A Note on the Modified Mild-Slope Equation

James T. Kirby and Shubhra K. Misra

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Center for Applied Coastal Research
Ocean Engineering Laboratory
University of Delaware
Newark, DE 19716, U.S.A.

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James T. Kirby and Shubhra K. Misra
Center for Applied Coastal Research
University of Delaware, Newark, DE 19716 USA
kirby@coastal.udel.edu

Abstract

We examine the formally small terms appearing in the modified mild-slope equation (MMSE) of Massel (1993) and Chamberlain & Porter (1995). By expanding the Hamiltonian for the problem with respect to small variations about a constant-depth reference state, we develop a leading-order mild-slope equation which retains some but not all of the previously identified additonal terms appearing in the MMSE. The results indicate that the usual leading-order mild-slope equation (MSE) of Berkhoff (1972) is incorrect as a consistent approximation for wave motion over a slowly-varying bed, which may account for some of it's apparent shortcomings. The results also differ from recent results of Miles and Chamberlain (1998), which identify the usual mild-slope equation as the first approximation in a heirarchy. Numerical examples which illustrate the effect of included and neglected terms are shown and indicate that the model suggested here is a robust predictor of wave reflection in relevant cases.

1 Introduction

The mild-slope equation (MSE) for surface wave propagation in water of slowly varying depth has a long and important history, both as a direct method for performing wave field calculations using the full elliptic boundary value problem, and as the basis for the derivation of asymptotic methods for forward-propagating waves in the parabolic approximation (Kirby, 1986a; Martin et al, 1997). The basic MSE model, developed initially by Berkhoff (1972) and elaborated by Smith & Sprinks (1975), has long been accepted as a means for obtaining a useful representation of topographic wave scattering processes, with Booij (1983) suggesting that the model is useful and accurate for predicting reflection from depth transitions having slopes up to 1:3. However, more recently, one class of bottom configuration has been found for which the original MSE proves to be completely inadequate: a bottom with undular, regularly spaced bed forms. For this case, Kirby (1986b) derived an extended mild-slope equation (EMSE, so named by O'Hare & Davies, 1993) by expanding the bottom boundary condition about a slowly-varying mean bed level, developing the model coefficients with respect to this mean level, and retaining the leading order effect of the perturbation representing the rapid portion of the bed undulation. This extension was found to provide an accurate picture of the resonant Bragg scattering process for a seabed consisting of a single dominant wavelength, but it was found to be inadequate for describing super- and subharmonic Bragg scattering in the case where the bed contains a superposition of more than one wavelength (O'Hare & Davies, 1993).

In their initial discussion of the MSE model, Smith & Sprinks (1975) obtained two residual terms which, in the final equation, are seen to produce terms proportional to $|\nabla h|^2$ and $\nabla^2 h$. The first term is typically argued to be small, since it is second order in the bottom slope and is thus taken to be a second order term in the expansion implied in the derivation of the equation. (Note that, in most cases, no explicit ordering or expansion technique is employed. The model is obtained essentially as a Galerkin approximation to the boundary value problem, using the depth dependence of a constant depth progressive wave as the trial shape function). Smith & Sprinks also argued that the second term, based on bed curvature, should be negligibly small in most cases. Historically this has been the standard assumption, with the only clear candidate for a topography violating the assumption being the undular bed case.

Recently, Massel (1993), Chamberlain & Porter (1995) and Porter & Staziker (1995) have rederived the MSE formulation and have essentially proceeded by retaining the extra terms described by Smith & Sprinks (1975) as an integral part of the model equation. (Massel and Porter & Staziker have also extended the model to include the effect of non-propagated modes. The effect of this extension is not considered in the work described below). As a result of testing the model equation with the extra terms retained, it has been clearly demonstrated that the so-called modified mild-slope equation (MMSE) provides a description of most of the cases in which either or both the original MSE and Kirby's EMSE fail: undular beds with one or several wavelengths, and in the steep transition asymptote of the depth-transition problem. Chamberlain & Porter further show that Kirby's EMSE can be obtained as an approximation to the MMSE model, and that the term describing the bottom curvature effect plays a dominant role in correcting the predictions of the original MSE.

The usefulness of the additional correction terms in the MMSE formulation has been clearly demonstrated from a practical point of view, and yet there is no clear basis for understanding why the additional terms (particularly the curvature term) should play a central role in correcting the model predictions. In this note, we attempt to provide such a basis. Using a Hamiltonian formulation together with the general notion of a canonical perturbation, we argue that the term appearing in the Hamiltonian which produces the final curvature term correction in the MMSE is actually of leading-order importance, if the unperturbed Hamiltonian is taken to be the Hamiltonian for the constant depth progressive wave problem. From this point of view, it is concluded that the added curvature term is a fundamental part of the leading-order MSE, and that the original model developed by Berkhoff (1972) (and used extensively in practice today) is actually incorrect without the retention of this term. We denote this new revision of the mild-slope formulation as the mMSE or slightly-modified mild-slope equation. We also provide additional computational examples illustrating the accuracy of the modified mild-slope equation in its various forms described below.

