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WAVE GENERATION IN BOUSSINESQ MODELS

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A One-Way Source Function Method for Wave Generation in Boussinesq Models

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Abstract

A source function method for the generation of waves internal to Boussinesq model grid boundaries (Wei *et al*, [7]) is modified to eliminate waves propagating backwards from the source region. The resulting modification to the technique greatly reduces the extent of sponge layers or other absorbing layers needed on the upwave open boundary in Boussinesq model applications. The method is also generalized to account for the presence of a strong current, for application to wave-current interaction problems.

1 Introduction

The need to propagate waves across the seaward boundary of a model grid in a Boussinesq model simulation presents us with a difficult problem, since the well-posedness of the open boundary condition is often not established and the treatment of radiating waves arriving from inside the domain is thus not well defined. Consequently, this problem is often handled quite empirically, using a source function method to generate waves internal to the grid, coupled with extensive absorbing sponge layers to remove both unwanted backward-propagating waves from the source as well as reflected waves radiating out of the numerically-simulated region. This method was pioneered by Larsen and Dancy [4], and can be made to work quite effectively if the wave-absorbing properties of the sponge layer seaward of the source are treated carefully. The method has recently been extended to the case of a spatially-distributed source of either mass or momentum, and a linearized solution has been provided for Boussinesq models with $O(kh)^2$ dispersion by Wei *et al* [7]. Gobbi and Kirby [2] (hereafter referred to as GK98) have provided the extension of the linearized result to models with $O(kh)^4$ dispersion and corresponding higher order spatial derivatives (up to 5th order).

In all the cases mentioned above, the wave source generates both forward and backward propagating waves, and the presence of an extensive and carefully designed sponge layer behind the source region is thus required. This problem can be handled quite adequately, but the resulting damping region can be large and thus adds measurably to the computational

effort required during model simulations. In this note, we suggest an alternate formulation which uses a mass and momentum source in tandem, and which eliminates the backward-propagating component of the generated wave in the linear approximation. The resulting generation mechanism is tested in the model of Wei *et al* [6] (hereafter referred to as WKGS). The absence of a backward-propagating mode greatly reduces the need for an absorbing layer behind the wave source.

As pointed out by Kirby [3], models such as WKGS which fall in the “fully-nonlinear” category are capable of correctly simulating the dispersion of short waves riding on strong currents, as the proper combination of terms giving a total derivative following the current \vec{U} ,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{U} \cdot \nabla \quad (1.1)$$

is retained in all higher-order dispersive terms. We therefore extend the method here to include the effect of a specified $O(1)$ current field, and test the resulting model for cases involving steady currents. In the following section, we present the extensions to the $O(kh)^2$ WKGS model. The more extensive development for the $O(kh)^4$ model of GK98 is given in the Appendix.

2 Theory for the $O(kh)^2$ model

The WKGS model consists of a set of fully nonlinear Boussinesq-type equations which simulate wave propagation. The governing equations in non-dimensional form are given by

$$\begin{aligned} \eta_t + \nabla \cdot ((h + \delta\eta)\vec{u}) + \nabla \cdot \left[\mu^2 \left\{ \left[\frac{z_\alpha^2}{2} - \frac{1}{6}(h^2 - h\delta\eta + (\delta\eta)^2) \right] \nabla(\nabla \cdot \vec{u}) \right. \right. \\ \left. \left. + \left[z_\alpha + \frac{1}{2}(h - \delta\eta) \right] \nabla(\nabla \cdot (h\vec{u})) \right\} \right] = 0 \end{aligned} \quad (2.1a)$$

$$\begin{aligned} \vec{u}_t + \delta(\vec{u} \cdot \nabla)\vec{u} + \nabla\eta + \mu^2 \left[\frac{z_\alpha^2}{2} \nabla(\nabla \cdot \vec{u}_t) + z_\alpha \nabla(\nabla \cdot (h\vec{u}_t)) \right. \\ \left. - \nabla \left[\frac{(\delta\eta)^2}{2} \nabla \cdot \vec{u}_t + \delta\eta \nabla \cdot (h\vec{u}_t) \right] \right] \\ + \delta\mu^2 \left[\nabla \left[(z_\alpha - \delta\eta)(\vec{u} \cdot \nabla)(\nabla \cdot (h\vec{u})) + \frac{1}{2}(z_\alpha^2 - (\delta\eta)^2)(\vec{u} \cdot \nabla)(\nabla \cdot \vec{u}) \right] \right. \\ \left. + \frac{1}{2} \nabla \left[(\nabla \cdot (h\vec{u}) + \delta\eta \nabla \cdot \vec{u})^2 \right] \right] = 0 \end{aligned} \quad (2.1b)$$

where

$$\nabla \equiv \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y}$$

$\mu = kh$, and $\delta = a/h$ are the dispersive and nonlinearity parameters respectively. Since the linear form of the governing equations can be explicitly solved (using Green's functions) to obtain a transfer function between the source and the desired wave, we assume linearity ($\delta \ll 1$) and constant water depth. Furthermore, since we are including $O(1)$ currents, we define the total velocity \vec{u} by

$$\vec{u} = \frac{1}{\delta} \vec{U} + \vec{u}_w \quad (2.2)$$

where \vec{u}_w is the velocity due to wave motion. The current velocity \vec{U} is assumed constant. Substituting (2.2) in (2.1) and keeping terms up to $O(\delta)$, we get

$$\frac{d\eta}{dt} + h\nabla \cdot \vec{u}_w + h^3\mu^2\nabla \cdot \left[\left(\alpha + \frac{1}{3} \right) \nabla(\nabla \cdot \vec{u}_w) \right] = 0 \quad (2.3a)$$

$$\frac{d\vec{u}_w}{dt} + \nabla\eta + \mu^2\alpha h^2 \frac{d}{dt}(\nabla(\nabla \cdot \vec{u}_w)) = 0 \quad (2.3b)$$

where

$$\alpha \equiv \frac{1}{2} \left(\frac{z_\alpha}{h} \right)^2 + \frac{z_\alpha}{h}$$

z_α is a reference depth where \vec{u}_w is defined, and is chosen to obtain appropriate dispersion characteristics even in deep water (see WKGS).

