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ON THE SPECTRAL ANALYSIS OF NONSTATIONARY RANDOM PROCESSES

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Abstract

Spectral design relations for random processes are re-examined in terms of the sampling properties of evolutionary spectral estimates. For a class of spectral windows optimal design relations and formal criteria for selecting optimal window forms are derived. It is shown that the optimal design relations eliminate inadequacies and subjectivity associated with the available forms of spectral design relations and bring out the salient features of the evolutionary spectral theory most effectively. Concepts are illustrated with artificially generated examples.

1. INTRODUCTION

The nature of various processes in economics, geophysics, and engineering such as stock market fluctuations, turbulence, wind-generated gravity waves, electrical noise, and mechanical vibration is randomly irregular. It has long been accepted that the most fruitful and effective approach to investigate such processes is through the theory of stochastic processes. In this approach the spectrum is a well-known concept, and the problem of its estimation is of fundamental importance. However, the development of the spectrum concept and the theory of spectral analysis has been exclusively for stationary random processes as a natural consequence of the fact that such processes are the easiest to understand in theory and applications. Realizing the truly non-stationary nature of the real world and motivated by the common objective to be able to study non-stationary processes in a manner as physically meaningful and mathematically rigorous as in the stationary cases, various attempts have been made to define a non-stationary spectrum concept. Among the currently available definitions, the most significant are those due to Fano (1950), Page (1952), Dubman (1965), Priestley (1965) and Marks (1970). Based on the reviews of the subject (see, e.g., Loynes, 1968; Priestley, 1965, 1971; Tayfun and Yang, 1972), and the theoretical and applied work stimulated by Priestley's definition (see, e.g., Brown, 1967; Hammond, 1968; Abdrabbo and Priestley, 1969; Priestley and Rao, 1969; Shinozuka, 1970; Shinozuka and Jan, 1971), it is evident that the theory of evolutionary spectrum is the most promising in terms of its fundamental approach and assumptions, mathematical rigor, interpretability and usefulness.

Priestley introduced the evolutionary spectrum and a spectral theory for estimating time-dependent spectra in a series of papers (Priestley, 1965, 1966, 1967, to be referred to subsequently as P-1,2,3). Evolutionary spectrum is a smooth generalization of the concepts associated with the orthogonal spectral representation of stationary random processes to a wider class of processes with implicit non-stationarity. Such processes, referred to as oscillatory or, semi-stationary processes, admit a spectral representation of the form

$$X(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega)\,, \qquad (1.1)$$

where $Z(\omega)$ is an orthogonal process with $E\{|dZ(\omega)|^2\} = dF(\omega)$, and $A(t,\omega)$ is deterministic function of t and ω. The evolutionary spectral density is defined to be

$$f(t,\omega) = |A(t,\omega)|^2 \, f(\omega) \quad , \tag{1.2}$$

where $f(\omega) = dF/d\omega$, assuming that $F(\omega)$ is differentiable. The particular case of (1.1), with $A(t,\omega) = 1$, is the well-known spectral representation of stationary processes with a mean-square spectral density $f(\omega)$. If the function $A(t,\omega)$ is normalized, say, at $t = t_0$, so that $A(t_0,\omega) = 1$, then $|A(t,\omega)|^2$ represents the change in spectral density at subsequent times while preserving the physically meaningful concept of frequency.

The evolutionary spectral density can be estimated from a sample of a single realization by a procedure which involves certain spectral windows or, simply filters, whose forms depend on various parameters. Priestley considered the problem of choosing these parameters to attain various desirable properties in the estimated spectra. In particular, choosing the relative mean-square error criterion as a figure of merit for the overall sampling quality of spectral estimates, he developed various design relations to determine the filter parameters. Under the condition that the mathematical forms of filters are chosen a priori from the available collection of windows, the relative mean-square error of the spectral estimates depends on two filter parameters. Several possibilities arise as to how these two parameters should be chosen. The first two require that estimates treated as a function of both time and frequency should have either a prescribed frequency-domain resolution or, a time-domain resolution, respectively. Therefore, these requirements yield one of the filter parameters explicitly, enabling us to determine the remaining one by minimizing the relative mean-square error. The third possibility is to require fixed resolution in both domains so that the two parameters are determined from these prescribed requirements without any specific regard to the overall sampling fluctuations of the spectral estimates. The last but probably the most natural possibility is to minimize the relative mean-square error jointly with respect to both of the parameters.

It is clear that all of the above possibilities have the serious shortcoming that the explicit mathematical forms of the filters must be a priori chosen while there is no formal criterion as to what forms are optimal. Furthermore, the first three possibilities have additional inadequacies mainly in the fact that requiring that estimates have a prescribed degree of resolution in a particular domain is a rather subjective decision. The parameters chosen in this manner can provide spectral estimates with too much sampling fluctuations to render any kind of a prescribed resolvability criterion meaningless. Clearly the most desirable criterion is to choose the design parameters by optimizing the overall sampling quality of spectral estimates expressed in terms of various properties of spectral windows. Therefore in this paper we discuss the possibilities to determine unique optimal design relations and formal optimality criteria for the mathematical forms of a class of spectral windows which conforms to the general character of the evolutionary spectral theory. Before doing so, we briefly summarize relevant parts of the evolutionary spectral theory employing the same notations used by Priestley except for some minor modifications.

2. EVOLUTIONARY SPECTRAL THEORY

We consider processes which admit a representation of the form (1.1). Given a sample record of $X(t)$ $(o \leq t \leq T)$, the spectral density $f(t,\omega)$ in the vicinity of time t and frequency ω is estimated in two steps. First we choose a filter $g_h(u)$, depending on a parameter h, and write

$$U(t,\omega) = \int_{t-T}^{t} g_h(u)\ X(t-u)\ e^{i\omega(t-u)}\ du \ . \tag{2.1}$$

Then we smooth the values of $|U(t,\omega)|^2$ over the neighboring values in time by using a weight function or, simply another filter $w_{T'}(u)$, depending on a parameter T', and estimate $f(t,\omega)$ by

$$\hat{f}(t,\omega) = \int_{t-T}^{t} w_{T'}(u)\,|U(t-u,\omega)|^2\,du \quad . \tag{2.2}$$

The filter $g_h(u)$ satisfies the following conditions.

