \frac{\delta}{2}\left((\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right) = 0; \quad z = \epsilon\eta \tag{27}$$

$$\eta_t + \alpha\left(\phi_x\eta_x + \phi_y\eta_y\right) - \frac{1}{\mu}\phi_z = 0; \quad z = \epsilon\eta \tag{28}$$

in which the scale of vertical velocity in the kinematic boundary condition is misrepresented, and where the distinction between horizontal and vertical contributions to the quadratic Bernoulli term and in Laplace's equation is absent altogether.

4

4 A revised scaling

The origin of the defects noted in the previous section stems from an unnatural choice for the vertical length scale, which should be approached with the same care as the nonlinear parameter. In particular, referring to (6)-(10), if we wish to unify the problem description using a single nonlinear parameter, it is clear that the ratio a/L_v should be set equal to α, leading to a choice of vertical scale

$$L_v = \frac{c^2}{g} \tag{29}$$

which has a continuous dependence on depth, and which is asymptotic to k^{-1} as $\mu \to \infty$ and to h as $\mu \to 0$. Also note that

$$L_v = \frac{c^2}{g} = k^{-1} \tanh \mu = k^{-1} \tilde{\mu} \tag{30}$$

where $\tilde{\mu} = \tanh \mu$ is introduced as a revised dispersion parameter. Using this choice for L_v then leads to the dimensionless system

$$\phi_{xx} + \phi_{yy} + \frac{1}{\tilde{\mu}^2} \phi_{zz} = 0; \qquad -\frac{\mu}{\tilde{\mu}} \leq z \leq \alpha \eta \tag{31}$$

$$\phi_z = 0; \qquad z = -\frac{\mu}{\tilde{\mu}} \tag{32}$$

$$\eta + \phi_t + \frac{\alpha}{2} \left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{\tilde{\mu}^2} (\phi_z)^2 \right) = 0; \qquad z = \alpha \eta \tag{33}$$

$$\eta_t + \alpha \left(\phi_x \eta_x + \phi_y \eta_y \right) - \frac{1}{\tilde{\mu}^2} \phi_z = 0; \qquad z = \alpha \eta \tag{34}$$

The resulting set of equations contains both the usual short and long wave scalings in the appropriate asymptotic limit, since the dispersion parameter $\tilde{\mu} \to 1$ as $\mu \to \infty$ and $\tilde{\mu} \to \mu$ as $\mu \to 0$. Equations (31)-(34) thus represent the rescaling of the governing equations proposed here.

5 Discussion

Given the natural appearance of Beji's wave Froude number in the initial scaling argument of section 2, it is surprising that the parameter has not been utilized previously in the literature. It bears some resemblance to the expansion parameter introduced by Cokelet (1977), which compares velocities in a coordinate system translating at the wave speed to the wave speed itself. Cokelet's parameter is defined according to

$$\epsilon_c^2 = 1 - \frac{q_c^2 q_t^2}{c^4} \tag{35}$$

where q_c and q_t denote fluid velocities at the wave crest and trough in the moving coordinate system, and c denotes the nonlinear phase speed of the wave. Note that $0 \leq \epsilon_c^2 \leq 1$ denotes

the range from lowest (with $q_c = q_t = c$) to highest (with $q_c = 0$) waves. For small amplitude waves, we may approximate c by the linear value, and

$$q_c = c - \frac{ga}{c} = c(1 - \alpha) \tag{36}$$

$$q_t = c + \frac{ga}{c} = c(1 + \alpha) \tag{37}$$

Substituting (36)-(37) in (35) and retaining leading order terms in α gives

$$\epsilon_c = \sqrt{2}\alpha \tag{38}$$

and the connection here to Cokelet's parameter is clear.

References

Beji, S., 1995, "Note on a nonlinearity parameter of surface waves", *Coastal Engineering*, **25**, 81-85.

Cokelet, E. D., 1977, "Steep gravity waves in water of arbitrary uniform depth", *Phil. Trans. Roy. Soc. London A*, **286**, 183-230.

Discussion of "Note on a Nonlinearity Parameter of Surface Waves" by S. Beji

James T. Kirby

To appear in *Coastal Engineering*

RESEARCH REPORT NO. CACR-98-03

April, 1998

Center for Applied Coastal Research
Ocean Engineering Laboratory
University of Delaware
Newark, DE 19716, U.S.A.