Writing (2.3) in dimensional form and introducing source functions in both the continuity and momentum equations we get

$$\frac{d\eta}{dt} + h\nabla \cdot \vec{u}_w + \alpha_1 h^3 \nabla^2(\nabla \cdot \vec{u}_w) = f(x, y, t) \quad (2.4a)$$

$$\frac{d\vec{u}_w}{dt} + g\nabla\eta + \alpha h^2 \nabla^2 \left(\frac{d\vec{u}_w}{dt} \right) = -g\nabla P(x, y, t) \quad (2.4b)$$

where $\alpha_1 \equiv \alpha + \frac{1}{3}$. Introducing a velocity potential $\phi(x, y, t)$ for \vec{u}_w , we can rewrite (2.4) as one equation

$$\frac{d^2\phi}{dt^2} - gh\nabla^2\phi - \alpha_1 gh^3 \nabla^2(\nabla^2\phi) + \alpha h^2 \nabla^2 \left(\frac{d^2\phi}{dt^2} \right) = -g \left(f + \frac{dP}{dt} \right) \quad (2.5)$$

Taking a Fourier transform in the y spatial direction

$$\phi(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(x, \lambda, t) \exp(i\lambda y) d\lambda \quad (2.6a)$$

$$f(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(x, \lambda, t) \exp(i\lambda y) d\lambda \quad (2.6b)$$

$$P(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{P}(x, \lambda, t) \exp(i\lambda y) d\lambda \quad (2.6c)$$

and subsequently a Fourier transform in t

$$\tilde{\phi}(x, \lambda, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(x, \lambda, \omega) \exp(-i\omega t) d\omega \quad (2.7a)$$

$$\tilde{f}(x, \lambda, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x, \lambda, \omega) \exp(-i\omega t) d\omega \quad (2.7b)$$

$$\tilde{P}(x, \lambda, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{P}(x, \lambda, \omega) \exp(-i\omega t) d\omega \quad (2.7c)$$

yields a fourth order ordinary differential equation

$$A\hat{\phi}^{(4)} + B\hat{\phi}^{(3)} + C\hat{\phi}^{(2)} + D\hat{\phi}^{(1)} + E\hat{\phi} = g(\hat{f} + U\hat{P}^{(1)} - i(\omega - \lambda V)\hat{P}) \quad (2.8)$$

where $()^{(n)} \equiv \frac{d^n}{dx^n}()$, and $\vec{U} = \vec{i}U + \vec{j}V$. The constants are given by

$$A = \alpha_1 gh^3 - \alpha h^2 U^2 \quad (2.9a)$$

$$B = 2i(\omega - \lambda V)U\alpha h^2 \quad (2.9b)$$

$$C = gh + (\omega - \lambda V)^2 \alpha h^2 + \lambda^2 (U^2 \alpha h^2 - 2\alpha_1 gh^3) - U^2 \quad (2.9c)$$

$$D = 2iU(\omega - \lambda V)[1 - \alpha(\lambda h)^2] \quad (2.9d)$$

$$E = (\omega - \lambda V)^2 [1 - \alpha(\lambda h)^2] - gh\lambda^2 [1 - \alpha_1(\lambda h)^2] \quad (2.9e)$$

The homogeneous solution for (2.8) is of the form

$$\hat{\phi} \sim \exp(ilx)$$

where l satisfies the Doppler-shifted dispersion relation of the Boussinesq equations

$$(\omega - \lambda V - lU)^2 = gh(\lambda^2 + l^2) \frac{[1 - \alpha_1 h^2(\lambda^2 + l^2)]}{[1 - \alpha h^2(\lambda^2 + l^2)]} \quad (2.10)$$

To obtain a particular solution for $\hat{\phi}$, and subsequently a relationship between wave amplitude and source function amplitudes, we use the method of Green's functions.

2.1 Green's function

Consider a Green's function $G(\zeta, x)$, such that

$$A \frac{\partial^4 G(\zeta, x)}{\partial x^4} - B \frac{\partial^3 G(\zeta, x)}{\partial x^3} + C \frac{\partial^2 G(\zeta, x)}{\partial x^2} - D \frac{\partial G(\zeta, x)}{\partial x} + EG(\zeta, x) = \delta(\zeta - x) \quad (2.11)$$

where $\delta(\zeta - x)$ is a delta function.