(a) $g_h(u)$ is square-integrable and normalized so that

$$2\pi \int_{-\infty}^{\infty} |g_h(u)|^2 du = \int_{-\infty}^{\infty} |\Gamma(\omega)|^2\,d\omega = 1, \tag{2.3}$$

where $\Gamma(\omega)$ is the Fourier transform of $g_h(u)$, and

(b) $g_h(u)$ has a finite width defined by

$$B_g = \int_{-\infty}^{\infty} |u|\ \ |g_h(u)|du \ . \tag{2.4}$$

Likewise the weight function $w_{T'}(u)$ satisfies, for all T',

(a) $w_{T'}(u) \geq 0$,

(b) $w_{T'}(u)$ decays to zero as $|u| \to \infty$,

(c) $\int_{-\infty}^{\infty} w_{T'}(u)\,du = 1,$ (2.5)

(d) $\int_{-\infty}^{\infty} \{w_{T'}(u)\}^2\,du < \infty,$

(e) $\lim_{T'\to\infty} \left[2\pi\, T' \int_{-\infty}^{\infty} \{w_{T'}(u)\}^2\,du\right] = C$ (a constant).

Clearly now, before the estimate $\hat{f}(t,\omega)$, can be evaluated from (2.1) and (2.2) by varying the value of ω, various decisions must be made as to the functional forms of $g_h(u)$ and $w_{T'}(u)$, the parameters h and T', and the minimum sample length T required so that no errors will be introduced in evaluating (2.1) and (2.2) due to filter end-effects. The figure of merit on which these decisions are based is the relative mean-square error of the estimate

$\hat{f}(t,\omega)$ at a particular time t and frequency ω defined as

$$\left[E\{\hat{f}(t,\omega) - f(t,\omega)\}^2\right] \Big/ f^2(t,\omega) = \left[\text{bias}^2\{\hat{f}(t,\omega)\} + \text{var}\{\hat{f}(t,\omega)\}\right] \Big/ f^2(t,\omega) \quad \cdot \qquad (2.6)$$

In the case of a normal process X(t), and for estimates of the form (2.2), the explicit form of the relative mean-square error is approximately given by (P-2)

$$M \simeq \frac{1}{4}\left\{\frac{B_w^2}{B_o^2(t,\omega)} + \frac{B_\Gamma^2}{B_f^2(t,\omega)}\right\}^2 + \frac{C(1+\delta_{0,\omega})}{T'} \int_{-\infty}^{\infty} |\Gamma(\omega)|^4 \, d\omega \ , \qquad (2.7)$$

where

$$B_\Gamma = \left\{\int_{-\infty}^{\infty} \omega^2 \ |\Gamma(\omega)|^2 \ d\omega\right\}^{\frac{1}{2}} , \qquad (2.8)$$

and

$$B_w = \left\{\int_{-\infty}^{\infty} u^2 \ w_{T'}(u) \ du\right\}^{\frac{1}{2}} , \qquad (2.9)$$

are measures of bandwidth of $\Gamma(\omega)$ and $w_{T'}(u)$, respectively,

$$B_o(t,\omega) = | \ f/\{\partial^2 f/\partial t^2\}|^{\frac{1}{2}} , \qquad (2.10)$$

and

$$B_f(t,\omega) = | \ f/\{\partial^2 f/\partial \omega^2\}| ^{\frac{1}{2}} \qquad (2.11)$$

are interpreted as the bandwidths of $f(t,\omega)$ over the time- and frequency-domains, respectively. These last two definitions are generalizations of the spectral bandwidth concept, and hence may be regarded as measure of how $f(t,\omega)$ varies as a spectral distribution over time and frequency. We note that in (2.7) the first group of terms corresponds to the contribution of bias of $\hat{f}(t,\omega)$ over both the time- and frequency-domains, and the last to the variance of $\hat{f}(t,\omega)$. The ratios $\gamma = B_w/B_o(t,\omega)$ and $\lambda = B_\Gamma/B_f(t,\omega)$ are defined as measures of resolution for the spectral estimate $\hat{f}(t,\omega)$ over time and frequency, respectively.

The particular case of stationary processes is also included in the above definitions. That is, if $A(t,\omega) = 1$, corresponding to stationarity, $B_f(t,\omega) \to B_f(\omega)$, $B_o(t,\omega) \to \infty$, and time-dependency smoothly disappears from the analysis.

We may regard the bias error as a consequence of the imperfections of the filters $g_h(u)$ and $w_{T'}(u)$, and variance as a consequence of employing a sample of only a single realization. Therefore, the relative mean-square error can be considered as a measure of the overall statistical errors involved in the estimation procedure. Any such estimation should, therefore, be based on the unique set of design relations which minimizes the relative mean-square error. In the next section we discuss how such a unique set of optimal design relations can be constructed for a class of filters which suggest themselves in a rational manner.