Discussion of "Note on a nonlinearity parameter of surface waves" by S. Beji

James T. Kirby

Center for Applied Coastal Research, University of Delaware

Newark, DE. 19716, kirby@coastal.udel.edu

Abstract

A recently proposed rescaling of the governing equations for inviscid, irrotational, free surface hydrodynamics is reconsidered. A redefinition of the proposed scaling parameters leads to a nondimensionalized problem which contains both the usual intermediate depth scaling and the shallow water scaling as asymptotic limits, thus eliminating the need for a separate development of scaling arguments for the two regimes as given in most standard references.

1 Introduction

Typical scalings of the water wave problem revolve around the use of three representative length scales given by water depth h, wave amplitude a and inverse wavenumber k^{-1}, and introduce three nondimensional parameters representing ratios of these scales, only two of which can be independent. In the following, we denote these ratios by

$$\mu = kh \quad \epsilon = ka \quad \delta = a/h \tag{1}$$

Of these, μ represents the ratio of water depth to wavelength and typically is used to distinguish between short wave ($\mu = O(1)$) and long wave ($\mu \ll 1$) regimes. The remaining two parameters characterize the importance of nonlinearity in the problem, with ϵ arising naturally in the short wave limit and δ arising naturally in the long wave limit, as discussed below.

Recently, Beji (1995) has suggested a scaling of the problem which introduces a so-called wave Froude number, which we denote in the following by

$$\alpha = \frac{ga}{c^2} \tag{2}$$

where g is gravitational acceleration and c denotes wave phase speed as dictated by linear theory,

$$c^2 = \frac{g}{k} \tanh kh = gh \frac{\tanh \mu}{\mu} \tag{3}$$

The resulting scaling identifies α as the principle parameter denoting the importance of nonlinearity within the governing equations. However, the formulation retains wave steepness ϵ as a distinct nonlinear parameter and characterizes one linear term by the ratio of the two nonlinear parameters ϵ and α.

In the following, we review the conventional scaling analysis of the short and long wave regimes for subsequent reference. We then review Beji's suggested rescaling and consider its asymptotic relation to the scaled equations obtained with conventional methods. A new scaling is then suggested and is shown to contain both short and long wave regimes as proper asymptotic limits. Finally, the relationship of Beji's wave Froude number to previous work is discussed.

2   A general framework and conventional scalings

Since the primary difference between short and long wave scaling resides in the different assumptions about vertical length scales, we proceed here by leaving the vertical length scale L_v unspecified during the initial stages of the analysis. Denoting dimensional variables by primes, we introduce dimensionless variables according to

$$x, y = kx', ky'; \quad z = z'/L_v; \quad t = kct'; \quad \eta = \eta'/a \tag{4}$$

We also leave the explicit expression for the scale of the velocity potential ϕ' somewhat open; instead, we note that its scale should be given by the product of a velocity scale, a horizontal length scale and a parameter characterising wave amplitude. We thus write

$$\phi = \phi'/\phi_0; \quad \phi_0 = ck^{-1}\alpha \tag{5}$$

where α is an unspecified parameter characterising amplitude. The resulting nondimensional Laplace equation and bottom boundary condition are given by

$$\phi_{xx} + \phi_{yy} + (kL_v)^{-2}\phi_{zz} = 0; \quad -(h/L_v) \leq z \leq (a/L_v)\eta \tag{6}$$

$$\phi_z = 0; \quad z = -(h/L_v) \tag{7}$$

The dynamic boundary condition is given by

$$\eta + \left[\frac{kc\phi_0}{ga}\right]\phi_t + \left[\frac{k^2\phi_0^2}{2ga}\right]\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{(kL_v)^2}(\phi_z)^2\right) = 0; \quad z = (a/L_v)\eta \tag{8}$$

Forcing a leading order balance between pressure gradient and local acceleration in the Bernoulli equation would cause the first square-bracketed coefficient in (8) to be unity, which leads to the choice $\alpha = ga/c^2$, as indicated in (2). Beji's parameter is thus a natural choice for a nonlinear parameter in the absence of any further specification of an asymptotic

form for c^2. Applying this choice then leads to a revised dimensionless dynamic condition given by

$$\eta + \phi_t + \frac{\alpha}{2}\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{(kL_v)^2}(\phi_z)^2\right) = 0; \qquad z = (a/L_v)\eta \tag{9}$$