Imposing radiating boundary conditions we can write $G(\zeta, x)$ in the form of the homogeneous solution for $\hat{\phi}$ in the region $x \neq \zeta$

$$G(\zeta, x) = \tilde{a} \exp[i\tilde{l}(x - \zeta)] \quad \text{for } x > \zeta \quad (2.12a)$$

$$G(\zeta, x) = a \exp[il(\zeta - x)] \quad \text{for } x < \zeta \quad (2.12b)$$

where l satisfies (2.10), and \tilde{l} satisfies the same equation but with the opposite sign for U . Substituting in (2.11) we get

$$A\tilde{l}^4 + iB\tilde{l}^3 - C\tilde{l}^2 - iD\tilde{l} + E = 0 \quad \text{for } x > \zeta \quad (2.13a)$$

$$Al^4 - iBl^3 - Cl^2 + iDl + E = 0 \quad \text{for } x < \zeta \quad (2.13b)$$

In the absence of currents (2.13) gives two roots

$$l_1 = \tilde{l}_1 = \left(\frac{C - \sqrt{C^2 - 4AE}}{2A} \right)^{\frac{1}{2}} \quad (2.14a)$$

$$l_2 = \tilde{l}_2 = \left(\frac{C + \sqrt{C^2 - 4AE}}{2A} \right)^{\frac{1}{2}} \quad (2.14b)$$

where l_1 is a real root and l_2 is an imaginary root.

On the other hand in the presence of currents there are two other distinct roots in each case. These roots correspond to waves arising due to reflection on a current and can be ignored (Peregrine [5]). Thus, in the presence of currents (2.13) is solved numerically using the Newton-Raphson technique with (2.14) as initial conditions to obtain a real and a complex root. We can therefore construct the Green's function as

$$G(\zeta, x) = \tilde{a}_1 \exp[i\tilde{l}_1(x - \zeta)] + \tilde{a}_2 \exp[i\tilde{l}_2(x - \zeta)] \quad \text{for } x > \zeta \quad (2.15a)$$

$$G(\zeta, x) = a_1 \exp[il_1(\zeta - x)] + a_2 \exp[il_2(\zeta - x)] \quad \text{for } x < \zeta \quad (2.15b)$$

Integrating (2.11) from $x = \zeta - 0$ to $x = \zeta + 0$, and assuming continuity of G , $\frac{\partial G}{\partial x}$, and $\frac{\partial^2 G}{\partial x^2}$, we get the following set of matching conditions

$$G|_{\zeta-0}^{\zeta+0} = 0 \quad (2.16a)$$

$$\left. \frac{\partial G}{\partial x} \right|_{\zeta=0}^{\zeta+0} = 0 \quad (2.16b)$$

$$\left. \frac{\partial^2 G}{\partial x^2} \right|_{\zeta=0}^{\zeta+0} = 0 \quad (2.16c)$$

$$\left. \frac{\partial^3 G}{\partial x^3} \right|_{\zeta=0}^{\zeta+0} = \frac{1}{A} \quad (2.16d)$$

Substituting (2.15) in (2.16) and solving gives

$$\tilde{a}_1 = \frac{i}{A(\tilde{l}_1 - \tilde{l}_2)(\tilde{l}_1 + l_1)(\tilde{l}_1 + l_2)} \quad (2.17a)$$

$$\tilde{a}_2 = \frac{i}{A(\tilde{l}_2 - \tilde{l}_1)(\tilde{l}_2 + l_1)(\tilde{l}_2 + l_2)} \quad (2.17b)$$

$$a_1 = \frac{i}{A(l_1 - l_2)(l_1 + \tilde{l}_1)(l_1 + \tilde{l}_2)} \quad (2.17c)$$

$$a_2 = \frac{i}{A(l_2 - l_1)(l_2 + \tilde{l}_1)(l_2 + \tilde{l}_2)} \quad (2.17d)$$

Eq. (2.15) together with (2.17) gives a Green's function $G(\zeta, x)$ which satisfies (2.11) over the entire domain.

2.2 Solution for $\hat{\phi}$

We can now solve for $\hat{\phi}$ in terms of the known function $G(\zeta, x)$ and consequently obtain the required source function such that waves of a desired wave amplitude move in one direction only.

Multiplying $G(\zeta, x)$ with (2.8), integrating from $x = -\infty$ to $x = \infty$, and using (2.11) together with the properties of a delta function we get

$$\begin{aligned} \hat{\phi}(\zeta, \omega, \lambda) = & g \int_{-\infty}^{\zeta} G(\hat{f} + U \frac{\partial \hat{P}}{\partial x} - i(\omega - \lambda V)\hat{P}) dx \\ & + g \int_{\zeta}^{\infty} G(\hat{f} + U \frac{\partial \hat{P}}{\partial x} - i(\omega - \lambda V)\hat{P}) dx \\ & - \hat{\phi} \frac{\partial^2 G}{\partial x^2} \Big|_{-\infty}^{\infty} - \hat{\phi} G \Big|_{-\infty}^{\infty} \end{aligned} \quad (2.18)$$

In the absence of a current the boundary terms in (2.18) cancel out. In our solution we shall neglect these boundary terms as they turn out to have negligible effects on the final solution.