3. OPTIMAL DESIGN RELATIONS

We consider now a class of filters $\{g(u)\}$ which are, in addition to satisfying the conditions (2.3) and (2.4), continuous functions, and for which we can find a positive constant h such that $g(u)=0$ for $|u| \geq h$. Similarly, consider the class of filters $\{w(u)\}$ which are, in addition to satisfying the conditions (2.5), piecewise continuous and of bounded variation, and for which we can find a positive constant T' such that $w(u)=0$ for $|u| \geq T'$. It is noted that the conditions $g(u)=0$ $(|u| \geq h)$ and $w(u)=0$ $(|u| \geq T')$ suggest themselves in a fairly natural way in (2.1) and (2.2) so that no errors will be introduced in spectral estimations due to filter end-effects. The continuity and piecewise continuity conditions on $g(u)$ and $w(u)$, respectively, guarantee that various properties of these functions are well-defined. In particular, if $g(u)$ has a point of discontinuity, it may be easily realized that B_Γ defined in (2.8) is infinite. Under these conditions, and noting that $\exp(i\omega t)$ is redundant in (2.1) on account of $|U|^2$ required in (2.2), we may rewrite (2.1) and (2.2) as

$$U(t,\omega) = \int_{-h}^{h} g(u)\ X(t-u)\ e^{-iu\omega}\ du \quad , \tag{3.1}$$

and

$$\hat{f}(t,\omega) = \int_{-T'}^{T'} w(u)\ |\ U(t-u,\omega)|^2\ du \ . \tag{3.2}$$

We note now that the minimum sample size required is clearly

$$T_{min} = 2\ (h + T') \quad , \tag{3.3}$$

assuming that t is centralized in the interval $(0,\ T_{min})$ so that $t = h + T'$.

Making use of the normality conditions (2.3) and (2.5c) for the filters g(u) and w(u), respectively, we may now define the "characteristic shape" functions {G(x)} and {W(x)} which satisfy

$$G(x) = \begin{cases} h^{1/2}\ g(xh), & |x| < 1 \\ 0 & , \text{otherwise} \ , \end{cases} \tag{3.4}$$

and

$$W(x) = \begin{cases} T'w(xT') \ , & |x| < 1 \\ 0 & , \text{otherwise} \ . \end{cases} \tag{3.5}$$

The characteristic shapes G(x) and W(x) defined in this manner enable us to rewrite various properties of the filters g(u) and w(u) which appear in (2.7) more explicitly in terms of the width parameters h and T', and a set of coefficients which depend only on the functional forms of the characteristic shapes.

In particular, we have

$$B_w = T'\left\{\int_{-1}^{1} x^2\, W(x) \quad dx\right\}^{\frac{1}{2}} \overset{def}{=} T'\, C_{w1} \,, \tag{3.6}$$

and

$$C = 2\pi \int_{-1}^{1} |W(x)|^2 \, dx \overset{def}{=} C_{w2} \,. \tag{3.7}$$

Similarly, it can be easily verified that

$$B_\Gamma = h^{-1}\left\{ 2\pi \int_{-1}^{1} \left|\frac{dG(x)}{dx}\right|^2 dx \right\}^{\frac{1}{2}} \overset{def}{=} h^{-1}\, C_{g1} \,, \tag{3.8}$$

and

$$\int_{-\infty}^{\infty} |\Gamma(\omega)|^4 \, d\omega = h\left\{4\pi \int_{o}^{2} \left| \int_{-1}^{1-y} G(x+y)G(x)dx\right|^2 dy\right\} \overset{def}{=} h\, C_{g2} \,. \tag{3.9}$$

We are now in a position to express (2.7) more explicitly in the form

$$M \cong \frac{1}{4}\left\{\frac{C_{w1}^2\, T'^2}{B_o^2(t,\omega)} + \frac{C_{g1}^2}{h^2\, B_f^2(t,\omega)}\right\}^2 + (1+\delta_{o,\omega})\,\frac{C_{g2}C_{w2}\, h}{T'} \,. \tag{3.10}$$

We recall that the objective is to determine the width parameters h, T', and possibly the optimal characteristic shapes G(x) and W(x), which minimize the relative mean-square error M. However, at this point let us assume that the set of coefficients (C_{g1}, C_{g2}, C_{w1}, C_{w2}) already corresponds to the optimal characteristic shapes, say, $G_o(x)$ and $W_o(x)$, and for simplicity denote it by $\overline{C}$. Also realizing that, for a given process, the spectral bandwidth measures $B_o(t,\omega)$ and $B_o(t,\omega)$ are fixed properties dependent only on t and ω, we can in principle regard M as a function of only h and T', with $\overline{C}$, t and ω treated as parameters, and concisely write

$$M = M(h,T';\, \overline{C},\, t,\, \omega) \,. \tag{3.11}$$

Consider now an optimization of M with respect to h and T'. For all values h, $T' \geq o$; we can easily show that the derivatives of M satisfy the conditions:

$$(\partial^2 M/\partial h^2)(\partial^2 M/\partial T'^2) - (\partial^2 M/\partial h \partial T')^2 > o,$$

$$\partial^2 M/\partial h^2 > o \quad \text{and} \quad \partial^2 M/\partial T'^2 > o.$$

Hence, a global minimum of M exists with respect to h and T'. Denoting the optimal values by the subscript zero, the necessary and sufficient conditions for h_o and T_o' to be optimal are,

$$\partial M/\partial h = 0, \text{ and } \partial M/\partial T' = 0, \text{ at } (h_o, T_o'),$$

by the usual differential calculus technique. However, before we proceed to the solutions of the two simultaneous equations resulting from the above conditions, we note that M, as a function of h and T', is a posynomial of the form

$$M = \sum_{j=1}^{4} k_j \; y_j \; (h, T'), \tag{3.12}$$

where

$$y_j(h,T') = h^{a_j} \; T'^{b_j} \quad (j = 1,2,3,4)$$

denote the product functions of h and T' with

$$\begin{aligned} a_1 &= 0, \; a_2 = -2, \; a_3 = -4, \; a_4 = 1, \\ b_1 &= 4, \; b_2 = 2, \; b_3 = 0, \; b_4 = -1, \end{aligned} \tag{3.13}$$