The corresponding kinematic surface boundary condition becomes

$$\eta_t + \alpha\left(\phi_x \eta_x + \phi_y \eta_y\right) - \frac{\alpha}{(kL_v)^2(a/L_v)}\phi_z = 0; \qquad z = (a/L_v)\eta \tag{10}$$

2.1 Short wave scaling

Conventional results for short and long wave scaling follow from a choice of the vertical length scale L_v and an asymptotic form for c^2. For short waves, we assume

$$L_v = k^{-1}; \qquad c^2 = gk^{-1}; \qquad \alpha = \epsilon \tag{11}$$

and obtain the usual dimensionless problem

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0; \qquad -\mu \leq z \leq \epsilon\eta \tag{12}$$

$$\phi_z = 0; \qquad z = -\mu \tag{13}$$

$$\eta + \phi_t + \frac{\epsilon}{2}\left((\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right) = 0; \qquad z = \epsilon\eta \tag{14}$$

$$\eta_t + \epsilon\left(\phi_x \eta_x + \phi_y \eta_y\right) - \phi_z = 0; \qquad z = \epsilon\eta \tag{15}$$

The problem is characterized by a single wave amplitude parameter ϵ. The dispersion parameter μ appears but only as a scale for the total water depth.

2.2 Long wave scaling

For the case of long waves, we adopt a fixed vertical scale for the wave and invoke shallow water asymptotes, and get

$$L_v = h; \qquad c^2 = gh; \qquad \alpha = \delta \tag{16}$$

The revised dimensionless problem is then given by

$$\phi_{xx} + \phi_{yy} + \frac{1}{\mu^2}\phi_{zz} = 0; \qquad -1 \leq z \leq \delta\eta \tag{17}$$

$$\phi_z = 0; \qquad z = -1 \tag{18}$$

$$\eta + \phi_t + \frac{\delta}{2}\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{\mu^2}(\phi_z)^2\right) = 0; \qquad z = \delta\eta \tag{19}$$

$$\eta_t + \delta\left(\phi_x \eta_x + \phi_y \eta_y\right) - \frac{1}{\mu^2}\phi_z = 0; \qquad z = \delta\eta \tag{20}$$

and the dynamical significance of the dispersion parameter μ^2 in the establishment of a distinction between horizontal and vertical motions is apparent.

3

3 Beji's scaling

Beji (1995) suggests a different approach which revolves around the retention of the parameter α. Beji then chooses a fixed vertical length scale k^{-1}. The general set of equations (6)-(10) then reduces to

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0; \quad -\mu \le z \le \epsilon\eta \tag{21}$$

$$\phi_z = 0; \quad z = -\mu \tag{22}$$

$$\eta + \phi_t + \frac{\alpha}{2}\left((\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right) = 0; \quad z = \epsilon\eta \tag{23}$$

$$\eta_t + \alpha\left(\phi_x\eta_x + \phi_y\eta_y\right) - \frac{\alpha}{\epsilon}\phi_z = 0; \quad z = \epsilon\eta \tag{24}$$

as given in Beji (1995). The resulting set of equations has two unusual features; the retention of two wave amplitude parameters, and the characterization of a linear term by the ratio of the two wave amplitude parameters. The first result is merely inconvenient, making the system somewhat more complex than any corresponding conventional form of the equations. The second result must be regarded as a defect of the proposed scaling, since there is no reason that the linear balance between vertical motion and surface movement should be influenced by any scale describing the wave amplitude. Beji recognized that this ratio constitutes a revision of the shallow water dispersion parameter, but the correct approach to writing it in such a way is not contained within Beji's proposed scaling.