2 Mild-Slope Equations

In order to provide context for the present derivations, we repeat the derivation of the MMSE, following Chamberlain & Porter but utilizing a Hamiltonian framework. The dependent variables in the problem are surface displacement $\eta(\mathbf{x}, t)$ and velocity potential $\phi(\mathbf{x}, z, t)$, where $\mathbf{x}$ denotes horizontal position, z denotes distance up from the still water level, and t denotes time. The Hamiltonian may then be defined as the sum of potential and kinetic energies,

$$H = \int_{\mathbf{x}} \mathcal{H} d\mathbf{x} \tag{1}$$

where

$$\mathcal{H} = \frac{1}{2}\rho g \eta^2 + \frac{\rho}{2}\int_{-h}^{\eta} \left\{ \nabla\phi \cdot \nabla\phi + (\phi_z)^2 \right\} dz \tag{2}$$

and where ∇ denotes a gradient in the horizontal plane. The local still water depth is denoted by $h(\mathbf{x})$. Following Miles (1977), H may be rendered in canonical form $H(\eta, \rho\zeta)$, where η represents the canonical coordinate and $\rho\zeta$ represents the conjugate momentum, with $\zeta(\mathbf{x}, t) \equiv \phi(\mathbf{x}, z = \eta, t)$. We invoke a linear approximation (or, restrict the Hamiltonian to quadratic terms) and introduce the approximation for the velocity potential ϕ

$$\phi(\mathbf{x}, z, t) = f_0(z; h(\mathbf{x}))\tilde{\phi}(\mathbf{x}, t) \tag{3}$$

with

$$f_0 = \cosh^{-1}(kh)\cosh[k(h+z)]; \qquad f_0(0) = 1 \tag{4}$$

This choice corresponds to an *a priori* elimination of non-propagating modes. Massel (1993) and Porter & Staziker (1995) have described model systems which retain non-propagating modes. It is clear from the linearization that the integral in (2) may be taken from $-h$ to 0, and that $\rho\tilde{\phi}$ is an adequate approximation for the canonical variable. The local contribution to the Hamiltonian may then be written as (retaining all small terms)

$$\mathcal{H} = \frac{1}{2}\rho g \eta^2 + \frac{\rho}{2}\left[I_1(\nabla\tilde{\phi})^2 + (I_2 + I_4)\tilde{\phi}^2 + 2\mathbf{I}_3 \cdot \tilde{\phi}\nabla\tilde{\phi} \right] \tag{5}$$

where

$$I_1 = \int_{-h}^{0} f_0^2 dz = g^{-1} C C_g \tag{6}$$

$$I_2 = \int_{-h}^{0} (f_{0,z})^2 dz = g^{-1}(\omega^2 - k^2 C C_g) \tag{7}$$

$$\mathbf{I}_3 = \int_{-h}^{0} f_0 \nabla f_0 dz = \int_{-h}^{0} f_0 f_1 dz \nabla h = u_1 \nabla h \tag{8}$$

$$I_4 = \int_{-h}^{0} (\nabla f_0)^2 dz = \int_{-h}^{0} f_1^2 dz |\nabla h|^2 = I_4' |\nabla h|^2 \tag{9}$$

where $f_1(z, h) = \partial f_0 / \partial h$. Expressions for f_1, u_1 and I_4' are given in the appendix. The results (6) and (7) follow from the dispersion relation

$$\omega^2 = gk \tanh(kh); \;\; C = \frac{\omega}{k}; \;\; C_g = \frac{\partial \omega}{\partial k} = \frac{C}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right) \tag{10}$$

where ω is the angular frequency, k is the wavenumber, C is the phase speed, and C_g is the group velocity.

2.1 The modified mild-slope equation

The modified mild-slope equation (MMSE), derived by Smith & Sprinks (1975) and developed more fully by Chamberlain and Porter (1995) and Massel (1993), is obtained by developing the canonical evolution equations given by

$$\eta_t = \frac{\delta \mathcal{H}}{\delta(\rho \tilde{\phi})} \tag{11}$$

$$(\rho \tilde{\phi})_t = -\frac{\delta \mathcal{H}}{\delta \eta} \tag{12}$$

where δ denotes variational derivatives. The resulting system of equations is given by

$$\eta_t = -\nabla \cdot (I_1 \nabla \tilde{\phi}) + (I_2 + I_4 - \nabla \cdot \mathbf{I}_3)\tilde{\phi} \tag{13}$$

$$\tilde{\phi}_t = -g\eta \tag{14}$$

or, after elimination of η and use of (6) and (7),

$$\tilde{\phi}_{tt} - \nabla \cdot (C C_g \nabla \tilde{\phi}) + (\omega^2 - k^2 C C_g + g I_4 - g \nabla \cdot \mathbf{I}_3)\tilde{\phi} = 0 \tag{15}$$