and k_j's are positive coefficients defined as

$$k_1 = \frac{1}{4} \left\{ C_{w1}/B_o(t,\omega) \right\}^4 ,$$

$$k_2 = \frac{1}{2} \left[\left\{ C_{w1} C_{g1} \right\} \Big/ \left\{ B_o(t,\omega) \; B_f(t,\omega) \right\} \right]^2 ,$$

$$k_3 = \frac{1}{4}\left\{C_{g1}/B_f(t,\omega)\right\}^4 ,$$

$$k_4 = (1 + \delta_{0,\omega})\, C_{w2}\, C_{g2} . \tag{3.14}$$

The posynomial form (3.11) is particularly amenable to an optimization by the geometric programming technique developed largely by Duffin, Peterson, and Zener (1967) in the 1960's. The application of the technique in this case (see Appendix) yields the following results:

$$M_o = 3\left\{\frac{(1 + \delta_{0,\omega})\, C_{w1}\, C_{w2}\, C_{g1}\, C_{g2}}{2\, B_o(t,\omega)\, B_f(t,\omega)}\right\}^{2/3} , \tag{3.15}$$

where

$$M_o = M_o\,(\overline{C},t,\omega) = \min_{h,T'} M(h,T';\overline{C},t,\omega),$$

with the optimal values of the parameters h and T' given by

$$h_o = (3/M_o)^{1/4}\left\{C_{g1}/B_f(t,\omega)\right\}, \tag{3.16}$$

$$T'_o = (M_o/3)^{1/4}\left\{B_o(t,\omega)/C_{w1}\right\}, \tag{3.17}$$

such that

$$(h_o/T'_o) = \left\{\frac{C^2_{w1}C^2_{g1}}{B^2_o(t,\omega)\, B^2_f(t,\omega)\, C_{w2}\, C_{g2}}\right\}^{1/3} . \tag{3.18}$$

The resolutions of the spectral estimate $\hat{f}(t,\omega)$ in time- and frequency-domains corresponding to the above optima are

$$\gamma_o = \lambda_o = (M_o/3)^{1/4} . \tag{3.19}$$

Finally, for the estimate $\hat{f}(t,\omega)$ in the vicinity of time t, the optimal minimum sample size required, from (3.3), is

$$T_{min} = 2\,(h_o + T'_o), \tag{3.20}$$

assuming that we have a sample $X(t)$ $(o \le t \le T)$ with $T > T_{min}$.

At this point several important remarks can be made about the nature of the preceding results as follows.

(1) Examining (3.15), we can immediately conclude that the optimal characteristic shapes $G_o(x)$ and $W_o(x)$ are those which minimize the coefficient products $C_{g1}C_{g2}$ and $C_{w1}C_{w2}$, respectively. Therefore

$$\min_{G(x)} \{C_{g1}\ C_{g2}\}, \qquad (3.21)$$

and

$$\min_{W(x)} \{C_{w1}\ C_{w2}\} \qquad (3.22)$$

constitute two formal criteria for choosing and, possibly constructing unique optimal characteristic shapes. We will elaborate on these possibilities in section 4.

(2) It is clear from the definition of the relative mean-square error that $M \leq 1$. Therefore, in (3.15), the condition

$$B_o(t,\omega)\ B_f(t,\omega) > (3^{3/2}/2)(1 + \delta_{o,\omega})\ C_{w1}\ C_{w2}\ C_{g1}\ C_{g2}$$

must be satisfied for the estimation procedure to be meaningful. The parameters $B_o(t,\omega)$ and $B_f(t,\omega)$ have the dimensions of t and t^{-1}, respectively. Therefore, the product $B_o(t,\omega)\ B_f(t,\omega)$ is a dimensionless quantity which can be regarded as an overall measure of "spectral characteristics" of a given process such that the larger this parameter is, the more feasible and accurate the estimation procedure becomes. Furthermore, the overall statistical quality of spectral estimates for two processes, one with a relatively smaller $B_f(t,\omega)$ and a relatively larger $B_o(t,\omega)$ than the other, would be the same so long as the product $B_o(t,\omega)\ B_f(t,\omega)$ is the same for both processes for all t and ω.

(3) The minimal error M_o consists of bias and variability contributions in a one to two ratio (see Appendix). This result, in retrospect with the nature of the geometric programming technique, or, simply an examination of group of terms in (3.10), indicates that errors due to bias in time- and frequency-domains are more sensitive to changes in the design parameters h and T' than those due to variability.

(4) For the particular case as $B_o(t,\omega) \to \infty$, corresponding to stationarity, we see that $M_o \to o$, and h_o, $T'_o \to \infty$ as the ratio $(h_o/T'_o) \to o$. This is very much consistent with the asymptotic sampling properties of spectral estimates in the classical analysis, and in principle it suggests that, as stationarity becomes dominant, we should take larger values h_o and T'_o, and therefore larger sample sizes while decreasing the ratio (h_o/T'_o) until estimates attain a convergent behavior.

(5) The above optimal results are based on the minimization of the relative mean-square error at a particular time t and frequency ω, and in that sense they are very general. That is, the optimal parameters h_o and T'_o, and therefore the filters g(u) and w(u) that we derive in terms of these parameters and optimal characteristic shapes through (3.4) and (3.5), will be dependent on t and ω in computing spectral estimates $\hat{f}(t,\omega)$ from (3.1) and (3.2). However, in practical spectral computations we may want to consider other possibilities as follows.