For large positive values of ζ the solution approaches

$$\hat{\phi}(\zeta, \omega, \lambda) = g \int_{-\infty}^{\zeta} G\left(\hat{f} + U \frac{\partial \hat{P}}{\partial x} - i(\omega - \lambda V) \hat{P}\right) dx \quad (2.19)$$

Substituting (2.15) in (2.19) and noting that l_2 being complex leads to exponentially decaying terms, we get

$$\hat{\phi}(\zeta, \omega, \lambda) \approx g a_1 \exp(i l_1 \zeta) \int_{-\infty}^{\infty} \left(\hat{f} + U \frac{\partial \hat{P}}{\partial x} - i(\omega - \lambda V) \hat{P} \right) \exp(-i l_1 x) dx \quad (2.20)$$

for $\zeta \rightarrow \infty$

Proceeding along similar lines we can also obtain $\hat{\phi}$ for large negative ζ

$$\hat{\phi}(\zeta, \omega, \lambda) \approx g \tilde{a}_1 \exp(-i \tilde{l}_1 \zeta) \int_{-\infty}^{\infty} \left(\hat{f} + U \frac{\partial \hat{P}}{\partial x} - i(\omega - \lambda V) \hat{P} \right) \exp(i \tilde{l}_1 x) dx \quad (2.21)$$

for $\zeta \rightarrow -\infty$

Setting (2.21) to zero we get

$$\int_{-\infty}^{\infty} \hat{f} \exp(i \tilde{l}_1 x) dx + \int_{-\infty}^{\infty} \left[U \frac{\partial \hat{P}}{\partial x} - i(\omega - \lambda V) \hat{P} \right] \exp(i \tilde{l}_1 x) dx = 0 \quad (2.22)$$

Eq. (2.22) gives the required relationship between the two sources $\hat{f}(x)$ and $\hat{P}(x)$ to generate forward propagating waves only.

To solve (2.20) and (2.22) we need to know the form of $\hat{f}(x)$ and $\hat{P}(x)$. The choice is arbitrary and we choose a smooth Gaussian shaped profile to obtain exact solutions for the integrals in (2.20) and (2.22).

$$\hat{f}(x) = D_1 \exp(-\beta x^2) \quad (2.23a)$$

$$\hat{P}(x) = D_2 x \exp(-\beta x^2) \quad (2.23b)$$

where β is an estimate of the source width and is determined in the same way as in Wei *et al* [7]. $D_1(\lambda, \omega, \beta)$, and $D_2(\lambda, \omega, \beta)$ are the unknown amplitudes of the source functions which have to be solved for. Figure 1 shows the shape of the two sources as a function of x . We have chosen a symmetric shape for $\hat{f}(x)$ and an antisymmetric shape for $\hat{P}(x)$. This symmetry-antisymmetry between the two sources is needed because otherwise, in the absence of a current, (2.22) would cancel out wave motion both in front of and behind the source region.

Substituting (2.23) in (2.22) and solving we get

$$D_2 = \frac{-2\beta D_1}{\tilde{l}_1(\omega - \lambda V + \tilde{l}_1 U)} \quad (2.24)$$

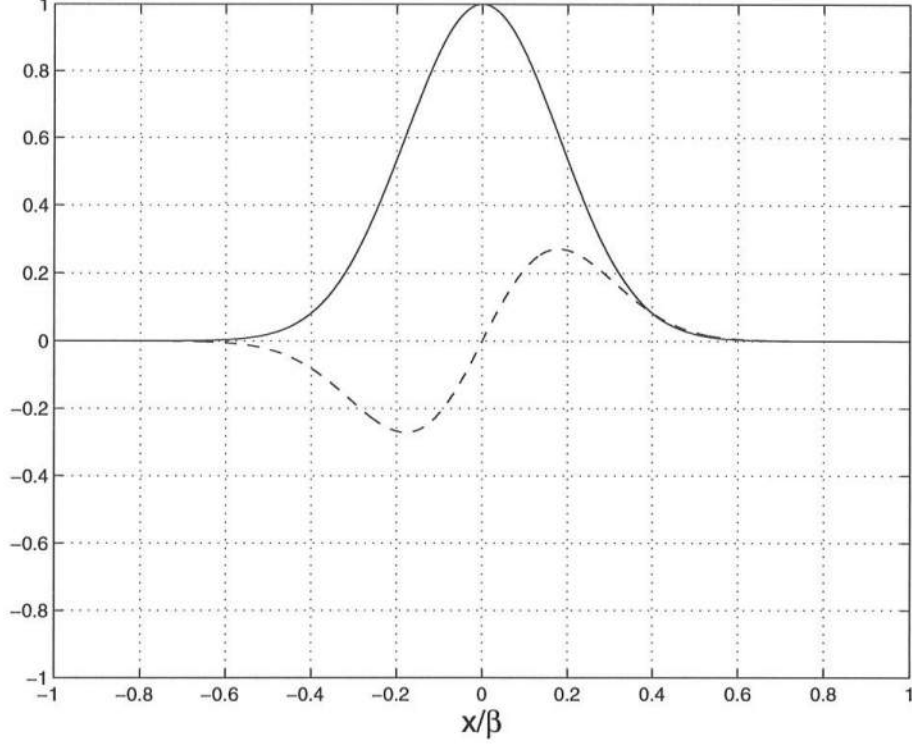


Figure 1: Source as a function of x . Solid line corresponds to $\frac{\hat{f}(x)}{D_1}$ and dashed line corresponds to $\frac{\hat{P}(x)}{D_2}$

To obtain a relation between the source function amplitude and wave motion amplitude, we Fourier transform the linear momentum equation of (2.3) in y and t

$$g\hat{\eta} = -\left[-i(\omega - \lambda V) + U\frac{\partial}{\partial x}\right]\left[1 + \alpha h^2\left(\frac{\partial^2}{\partial x^2} - \lambda^2\right)\right]\hat{\phi} \quad (2.25)$$

Since we are considering progressive wave motion, we take

$$\hat{\eta}(x) = \eta_0 \exp(il_1 x)$$

where η_0 is the wave amplitude. Substituting for $\hat{\phi}$ from (2.20) and for the source functions from (2.23) and (2.24), we get

$$D_1 = \frac{-i\eta_0 \exp(\frac{l_1^2}{4\beta})}{a_1 \sqrt{\frac{\pi}{\beta}} \left[1 + \frac{l_1(\omega - \lambda V - l_1 U)}{\tilde{l}_1(\omega - \lambda V + \tilde{l}_1 U)}\right] (\omega - \lambda V - l_1 U) (1 - \alpha h^2(l_1^2 + \lambda^2))} \quad (2.26)$$

where a_1 is obtained from (2.17).