This form of the model equation is equivalent to the time-dependent model presented by Smith & Sprinks, and to the elliptic model presented by Chamberlain & Porter in the case of purely time-harmonic motions. The connection may be established by noting that

$$\nabla \cdot \mathbf{I}_3 - I_4 = r(h) = \int_{-h}^{0} f_0 \nabla^2 f_0 dz - \nabla h \cdot (f_0 \nabla f_0) \,|_{-h} \tag{16}$$

where the notation $r(h)$ is taken directly from Chamberlain & Porter. Restricting attention to time-harmonic motion

$$\tilde{\phi} = \hat{\phi}(\mathbf{x})\mathrm{e}^{-i\omega t} \tag{17}$$

gives the reduced wave equation

$$\nabla \cdot (CC_g \nabla \hat{\phi}) + \left(k^2 CC_g + gr(h)\right) \hat{\phi} = 0 \tag{18}$$

which is (2.12) in Chamberlain & Porter. Utilizing Chamberlain and Porter's notation further, we may write $r(h)$ as

$$r(h) = \nabla \cdot (u_1 \nabla h) - I_4 \tag{19}$$

Chamberlain & Porter and Massel have shown that a retention of the terms involving $\mathbf{I}_3$ and I_4 provides a much more robust model equation for studying either waves over fairly large slopes or waves over undular beds, where terms in the square of the bed slope or terms in the bed curvature may be respectively large. However, as mentioned in the introduction, it is unsettling that both the evolution equations and the Hamiltonian retain terms to second order in a small bottom slope parameter, when the model wave potential is only accurate to zeroth order in the same parameter. This concern leads to the truncation usually employed to obtain the original mild-slope equation.

2.2 Original mild-slope equation

The original mild slope equation follows by neglecting terms involving $\mathbf{I}_3$ and I_4 in (13)-(14) or (15), or, correspondingly, the same terms in the Hamiltonian (5). The resulting evolution equations are

$$g\eta_t = -\nabla \cdot (CC_g \nabla \tilde{\phi}) + (\omega^2 - k^2 CC_g)\tilde{\phi} \tag{20}$$

$$\tilde{\phi}_t = -g\eta \tag{21}$$

which reduce to the time-dependent model of Smith & Sprinks

$$\tilde{\phi}_{tt} - \nabla \cdot (CC_g \nabla \tilde{\phi}) + (\omega^2 - k^2 CC_g)\tilde{\phi} = 0 \tag{22}$$

and the elliptic model of Berkhoff

$$\nabla \cdot (CC_g \nabla \hat{\phi}) + k^2 CC_g \hat{\phi} = 0 \tag{23}$$

after assuming a purely periodic motion. As will be seen below, this is too severe a truncation of the original Hamiltonian.

2.3 Model based on a truncated Hamiltonian

The reduction to the usual mild-slope equation follows from eliminating all terms which are apparently second order in a bottom slope parameter (i.e., $\mathbf{I}_3$ and I_4) in the evolution equations (13)-(14) or (15), or correspondingly, the term $r(h)$ in (18), following Smith & Sprinks (1975). In derivations of the model equations using Galerkin methods or Green's Identities, there is little or no guidance which would tell us if there is any basic distinction between the added small terms. However, we note that the term $\mathbf{I}_3$ results from a term in the Hamiltonian which is only first order in the small bottom slope parameter, and is hence a possibly important contribution to the estimate of the Hamiltonian at the order of approximation retained in the final evolution equations.

We approach this problem by proceeding in the spirit of a canonical perturbation expansion of $\mathcal{H}$ (see Goldstein (1980), Chapter 11), but without proceeding to the actual averaging problem or a determination of the action-angle variables. The usual approach is to write the Hamiltonian as the sum of an unperturbed and a perturbation Hamiltonian,

$$\mathcal{H} = \mathcal{H}_0 + \delta\mathcal{H} \tag{24}$$

where $\mathcal{H}_0$ is the Hamiltonian for a solvable problem and $\delta\mathcal{H}$ represents deviations from that solvable case. The simplest solvable case in the mild-slope context is the case of wave propagation over some region with a constant characteristic depth h_0 at position $\mathbf{x}_0$. Denoting quantities evaluated at depth h_0 with subscript 0, we write

$$I_1(\mathbf{x}) = I_{10} + \left(\frac{dI_1}{dh}\right)_0 \nabla h\cdot(\mathbf{x}-\mathbf{x}_0) + O(\nabla h)^2; \quad I_2(\mathbf{x}) = I_{20} + \left(\frac{dI_2}{dh}\right)_0 \nabla h\cdot(\mathbf{x}-\mathbf{x}_0) + O(\nabla h)^2 \tag{25}$$