(a) We can consider a spectral design minimizing the maximum possible relative mean-square error over either time or, frequency, respectively. More formally then, we seek the values h_o, T'_o which minimize

(i) $$\sup_{0 \leq t \leq T} M(h, T'; \overline{C}, t,\omega) \qquad (3.23)$$

over time, or

(ii) $$\sup_{\omega} M(h, T'; \overline{C}, t,\omega) \qquad (3.24)$$

over frequency. These results are readily included in the optimal design relations given above, and they correspond to simply replacing $B_o(t,\omega)$ and $B_f(t,\omega)$ with, say, $B_o(\omega)$ and $B_f(\omega)$ such that

$$B_o(\omega)\ B_f(\omega) = \inf_{0 \leq t \leq T} \{B_o(t,\omega)\ B_f(t,\omega)\} \qquad (3.25)$$

for the (i) case, and with, say, $B_o(t)$ and $B_f(t)$ such that

$$B_o(t)\ B_f(t) = \inf_{\omega} \{B_o(t,\omega)\ B_f(t,\omega)\} \qquad (3.26)$$

for the (ii) case, respectively, in the optimal relations (3.15) through (3.19).

(b) As the simplest possibility we may consider the design minimizing the maximum possible error over both time and frequency. That is, we seek the parameters h_o, T'_o which minimize

$$\sup_{\substack{\omega \\ o \leq t \leq T}} M(h, T'; \overline{C}, t, \omega) \tag{3.27}$$

The design relations corresponding to this case is similarly given by replacing $B_o(t,\omega)$ and $B_f(t,\omega)$ in (3.15) through (3.19) simply with, say, B_o and B_f such that

$$B_o B_f = \inf_{\substack{\omega \\ 0 \leq t \leq T}} B_o(t,\omega)\, B_f(t,\omega) \tag{3.28}$$

This possibility therefore provides the optimal design relations which are independent of t and ω.

4. OPTIMALITY CRITERIA FOR SPECTRAL WINDOWS

As we have mentioned in the preceding section, it is now possible to consider formal optimality criteria for the class of filters {w(u)} and {g(u)} on the basis of the optimal design relations. The optimal filters w(u) and g(u) are evidently those with the characteristic functions W(x) and G(x) which minimize the product functionals $C_{w1}C_{w2}$ and $C_{g1}C_{g2}$, respectively. Using the definitions of these coefficients given in (3.6) through (3.9), we may rewrite the optimality criteria more explicitly as,

$$\min_{W(x)} \left\{ \int_{-1}^{1} x^2 W(x)\, dx \right\}^{\frac{1}{2}} \left\{ 2\pi \int_{-1}^{1} |W(x)|^2 dx \right\} , \tag{4.1}$$

subject to the constraints that W(x) is a real, non-negative, piecewise continuous function of bounded variation, square-integrable, and identically zero for values $|x| \geq 1$, and properly normalized so that

$$\int_{-1}^{1} W(x)\, dx = 1 ,$$

and

$$\min_{G(x)} \left\{ 2\pi \int_{-1}^{1} \left|\frac{dG(x)}{dx}\right|^2 dx \right\}^{\frac{1}{2}} \left\{ 4\pi \int_{o}^{2} \left| \int_{-1}^{1-y} G(x+y)G(x)dx \right|^2 dy \right\} , \quad (4.2)$$

subject to the constraints that G(x) is a continuous function, identically zero for values $|x| \geq 1$, and properly normalized so that

$$2\pi \int_{-1}^{1} |G(x)|^2 \, dx = 1 \ .$$

The solution of (4.1) and (4.2), if they uniquely exist, provide the optimal characteristic shapes W(x) and G(x), from which the filters w(u) and g(u) can be constructed, using (3.4) and (3.5), as

$$w(u) = (T')^{-1} \, W(u/T'),$$

and (4.3)

$$g(u) = (h)^{-\frac{1}{2}} \, G(u/h) \ .$$

One possibility for finding the solutions of (4.1) and (4.2) is to expand W(x) and G(x) in terms of complete orthogonal functions such as Fourier series or, Legendre functions of the first kind with unknown coefficients, and to attempt to determine these coefficients by minimizing (4.1) and (4.2) subject to the relevant constraints. Such an approach turns out to be especially feasible for (4.1) by virtue of the relatively simple functional form. In particular, if we let

$$W(x) = \sum_{k=o}^{\infty} A_k \, P_k(x) \ ,$$

where A_k and $P_k(x)$ are the set of unknown coefficients and Legendre functions of the first kind, respectively, we can easily show that the unique optimal solution to (4.1) is obtained, with $A_o = 1/2$, $A_2 = (1/2)$, and $A_k = 0$ for $k \neq 0,2$, as

$$W(x) = \begin{cases} \frac{3}{4}(1 - x^2) , & |x| < 1 \\ 0 , & |x| \geq 1 . \end{cases} \quad (4.4)$$

Therefore, we have $C_{w1} = \sqrt{0.2}$, $C_{w2} = (6\pi)/5$, and the minimal product $C_{w1}C_{w2} = 1.686$. It is very interesting to note that (4.4) corresponds to the well-known Parzen filter. It might be also useful to compare the filter (4.4) with others from the standard collection of windows (see for example Parzen, 1961). Table I illustrates this comparison for various filters which conform to the constraints of the class of filters {w(u)} considered in this paper.

TABLE I

Filter W(x)	C_{w1}	C_{w2}	$C_{w1}C_{w2}$
$\frac{3}{4}(1-x^2)$	$(0.2)^{\frac{1}{2}}$	$(6\pi)/5$	1.686
$(\pi/4)\cos\frac{1}{2}(\pi x)$	$(1 - \frac{8}{\pi^2})^{\frac{1}{2}}$	$\pi^3/8$	1.687
$\frac{1}{2}(1 + \cos\pi x)$	$(\frac{1}{3} - \frac{2}{\pi^2})^{\frac{1}{2}}$	$(3\pi)/2$	1.704
$1 - \lvert x \rvert$	$(1/6)^{\frac{1}{2}}$	$(4\pi)/3$	1.710
1 (rectangular)	$(1/3)^{\frac{1}{2}}$	π	2.178