Thus, with the help of (2.26) and (2.24) we can obtain the amplitudes of the source functions in the mass and momentum equations such that they cancel all wave motion

behind the source region, and add up in front of the source region to give the desired wave motion.

3 Numerical examples

We shall now test our modified source function model derived in the previous section with the help of the 1-D version of the fully nonlinear WKGS model. A schematic view of the test condition is shown in Figure 2. The total length of the domain is $L_x = 30m$ and water depth $h = 0.5m$. The center of the source is at $x_s = 5m$. A fairly large sponge layer has been provided at one end of the domain to absorb the wave energy.

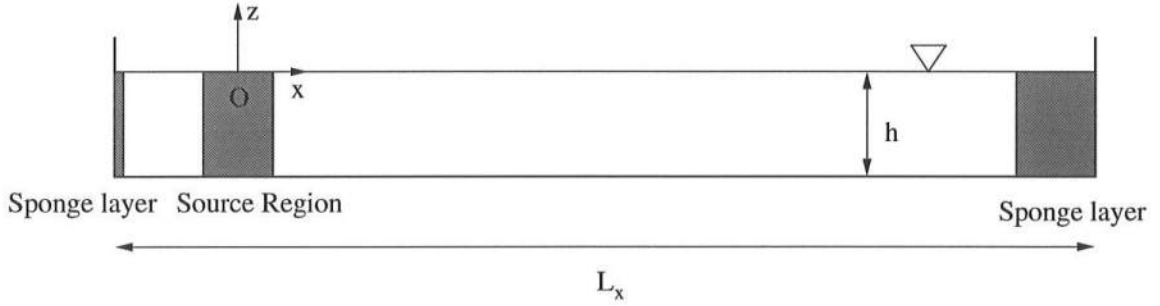


Figure 2: Computational domain

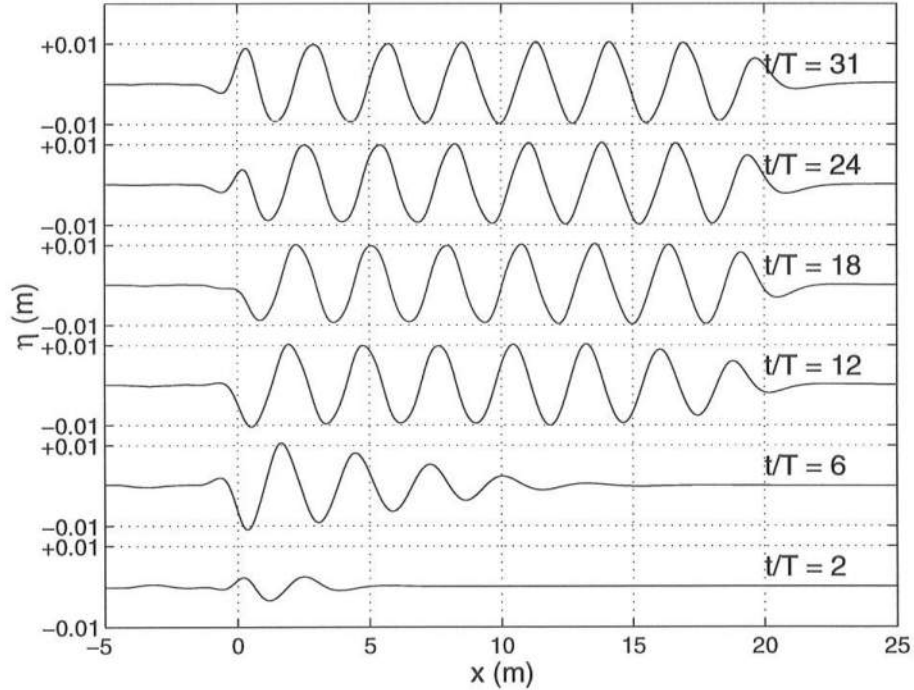


Figure 3: Snapshot of surface elevation η , $T = 1.5$ sec, $H = 0.02m$, $Fr = 0$

We consider monochromatic waves with $T = 1.5sec$ and $H = 0.02m$. These are small

waves and the aim is to see whether in the linear limit our source function method is able to reproduce the desired wave height. The grid size used in the model is $\Delta x = 0.025m$ and $\Delta t = 0.00458sec$. Figure 3 shows the snapshots of the surface elevation $\eta(x)$ at various times. The two sources method works really well in creating waves of a desired wave height along one direction only. There are no currents in this simulation.