We then take the unperturbed Hamiltonian to be

$$\mathcal{H}_0 = \frac{1}{2}\rho g\eta^2 + \frac{\rho}{2}\left(I_{10}(\nabla\tilde{\phi})^2 + I_{20}\tilde{\phi}^2\right) \tag{26}$$

which is the Hamiltonian for the Klein-Gordon equation

$$\tilde{\phi}_{tt} - (CC_g)_0\nabla^2\tilde{\phi} + \left(\omega^2 - (k^2CC_g)_0\right)\tilde{\phi} = 0 \tag{27}$$

or, in the case of purely harmonic motion, the Helmholtz equation

$$\nabla^2\hat{\phi} + k_0^2\hat{\phi} = 0 \tag{28}$$

If we now consider perturbations to this motion in the neighborhood of the depth h_0, we may take the perturbation potential to be

$$\delta\mathcal{H} = \nabla h\cdot\left\{\left[\left(\frac{dI_1}{dh}\right)_0(\nabla\tilde{\phi})^2 + \left(\frac{dI_2}{dh}\right)_0\tilde{\phi}^2\right](\mathbf{x}-\mathbf{x}_0) + 2u_{10}\tilde{\phi}\nabla\tilde{\phi}\right\} + O(\nabla h)^2 \tag{29}$$

It is apparent then that the $\mathbf{I}_3$ term will contribute to the distortion of the wave field at the same order as the effect of local variations in the original coefficients I_1 and I_2, and that contributions to this local distortion from I_4 terms are an order smaller. The contributions of $\mathbf{I}_3$ are thus of leading-order importance in the extension from the constant-depth to the mild-slope problem, and must be retained in order to provide a consistant leading-order approximation for variable water depth. Retaining the $\mathbf{I}_3$ term but reverting back to a more general expression retaining the full expressions for I_1, I_2 and $\mathbf{I}_3$ yields the approximate Hamiltonian

$$\mathcal{H}_1 = \frac{1}{2}\rho g \eta^2 + \frac{\rho}{2}\left[I_1(\nabla\tilde{\phi})^2 + I_2\tilde{\phi}^2 + 2\mathbf{I}_3 \cdot \tilde{\phi}\nabla\tilde{\phi}\right] \tag{30}$$

which should account for all leading order local effects distorting the wave field at each depth. The corresponding evolution equations are given by

$$\eta_t = -\nabla \cdot (I_1 \nabla\tilde{\phi}) + (I_2 - \nabla \cdot \mathbf{I}_3)\tilde{\phi} \tag{31}$$

$$\tilde{\phi}_t = -g\eta \tag{32}$$

or, after elimination of η,

$$\tilde{\phi}_{tt} - \nabla \cdot (CC_g \nabla\tilde{\phi}) + (\omega^2 - k^2 CC_g - g\nabla \cdot \mathbf{I}_3)\tilde{\phi} = 0 \tag{33}$$

The corresponding time-harmonic equation in a notation similar to Chamberlain & Porter's would then be

$$\nabla \cdot (CC_g \nabla\hat{\phi}) + (k^2 CC_g + g r_1(h))\hat{\phi} = 0 \tag{34}$$

where $r_1(h)$ is a truncated $r(h)$ given by

$$r_1(h) = r(h) + I_4 = \nabla \cdot (u_1 \nabla h) \tag{35}$$

Expanding the expressions (19) and (35), we obtain

$$r(h) = u_1\nabla^2 h + u_2(\nabla h)^2; \quad u_2(h) = u_1' - \int_{-h}^{0} f_1^2 dz \tag{36}$$

$$r_1(h) = u_1\nabla^2 h + u_1'(\nabla h)^2 \tag{37}$$

where u_1' denotes the derivative of u_1 with respect to h. The model of the present section, denoted subsequently as the mMSE or slightly-modified mild-slope equation, is identical to the MMSE in the term proportional to the curvature of the bed. Following the development in Chamberlain & Porter (1995), it follows that the mMSE also contains the EMSE of Kirby (1986) as an appropriate limit. We thus do not expect the present simplification of the theory to change any existing estimates of reflection from an undular bed obtained using the MMSE.