Unfortunately, the problem of finding the optimal G(x) in a similar manner is not satisfactorily solved on account of the more complicate nature of the functional (4.2). Nevertheless, we can use (4.2) as a figure of merit for choosing an optimal form G(x) from the available collection of spectral windows. With this purpose in mind, Table II was prepared to illustrate the relevant properties of most of the well-known window shapes consistent with the class of functions $\{g(u)\}$. It is evident that among the four different shapes examined the optimal filter is the "hanning" window given by

$$G(x) = \begin{cases} (6\pi)^{-\frac{1}{2}} (1 + \cos\pi x), & |x| < 1 \\ 0 & , |x| \geq 1 . \end{cases} \tag{4.5}$$

TABLE II

Filter $G(x)$	C_{g1}	C_{g2}	$C_{g1}C_{g2}$
$(6\pi)^{-\frac{1}{2}} (1 + \cos\pi x)$	$\pi/\sqrt{3}$	.1528	.2778
$(2\pi)^{-\frac{1}{2}} \cos \frac{1}{2} (\pi x)$	$\pi/2$	.1875	.2946
$(\frac{3}{4}\pi)^{\frac{1}{2}} (1 - \lvert x \rvert)$	$\sqrt{3}$	.1725	.2988
$\{15/(32\pi)\}^{\frac{1}{2}} (1 - x^2)$	$\sqrt{(5/2)}$	.1926	.3045

5. ANALYSIS OF ARTIFICIAL PROCESSES

It is worthwhile to illustrate the application of the optimal design relations and to re-examine the validity of the evolutionary spectral theory by studying artificial processes. With these objectives in mind, we consider here the uniformly modulated processes, quite similar to the example given by Priestley (P-1,2), generated from the model in continuous time

$$X(t) = \left\{ e^{-(t^2/2\beta^2)} \right\} Y(t), \tag{5.1}$$

where $Y(t)$ is a stationary normal process with the mean-square spectral density function

$$f_\alpha(\omega) = \left\{ (|w| - 1)^2 + \alpha^2 \right\}^{-1}, \quad |\omega| \leq \pi , \tag{5.2}$$

and, α and β are constants to be specified. The evolutionary spectral density function of X(t) is given by

$$f_{\alpha\beta}(t,\omega) = e^{-t^2/\beta^2} f_{\alpha}(\omega) . \tag{5.3}$$

In so much as $f_{\alpha\beta}(t,\omega)$ is symmetric with respect to t and ω, we may restrict the analysis to the values $t,\omega \geq o$, and write, using the definitions (2.10), (2.11), and from (5.3),

$$B_o(t,\omega) = B_o(t) = (\beta^2/\sqrt{2})\left|2t^2 - \beta^2\right|^{-\frac{1}{2}},$$

and

$$B_f(t,\omega) = B_f(\omega) = \left\{(\omega-1)^2 + \alpha^2\right\} \left|6(\omega-1)^2 - 2\alpha^2\right|^{-\frac{1}{2}}.$$

For the simplicity of illustration, let us consider the spectral design minimizing the maximum estimation error over frequency and time in the interval, say, $0 \leq t \leq 2\beta$. Hence, we have

$$B_o = \inf_{0 \leq t \leq 2\beta} B_o(t) = \beta/\sqrt{14} , \tag{5.4}$$

and

$$B_f = \inf_{\omega} B_f(\omega) = \alpha/\sqrt{2} , \tag{5.5}$$

corresponding to the values of $B_o(t)$ and $B_f(\omega)$ at $t = 2\beta$ and $\omega = 1$, respectively. It is clear now that the parameters α and β directly control how $f_{\alpha\beta}(t,\omega)$ varies over frequency and time. That is, as α and β become larger $f_{\alpha\beta}(t,\omega)$ varies more smoothly as a function of frequency and time, and vice versa.

Table III illustrates the approximate values of the optimal design parameters based on two different sets of constants α and β, and corresponding to the filters

$$g(u) = \begin{cases} (6\pi h_o)^{-\frac{1}{2}} \left(1 + \cos \frac{u\pi}{h_o}\right) & , |u| < h_o \\ 0 & , |u| \geq h_o , \end{cases} \tag{5.6}$$

with $C_{g1} = \pi/\sqrt{3}$, $C_{g2} = .1528$, and

$$w(u) = \begin{cases} \frac{3}{4}(T_o')^{-1} \left\{1 - (u/T_o')^2\right\} & , |u| < T_o' \\ 0 & , |u| \geq T_o' , \end{cases} \tag{5.7}$$

with $C_{w1} = \sqrt{0.2}$, and $C_{w2} = (6\pi)/5$.

TABLE III

α	β	B_o	B_f	$M_o(\%)$	h_o	T_o'	$\gamma_o = \lambda_o$
0.5	600	160	.354	7.5	13	145	0.40
0.3	360	96	.212	15.5	18	103	0.48

The estimates of $f_{\alpha\beta}(t,\omega)$ can now be evaluated in a digital manner by generating samples of X(t) from (5.1) and (5.2). The techniques for digitally simulating stationary and non-stationary processes with prescribed spectral density functions are described elsewhere (see, e.g., Shinozuka and Jan, 1971), and therefore, will not be discussed here. However, it might be useful to indicate various modifications that can be made in the digital spectral analysis of a continuous parametered process. In particular, if we have a continuous sample X(u) ($0 \leq u < T$) digitized at a periodic sampling interval of Δt sec., we can form the discrete parametered sequence $X_1, X_2, \ldots, X_N$ where $X_t = X(t\Delta t)$, and N is the largest integer smaller than $T/\Delta t$. When Δt is chosen (Nyquist interval) so that the spectral density, say, $f(u,\omega)$ of the continuous parametered process $X(u)$ satisfies, for all u, the condition