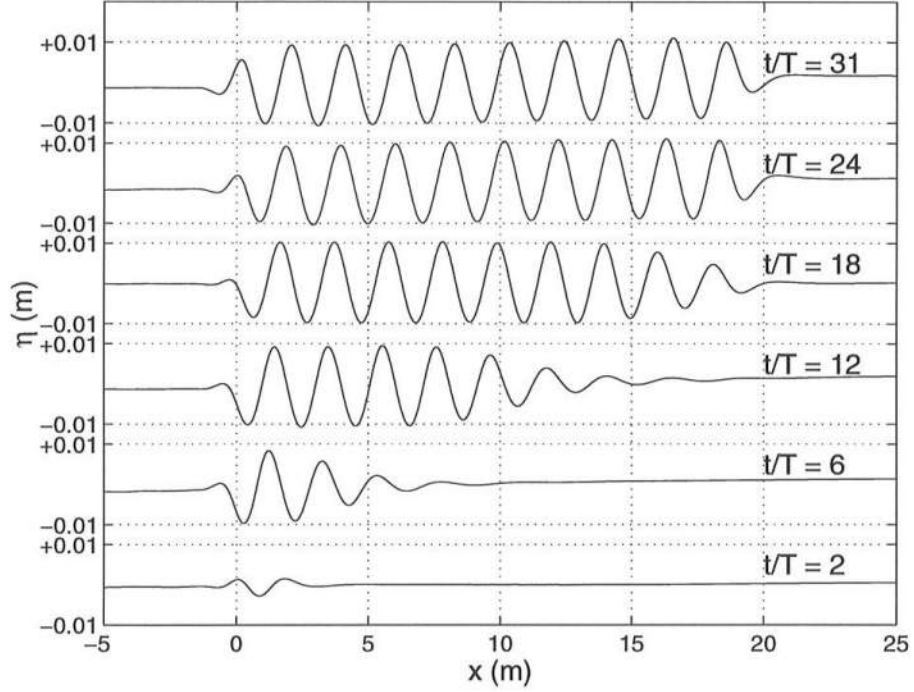


Figure 4: Snapshot of surface elevation η , $T = 1.5$ sec, $H = 0.02m$, $Fr = -0.15$

While developing our model we had claimed that the boundary terms in (2.18), which do not cancel out in the presence of a current, are small and can be neglected. To test this claim we considered the case of the monochromatic wave moving over a strong opposing current (Figure 4), and also a strong following current (Figure 5). In both cases the desired wave is reproduced very well. Thus the boundary terms do turn out to have negligible effects.

The source function method has been developed using linear wave theory and is expected to exhibit errors for large wave heights. To test the limits of this theory a comparison between the measured wave height (obtained from the WKGS model at $x = 10m$) and the desired wave height, for different values of δ is shown in Figure 6. The error is very small ($\pm 2\%$) up to $\delta = 0.11$, and increases for larger values of δ due to nonlinear effects. Note that no attempt is made to directly generate higher harmonics in this simulation.

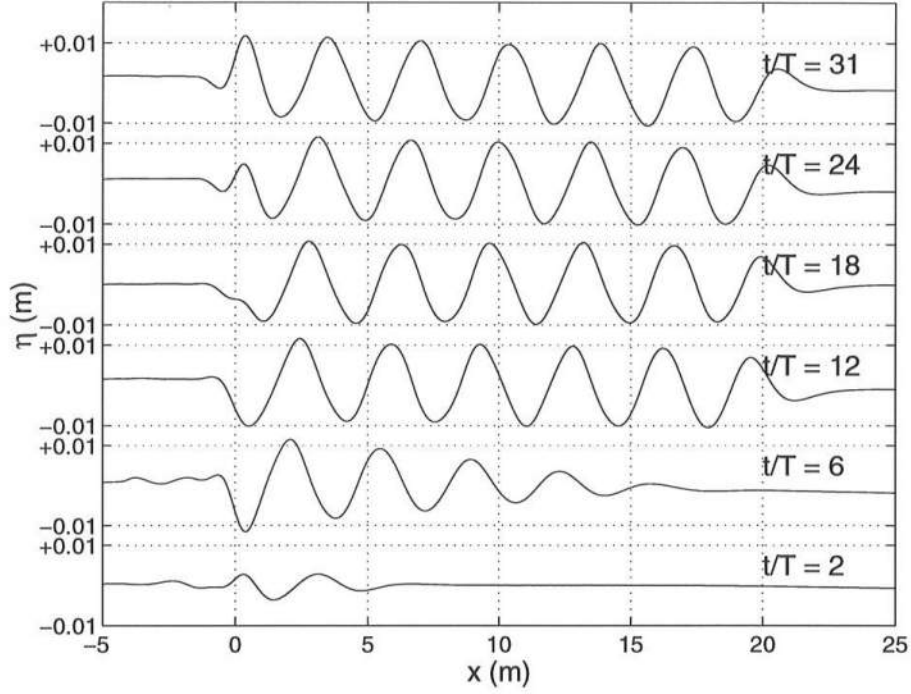


Figure 5: Snapshot of surface elevation η , $T = 1.5$ sec, $H = 0.02$ m, $Fr = 0.15$

4 Conclusions

In this note we have suggested an alternate formulation for generating waves using two spatially distributed source functions. This formulation generates waves traveling in one direction only as opposed to the method used by Wei *et al* [7] and GK98. The formulation also allows for the existence of an underlying $O(1)$ current field.

The method was tested with the WKGS model both in the presence and absence of strong currents and found to work very well. Even though the method uses the linear wave approximation it reproduces the larger wave heights with reasonable accuracy.

The advantage of this method is that it eliminates non-physical backward propagating waves, thus greatly reducing the required damping layer behind the source region. This potentially saves a considerable amount of computational effort specially when working with 2D Boussinesq models.