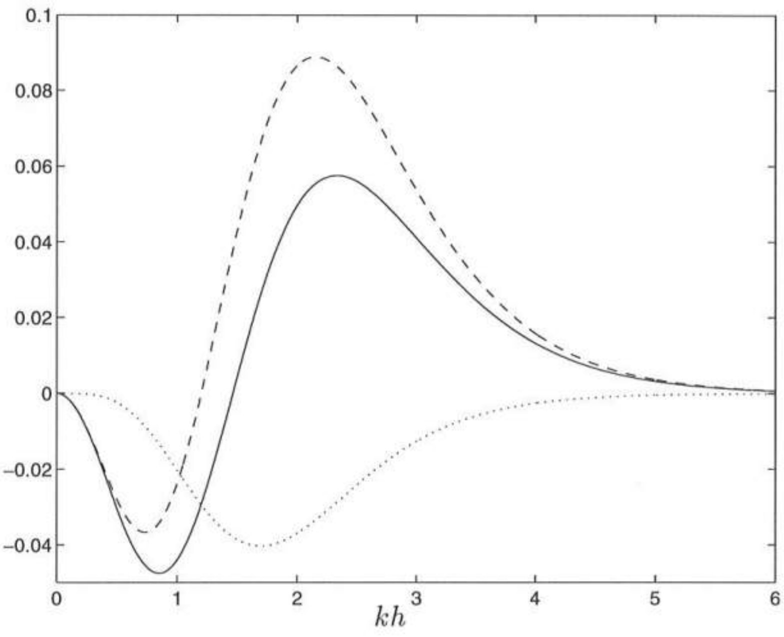

Figure 1: Variation of second order coefficients in modified mild-slope equations. Solid line, hu_2 in Chamberlain & Porter (1995). Dashed line, hu_1'. Dotted line, $-h\int_{-h}^{0} f_1^2 dz$.

In contrast, the two models differ in the term related to the square of the bottom slope. Numerical values of the original model coefficient u_2 and the two contributions u_1' and $\int_{-h}^{0} f_1^2 dz$ are shown in Figure 1 for a range of kh values. It is seen that the two contributions to the term in the MMSE tend to cancel each other to a certain extent, and that the contribution from the I_4 term in the original Hamiltonian is not terribly small relative to the contribution from the I_3 term.

3 Numerical Examples

3.1 Linear Depth Transition

Since the reduction of the MMSE to the mMSE obtained in the previous section changes the term related to $|\nabla h|^2$, we consider here whether this change has a marked effect on prediction of wave reflection for wave propagation over a relatively steep transition. The example considered is the linear depth transition studied by Booij (1983) and further considered by Massel (1993), Chamberlain & Porter (1995) and Porter & Staziker (1995). We use the geometry and parameters considered by Suh et al (1997), who performed additional finite element calculations in order to provide greater detail in the mild-slope limit.

The bottom geometry is defined as

$$h(x) = \begin{cases} h_1; & x < 0 \\ h_1 - mx; & 0 \le x \le B \\ h_2; & x > B \end{cases} \tag{38}$$

where B is the horizontal length of the transition. We take $h_1 = 0.6m$ and $h_2 = 0.2m$ in the present calculations. The bed slope m in the transition region is given by

$$m = \frac{h_1 - h_2}{B} \tag{39}$$

and ranges from $0.04 \to \infty$ in the calculations discussed below. The bed slope may be represented in this case by the piecewise-continuous function

$$h_x = -m\left(H(x) - H(x - B)\right) \tag{40}$$

where $H(x)$ denotes a step function, while the bed curvature is represented by a set of delta functions

$$h_{xx} = \delta(x) - \delta(x - B) \tag{41}$$

This represents a somewhat pathological choice of geometries for evaluating the present model system, as the critical model term representing bed curvature at the ends of the slope takes on finite values only in the finite-difference approximations used for the numerical treatment. This problem may be circumvented by solving separately for each region indicated in (38) and then matching across the boundaries between regions (Chamberlain and Porter, 1995; Porter and Staziker, 1995). However, in practice, finite difference solutions treating the entire domain without regard for the slope and curvature discontinuities are found to converge well, and we thus do not use any special treatments for the discontinuities in the calculations below.

The wave period for the present computations is taken to be $T = 2s$ and is the same as in Booij's and Suh et al's finite element calculations. This leads to dimensionless depth values of $k_1 h_1 = 0.8646$ in the incident wave region with depth h_1 and $k_2 h_2 = 0.4642$ in the transmitted region with depth h_2. Referring to Figure 1, it is seen that this choice of parameters minimizes the difference between the MMSE and mMSE in this example, since the slope-squared terms have similar magnitude. The models could be made more different by choosing relatively larger depths, but we would expect to get correspondingly smaller reflections. We retain the present parameters in order to compare to previously published results.

Slope lengths B from $10m$ to $0.05m$ (corresponding to slopes ranging from 1:25 to the steepest case of 8:1) were used for numerical calculations based on the MSE, MMSE, mMSE, and an arbitrarily truncated mMSE which neglects all $|\nabla h|^2$ effects. All calculations were done using a centered, second-order accurate finite-difference approach, leading to a simple tridiagonal system. Model results are plotted along with the finite element results of Suh et al (1997) in Figure 2. The abscissa and vertical axes represent the dimensionless horizontal length of the plane slope $k_1 B$ and a reflection coefficient R, respectively. Circles denote finite-element results, whereas the solid line, the dashed line and the chain dotted line represent the MSE, MMSE (with all the $\mathbf{I}_3$ and I_4 contributions), and the mMSE (retaining only the $\mathbf{I}_3$ contributions), respectively. The dotted line represents results with only the curvature term retained in the mMSE.