$$f(u,\omega) = 0, \quad |\omega| \geq \pi/\Delta t\,,$$

then it is convenient to regard the sequence $\{X_t\}$ as if it consisted of points at unit time intervals. This is equivalent to transforming the original frequency scale into a standardized dimensionless frequency, say ω^*, defined in $(-\pi,\pi)$, and such that $\omega^* = \omega\Delta t$. Consequently, the spectral density, say $f_d(t,\omega^*)$, of the discrete sequence $\{X_t\}$ and that of the actual process $X(t\Delta t)$ are related to one another in the form

$$f(t\Delta t,\, \omega^*/\Delta t) = \Delta t\; f_d(t,\omega^*), \quad |\omega^*| < \pi\,, \tag{5.8}$$

where

$$f_d(t,\omega^*) = \sum_{j=-T_o^*}^{T_o^*} w_j \left| U_{t-j}(\omega^*) \right|^2 . \tag{5.9}$$

and

$$U_t(\omega^*) = \sum_{j=-h_o^*}^{h_o^*} g_j\, X_{t-j}\, e^{-ij\omega^*} \tag{5.10}$$

are the discrete time analogues of (3.1) and (3.2), with $g_j = g(j\Delta t)$ and $w_j = w(j\Delta t)$ derived from the continuous versions such that

$$2\pi \sum_{j=-h_o^*}^{h_o^*} |g_j|^2 = 1, \quad \text{and} \quad \sum_{j=-T_o^*}^{T_o^*} w_j = 1\ .$$

The parameters h_o^* and T_o^* are now interpreted as the largest integers smaller than $(h_o/\Delta t)$ and $(T_o'/\Delta t)$, respectively, where h_o and T_o' are defined as before. In general then, we may say that the optimal design relations remain invariant except for a minor modification by the scaling factor Δt.

In the examples considered here, we have $\Delta t = 1$, and, therefore, the processes $\{X(t)\}$ generated from (5.1) have a dimensionless character. With $h_o^* = h_o$ and $T_o^* = T_o'$, we can now write

$$g_j = (6\pi h_o)^{-\frac{1}{2}} \left(1 + \cos \frac{j\pi}{h_o}\right), \quad (j = -h_o, \ldots, -1, 0, 1, \ldots, h_o) \;, \tag{5.11}$$

and, for $T_o' >> 1$,

$$w_j \sim \frac{3}{4} (T_o')^{-1} \left\{ 1 - (j/T_o')^2 \right\} \;, \quad (j = -T_o', \ldots, -1, 0, 1, \ldots, T_o') \;. \tag{5.12}$$

The estimated forms of $f_{\alpha\beta}(t,\omega)$ at various times in the interval $0 \leq t \leq 2\beta$ were obtained from the two simulated samples corresponding, respectively, to ($\alpha = 0.5$, $\beta = 600$) and ($\alpha = 0.3$, $\beta = 360$), using equations (5.8) through (5.12). These estimates are shown in Figs. 1 and 2 together with the corresponding theoretical forms for comparison. It is noted that the scale of ordinates for the theoretical and estimated forms at $t = 600$ in Fig. 2 is magnified by ten times to achieve a better resolution.