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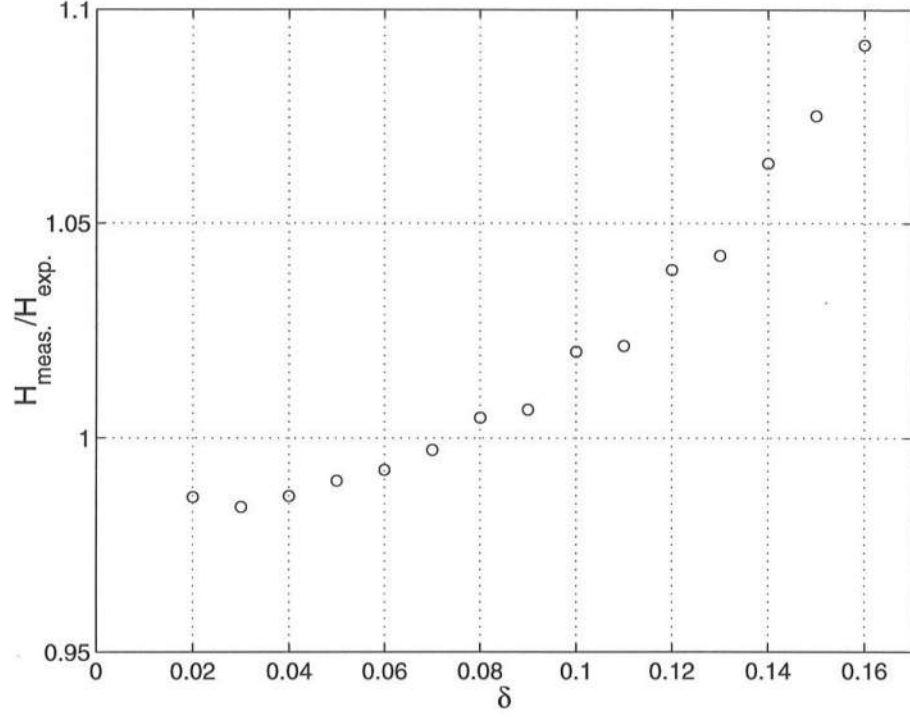


Figure 6: Comparison between the measured and expected wave height for different values of $\delta = \frac{H_{exp}}{2h}$ ($T = 1.5$ sec, $Fr = -0.15$)

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A Appendix: Theory for the $O(kh)^4$ model of GK98

The set of governing equations in non-dimensional form for the linear version of the $O(kh)^4$ model of GK98 is given by

$$\begin{aligned} & \frac{d\eta}{dt} + h\nabla \cdot \vec{u}_w + \mu^2 \frac{h^3}{2} (\gamma_1 - 1/3) \nabla^2 (\nabla \cdot \vec{u}_w) \\ & + \mu^4 \frac{h^5}{4} \left[\gamma_1 (\gamma_1 - 1/3) - \frac{(\gamma_2 - 1/5)}{6} \right] \nabla^2 \left(\nabla^2 (\nabla \cdot \vec{u}_w) \right) = 0 \end{aligned} \quad (A.1a)$$

$$\begin{aligned} & \frac{d\vec{u}_w}{dt} + \nabla \eta + \mu^2 \frac{h^2}{2} (\gamma_1 - 1) \frac{d}{dt} \left\{ \nabla (\nabla \cdot \vec{u}_w) \right\} \\ & + \mu^4 \frac{h^4}{4} \left[\gamma_1 (\gamma_1 - 1) - \frac{(\gamma_2 - 1)}{6} \right] \frac{d}{dt} \left\{ \nabla \left(\nabla^2 (\nabla \cdot \vec{u}_w) \right) \right\} = 0 \end{aligned} \quad (A.1b)$$

where γ_1 and γ_2 are given by

$$\gamma_1 \equiv \frac{1}{h^2} \left[\beta (h + z_a)^2 + (1 - \beta) (h + z_b)^2 \right] \quad (A.2a)$$

$$\gamma_2 \equiv \frac{1}{h^4} \left[\beta(h + z_a)^4 + (1 - \beta)(h + z_b)^4 \right] \quad (\text{A.2b})$$

and \vec{u}_w is defined by weighting the velocities at the reference depths z_a and z_b

$$\vec{u}_w = \beta \vec{u}_a + (1 - \beta) \vec{u}_b \quad (\text{A.3})$$

β, z_a and z_b are chosen to obtain appropriate dispersion characteristics even in deeper waters (see GK98).

Writing the equations in dimensional form, using a potential function $\phi(x, y, t)$ and introducing source functions we get a combined equation of the form

$$\begin{aligned} \frac{d^2 \phi}{dt^2} - gh \nabla^2 \phi - g \frac{h^3}{2} C_1 \nabla^2 \nabla^2 \phi - g \frac{h^5}{4} C_2 \nabla^2 \nabla^2 \nabla^2 \phi + \frac{h^2}{2} C_3 \nabla^2 \left(\frac{d^2 \phi}{dt^2} \right) \\ + \frac{h^4}{4} C_4 \nabla^2 \nabla^2 \left(\frac{d^2 \phi}{dt^2} \right) = -g \left(f + \frac{dP}{dt} \right) \end{aligned} \quad (\text{A.4})$$

where $C_1 \equiv \gamma_1 - 1/3, C_2 \equiv \gamma_1(\gamma_1 - 1/3) - \frac{(\gamma_2 - 1/5)}{6}, C_3 \equiv \gamma_1 - 1$ and $C_4 \equiv \gamma_1(\gamma_1 - 1) - \frac{(\gamma_2 - 1)}{6}$.