Another defect of the proposed scaling lies in its inability to reproduce the shallow water theory. This fault results from the apparent uniform choice of k^{-1} as a vertical length scale. Taking the asymptotic limit $c^2 \to gk^{-1}$ in (21)-(24) reproduces the short-wave analysis presented above. However, taking the long wave limit $c^2 \to gh$ produces the defective system

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0; \quad -\mu \le z \le \epsilon\eta \tag{25}$$

$$\phi_z = 0; \quad z = -\mu \tag{26}$$

$$\eta + \phi_t + \frac{\delta}{2}\left((\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right) = 0; \quad z = \epsilon\eta \tag{27}$$

$$\eta_t + \alpha\left(\phi_x\eta_x + \phi_y\eta_y\right) - \frac{1}{\mu}\phi_z = 0; \quad z = \epsilon\eta \tag{28}$$

in which the scale of vertical velocity in the kinematic boundary condition is misrepresented, and where the distinction between horizontal and vertical contributions to the quadratic Bernoulli term and in Laplace's equation is absent altogether.

4

4 A revised scaling

The origin of the defects noted in the previous section stems from an unnatural choice for the vertical length scale, which should be approached with the same care as the nonlinear parameter. In particular, referring to (6)-(10), if we wish to unify the problem description using a single nonlinear parameter, it is clear that the ratio a/L_v should be set equal to α, leading to a choice of vertical scale

$$L_v = \frac{c^2}{g} \tag{29}$$

which has a continuous dependence on depth, and which is asymptotic to k^{-1} as $\mu \to \infty$ and to h as $\mu \to 0$. Also note that

$$L_v = \frac{c^2}{g} = k^{-1} \tanh \mu = k^{-1} \tilde{\mu} \tag{30}$$

where $\tilde{\mu} = \tanh \mu$ is introduced as a revised dispersion parameter. Using this choice for L_v then leads to the dimensionless system

$$\phi_{xx} + \phi_{yy} + \frac{1}{\tilde{\mu}^2} \phi_{zz} = 0; \qquad -\frac{\mu}{\tilde{\mu}} \le z \le \alpha\eta \tag{31}$$

$$\phi_z = 0; \qquad z = -\frac{\mu}{\tilde{\mu}} \tag{32}$$

$$\eta + \phi_t + \frac{\alpha}{2}\left((\phi_x)^2 + (\phi_y)^2 + \frac{1}{\tilde{\mu}^2}(\phi_z)^2 \right) = 0; \qquad z = \alpha\eta \tag{33}$$

$$\eta_t + \alpha\left(\phi_x \eta_x + \phi_y \eta_y \right) - \frac{1}{\tilde{\mu}^2}\phi_z = 0; \qquad z = \alpha\eta \tag{34}$$

The resulting set of equations contains both the usual short and long wave scalings in the appropriate asymptotic limit, since the dispersion parameter $\tilde{\mu} \to 1$ as $\mu \to \infty$ and $\tilde{\mu} \to \mu$ as $\mu \to 0$. Equations (31)-(34) thus represent the rescaling of the governing equations proposed here.

5 Discussion

Given the natural appearance of Beji's wave Froude number in the initial scaling argument of section 2, it is surprising that the parameter has not been utilized previously in the literature. It bears some resemblance to the expansion parameter introduced by Cokelet (1977), which compares velocities in a coordinate system translating at the wave speed to the wave speed itself. Cokelet's parameter is defined according to

$$\epsilon_c^2 = 1 - \frac{q_c^2 q_t^2}{c^4} \tag{35}$$

where q_c and q_t denote fluid velocities at the wave crest and trough in the moving coordinate system, and c denotes the nonlinear phase speed of the wave. Note that $0 \le \epsilon_c^2 \le 1$ denotes

the range from lowest (with $q_c = q_t = c$) to highest (with $q_c = 0$) waves. For small amplitude waves, we may approximate c by the linear value, and

$$q_c = c - \frac{ga}{c} = c(1 - \alpha) \tag{36}$$

$$q_t = c + \frac{ga}{c} = c(1 + \alpha) \tag{37}$$

Substituting (36)-(37) in (35) and retaining leading order terms in α gives

$$\epsilon_c = \sqrt{2}\alpha \tag{38}$$

and the connection here to Cokelet's parameter is clear.

References

Beji, S., 1995, "Note on a nonlinearity parameter of surface waves", *Coastal Engineering*, **25**, 81-85.

Cokelet, E. D., 1977, "Steep gravity waves in water of arbitrary uniform depth", *Phil. Trans. Roy. Soc. London A*, **286**, 183-230.