It is seen that the MMSE and mMSE results agree well with the finite element results. For steeper slopes, mMSE results obtained using the added curvature term alone deviate markedly from the MMSE results, while the full mMSE predicts results which are indistinguishable from the MMSE model results. The MSE is unable to correctly estimate reflection even from the very mildest of slopes, as is evident from the deviation of MSE results from both the finite-element and MMSE or mMSE model results over the entire range of water depths. Note that the values near the right edge of the plot correspond to bed slopes on the order of .04 or 1:25, which is well within the range of slopes that would normally be thought of as being applicable in the original MSE formulation. The original finite-element calculations of Booij (1983) did not cover this range of mild slopes, and so the inability of the model to predict accurate results here was not uncovered.

3.2 Smooth Depth Transition

In order to avoid the singularities in model coefficients associated with the presence of slope discontinuities in the previous example, we consider here a smooth depth transition given by

$$h(x) = \begin{cases} h_1; & x < 0 \\ h_1 - \frac{h_1 - h_2}{B}\left(x - \frac{B}{2\pi}\sin\frac{2\pi x}{B}\right); & 0 \leq x \leq B \\ h_2; & x > B \end{cases} \tag{42}$$

where h_1, h_2 and B are defined as before. The bottom slope and curvature are shown in Figure 3, and represent continuous and piecewise continuous functions of x respectively. The resulting transition is thus relatively smooth compared to the linear transition studied previously, and reflection coefficients are expected to decay nearly exponentially rather than algebraically in the limit as the bed slope goes to zero. Results for this case are shown in Figure 4 for the same set of model configurations as in Figure 2. The uniform deviation between MSE and either the MMSE or mMSE results are again apparent over the full range of water depths, although it is less severe in the mild-slope limit than in the previous example. The influence of terms proportional to $|\nabla h|^2$ is again apparent in the short transition limit, as is the overall agreement between MMSE and mMSE results.

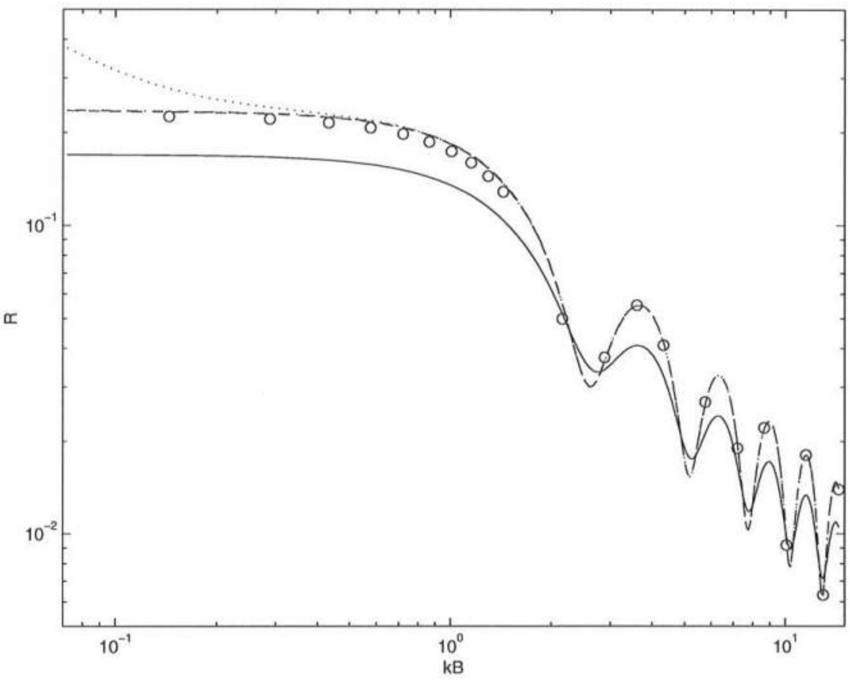

Figure 2: Comparison of model predictions of reflection from a linear depth transition (38). Solid line - MSE; Dash line - MMSE; Dash-dot line, mMSE, Dotted line, mMSE without slope-squared terms.