Taking a Fourier transform along both y and t yields the following ordinary differential equation

$$\begin{aligned} A \hat{\phi}^{(6)} + B \hat{\phi}^{(5)} + C \hat{\phi}^{(4)} + D \hat{\phi}^{(3)} + E \hat{\phi}^{(2)} + F \hat{\phi}^{(1)} \\ + K \hat{\phi} = g(\hat{f} + U \hat{P}^{(1)} - i(\omega - \lambda V) \hat{P}) \end{aligned} \quad (\text{A.5})$$

where

$$A \equiv C_2 g \frac{h^5}{4} - C_4 U^2 \frac{h^4}{4} \quad (\text{A.6a})$$

$$B \equiv 2i(\omega - \lambda V) U C_4 \frac{h^4}{4} \quad (\text{A.6b})$$

$$C \equiv g \frac{h^3}{2} (C_1 - \frac{3}{2} C_2 \lambda^2 h^2) - C_3 U^2 \frac{h^2}{2} + C_4 \frac{h^4}{4} ((\omega - \lambda V)^2 + 2\lambda^2 U^2) \quad (\text{A.6c})$$

$$D \equiv i(\omega - \lambda V) U h^2 (C_3 - \lambda^2 h^2 C_4) \quad (\text{A.6d})$$

$$\begin{aligned} E \equiv gh - U^2 - \lambda^2 g h^3 (C_1 - \frac{3}{4} \lambda^2 h^2 C_2) + C_3 \frac{h^2}{2} ((\omega - \lambda V)^2 + \lambda^2 U^2) \\ - C_4 \frac{h^4}{4} \lambda^2 (2(\omega - \lambda V)^2 + \lambda^2 U^2) \end{aligned} \quad (\text{A.6e})$$

$$F \equiv 2i(\omega - \lambda V) U (1 - \lambda^2 + \lambda^4) \quad (\text{A.6f})$$

$$K \equiv (\omega - \lambda V)^2 - gh\lambda^2 + \lambda^4 g \frac{h^3}{2} (C_1 + \lambda^2 \frac{h^2}{2} C_2) - \lambda^2 (\omega - \lambda V)^2 \frac{h^2}{2} (C_3 - C_4 \lambda^2 \frac{h^2}{2}) \quad (\text{A.6g})$$

Once again using the method of Green's function and following the steps given before, we obtain

$$G(\zeta, x) = \tilde{a}_1 \exp[i\tilde{l}_1(x - \zeta)] + \tilde{a}_2 \exp[i\tilde{l}_2(x - \zeta)] + \tilde{a}_3 \exp[i\tilde{l}_3(x - \zeta)] \quad \text{for } x > \zeta \quad (\text{A.7a})$$

$$G(\zeta, x) = a_1 \exp[i l_1(\zeta - x)] + a_2 \exp[i l_2(\zeta - x)] + a_3 \exp[i l_3(\zeta - x)] \quad \text{for } x < \zeta \quad (\text{A.7b})$$

where \tilde{l}_1 and l_1 are real and $\tilde{l}_2, \tilde{l}_3, l_2, l_3$ are complex. The coefficients are obtained from a set of matching conditions similar to the $O(kh)^2$ solution and are given by

$$\tilde{a}_1 = \frac{i}{A(\tilde{l}_1 - \tilde{l}_3)(\tilde{l}_2 - \tilde{l}_1)(\tilde{l}_1 + l_1)(\tilde{l}_1 + l_2)(\tilde{l}_1 + l_3)} \quad (\text{A.8a})$$

$$\tilde{a}_2 = \frac{i}{A(\tilde{l}_2 - \tilde{l}_1)(\tilde{l}_3 - \tilde{l}_2)(\tilde{l}_2 + l_1)(\tilde{l}_2 + l_2)(\tilde{l}_2 + l_3)} \quad (\text{A.8b})$$

$$\tilde{a}_3 = \frac{i}{A(\tilde{l}_3 - \tilde{l}_1)(\tilde{l}_2 - \tilde{l}_3)(\tilde{l}_3 + l_1)(\tilde{l}_3 + l_2)(\tilde{l}_3 + l_3)} \quad (\text{A.8c})$$

$$a_1 = \frac{i}{A(l_1 - l_2)(l_3 - l_1)(l_1 + \tilde{l}_1)(l_1 + \tilde{l}_2)(l_1 + \tilde{l}_3)} \quad (\text{A.8d})$$

$$a_2 = \frac{i}{A(l_2 - l_3)(l_1 - l_2)(l_2 + \tilde{l}_1)(l_2 + \tilde{l}_2)(l_2 + \tilde{l}_3)} \quad (\text{A.8e})$$

$$a_3 = \frac{i}{A(l_3 - l_1)(l_2 - l_3)(l_3 + \tilde{l}_1)(l_3 + \tilde{l}_2)(l_3 + \tilde{l}_3)} \quad (\text{A.8f})$$

From here on the solution follows the $O(kh)^2$ solution very closely, and thus, the intermediate steps have been omitted. The final relationship between the wave amplitude η_0 and source function amplitude D_1 is given by

$$D_1 = \frac{-i\eta_0 \exp(\frac{l_1^2}{4\beta})}{a_1 \sqrt{\frac{\pi}{\beta}} \left[1 + \frac{l_1(\omega - \lambda V - l_1 U)}{\tilde{l}_1(\omega - \lambda V + \tilde{l}_1 U)} \right] (\omega - \lambda V - l_1 U) \left[1 - \frac{h^2}{2} (l_1^2 + \lambda^2) (C_3 - \frac{h^2}{2} C_4 (l_1^2 + \lambda^2)) \right]} \quad (\text{A.9})$$

with (2.24) remaining unchanged.

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