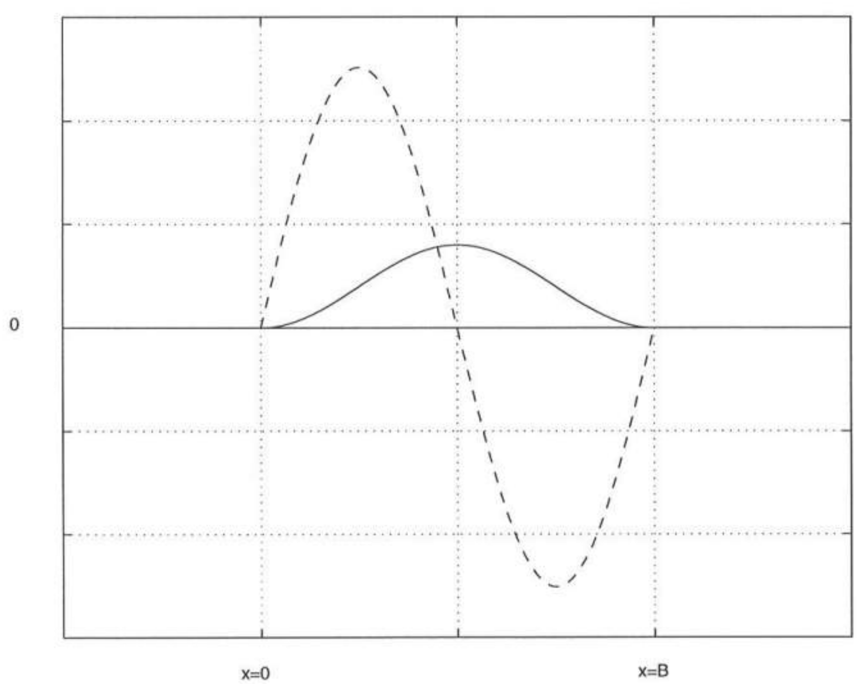

Figure 3: Bottom slope (solid line) and bed curvature (dashed line) for the smooth transition.

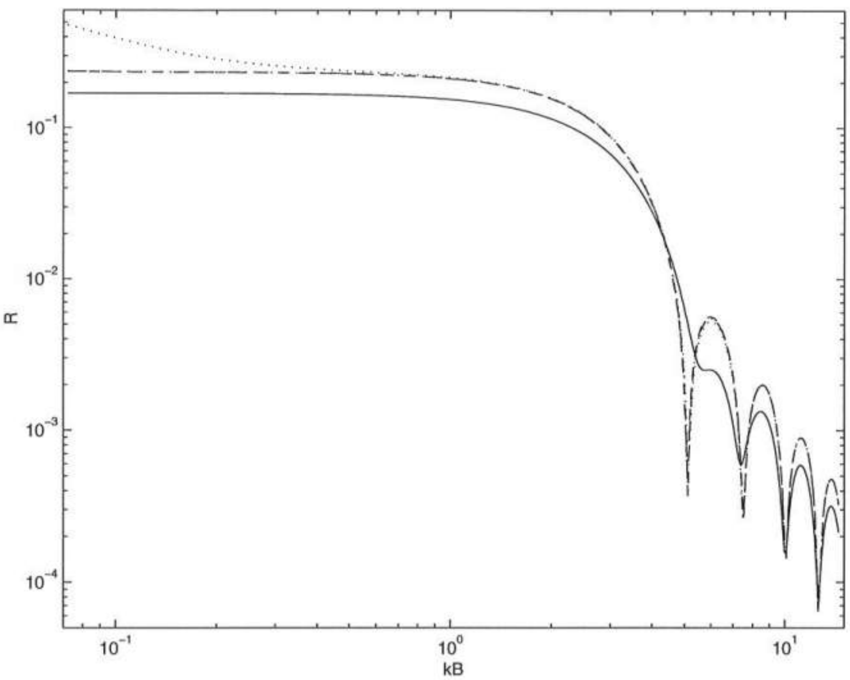

Figure 4: Comparison of model predictions of reflection from a smooth depth transition (42). Solid line - MSE; Dash line - MMSE; Dash-dot line - mMSE; Dotted line - mMSE without slope-squared terms.

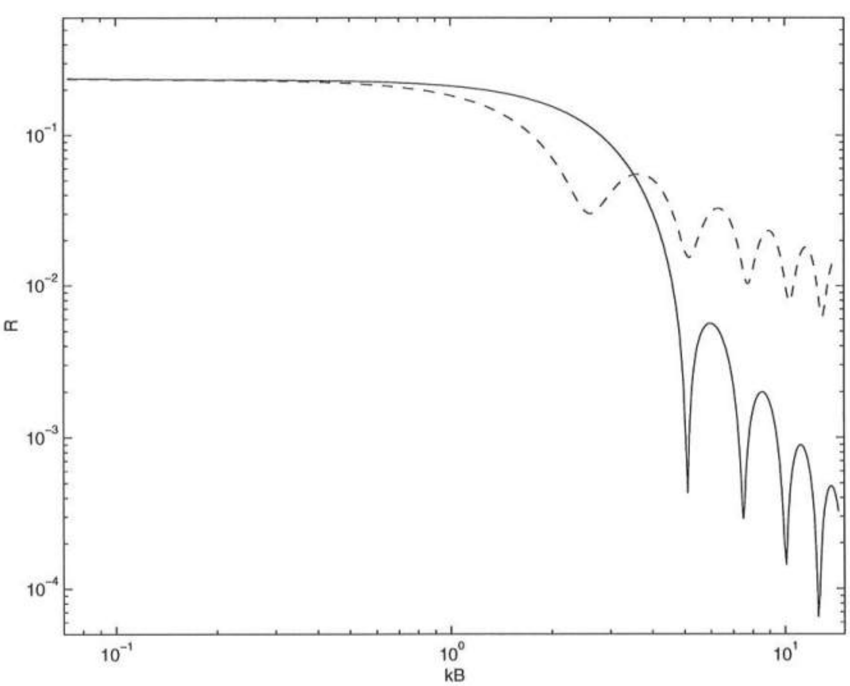

Figure 5: Comparison of mMSE model predictions of reflection R from the linear (dashed line) and smooth (solid line) depth transitions.

3.3 The Asymptotic Limit : A Vertical Step

Both MMSE and mMSE computations for the previous examples apparently have equivalent asymptotes at the left of Figures 2, 4 or 5. This corresponds to an approach to a sudden transition in depth from h_1 to h_2 or a slope $m = \infty$, although the largest slope studied numerically was $m = 8$. In order to investigate the validity of the MMSE and mMSE in the limit of large bottom slope, reflection from a vertical step was studied. The reflection coefficient R was calculated using a matched-eigenfunction method (Takano, 1960; Kirby & Dalrymple, 1983). In this limiting case, R was found to be 0.2291, whereas the mMSE, for a slope length of $B = 0.05$ or slope $m = 8$ gives R = 0.2346 for the linear transition, corresponding to an error of 2.4%. The asymptotic limit of the mMSE prediction thus matches the sudden depth transition result very well, indicating that the MMSE and mMSE models are able to predict reflection even from very steep slopes quite satisfactorily.

4 Discussion

The present work does not provide any new results illustrating the difference between classical MSE and more recent MMSE predictions. Rather, the goal here has been to suggest that the classical MSE formulation is not the correct leading-order approximation for the variable depth problem, and that the mMSE model (31)-(32), (33) or (34) should be taken to be the leading order approximation based on a derivation that retains leading-order corrections to the flat bottom problem in a truncated Hamiltonian. The present mMSE model (31)-(32) or it's variants (33) or (34) is capable of making correct predictions of reflection in situations where the classical MSE loses accuracy, and reproduces essentially all the predictions of the newer MMSE of Chamberlain and Porter and others, which we argue consists of the mMSE plus additional small terms.

Recently, Miles and Chamberlain (1998) have considered the derivation of the MSE heirarchy, and have considered approximate models resulting from various levels of truncation of an averaged Lagrangian. In addition to approximations resulting from a trial function of the form (3), they have considered a further-extended form corresponding to a potential which satisfies the bottom boundary condition to leading order in ∇h. In their results, the resulting system would lead to an elliptic mild-slope formulation including derivative terms to fourth order. Neglecting the terms in the trial function proportional to ∇h leads to the Lagrangian formulation giving the MMSE, as in Chamberlain and Porter (1995). A further truncation of the system to give a Lagrangian for the classical MSE (see their derivation from (2.13) - (2.16)) involves the neglect of terms which are proportional to both $(\nabla h)^2$ and $\nabla^2 h$ and are not of the same apparent order as in the present derivation. It is noted here that similar extensions of the mild-slope equation to include slope-dependent effects in the trial function have been attempted by Kirby (1983) and Zhang (1996). It is not known whether these extensions improve the predictive capability of the mMSE or MMSE relative to the fairly typical results shown here.

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Appendix. Expressions for various model equation coefficients

The term $f_1(h)$ defined in (20) is given by

$$f_1(h) = \frac{\partial f_0(z,h)}{\partial h} = k'(h)(\cosh(kh))^{-1}(z\sinh(k[h+z]) - k^{-1}\sinh(kh)\sinh(kz)) \quad (43)$$

where $k'(h)$ is obtained from the dispersion relation (10) and is given by

$$k'(h) = \frac{dk(h)}{dh} = -2k^2(K+\sinh(K))^{-1} \quad (44)$$

where $K = 2kh$. The coefficient u_1 defined in (20) is given by

$$u_1(h) = \frac{(\sinh(K) - K\cosh(K))}{4\cosh^2(kh)(K+\sinh(K))} \quad (45)$$

The derivative u_1' appearing in (28) and (29) is given by

$$\begin{aligned} hu_1' &= h\frac{du_1(h)}{dh} = -(4\cosh^2(kh)(K+\sinh K)^3)^{-1}(K\sinh K)\left[K\sinh K(K+\sinh K)+\right. \\ &\left.(\sinh K - K\cosh K)\left\{\sinh K(K+\sinh K)/(1+\cosh K)+1+\cosh K\right\}\right] \quad (46) \end{aligned}$$

Finally, we note that

$$I_4 = \int_{-h}^{0} f_1^2 dz (\nabla h)^2 \tag{47}$$

The integral is given by

$$\begin{aligned}\int_{-h}^{0} f_1^2 dz &= k(\cosh^2(kh)(K+\sinh K)^2)^{-1}\left(\frac{3\sinh K - 3K - K^2(kh)}{6}\right.\\ &\left. -\sinh(kh)\left[2(kh)^2\cosh(kh) - K\sinh(kh)\right] + \sinh^2(kh)(\sinh K - K)\right)\end{aligned} \tag{48}$$

The expression $u_1' - \int_{-h}^{0} f_1^2 dz$ is equivalent to the expression for u_2 given below (2.16) in Chamberlain & Porter.