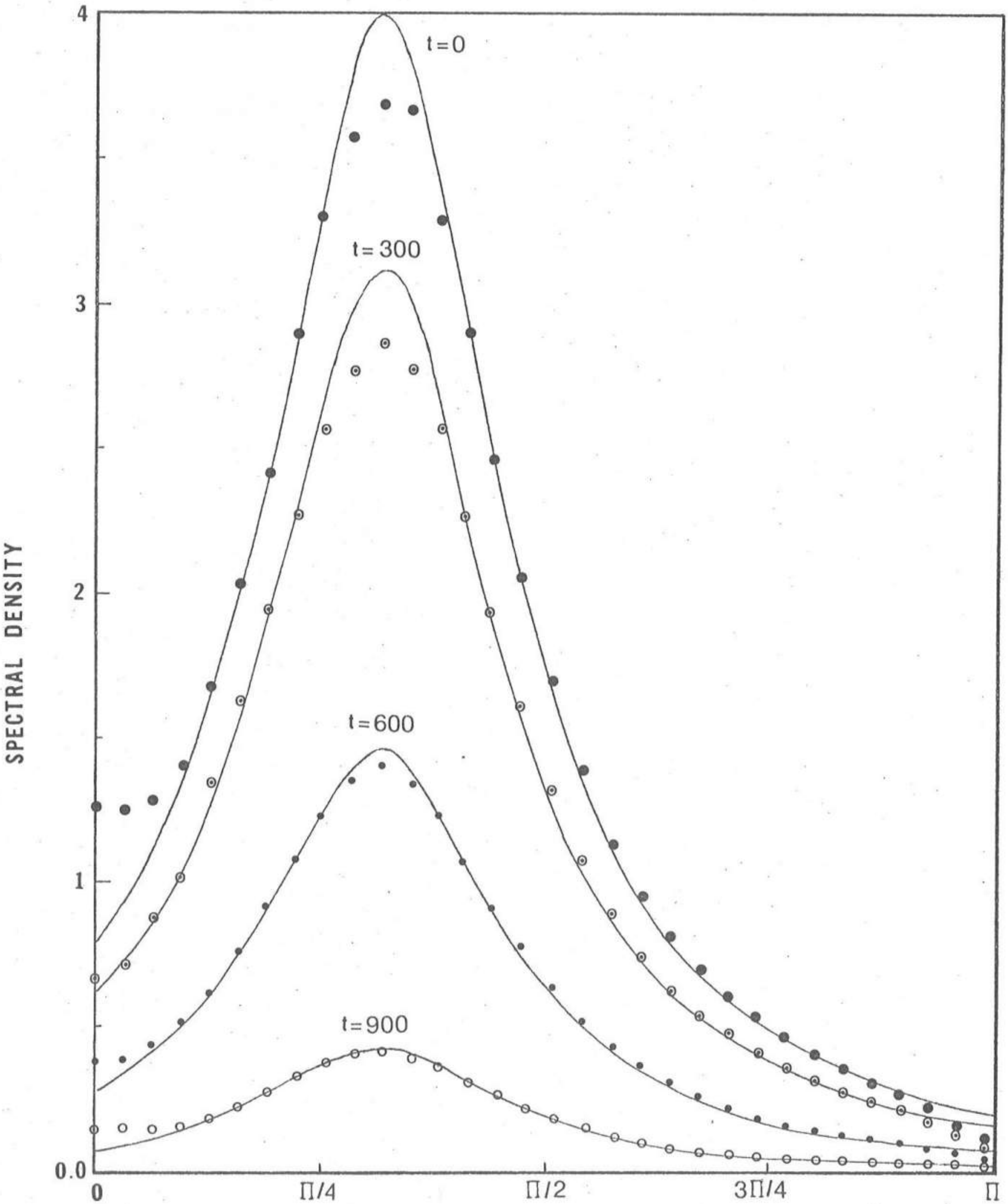

FIG. 1 Theoretical evolutionary spectra (continuous curves) and corresponding estimates at t=0, 300, 600, 900, (α = 0.5, β = 600).

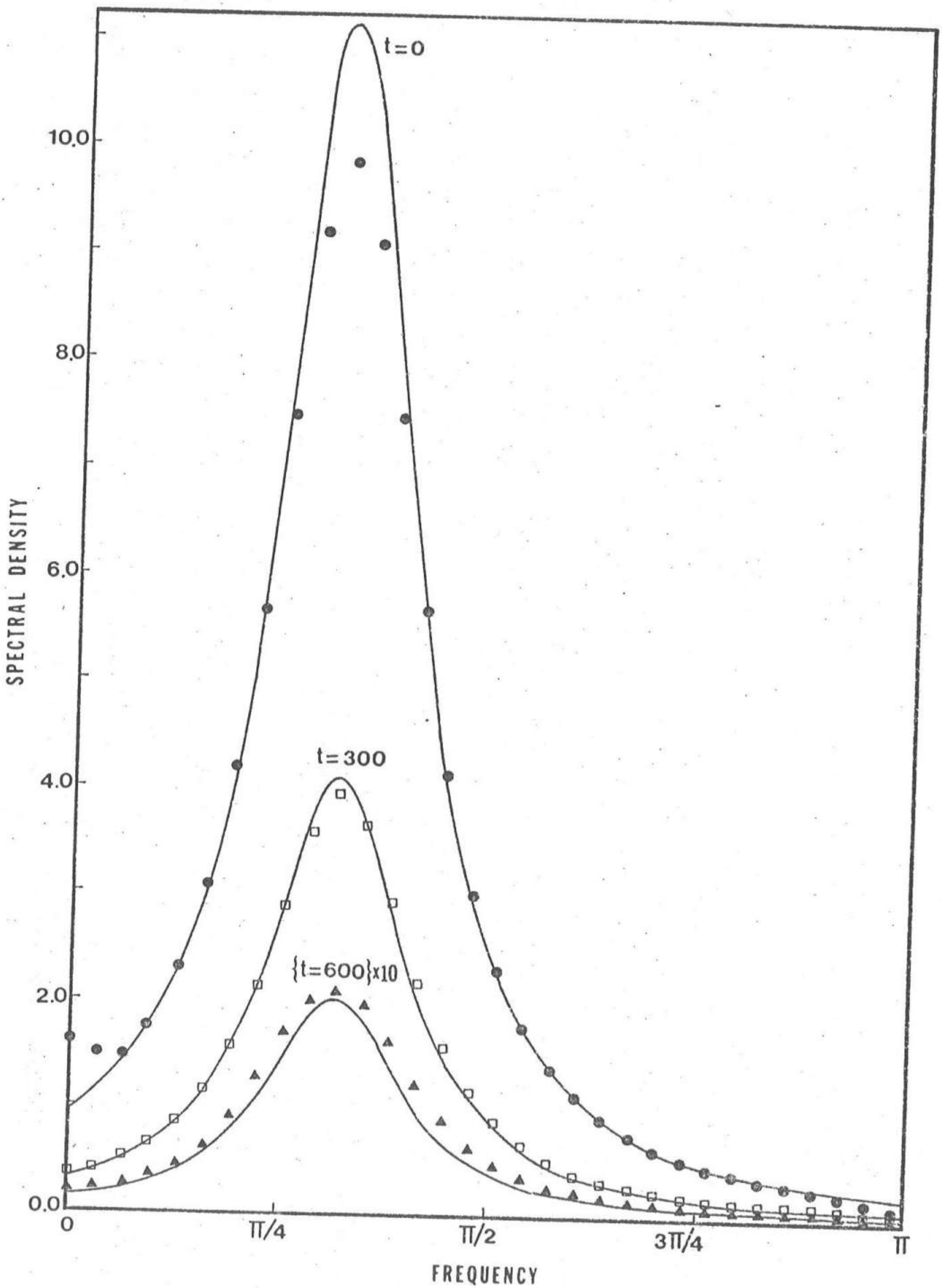

FIG. 2. Theoretical evolutionary spectra (continuous curves) and corresponding estimates at t = 0, 300, 600, ($\alpha = 0.3$, $\beta = 360$).

CONCLUSIONS

Sampling properties of the evolutionary spectral estimates were re-examined with respect to a class of filters with finite widths. The finiteness restriction on the filter widths is consistent with the practical objective to compute spectra from a limited record length, and with the inherent requirement that the filters used in estimating time-dependent spectra have a locally instantaneous character (see P-1,2,3). This approach consequently led to the construction of a set of uniquely determined design relations in terms of the filter width and shape parameters and the spectral bandwidth characteristics of a process, and provided formal criteria for choosing or constructing optimal filter shapes.

In the minimal mean-square error of spectral estimates, the importance of the product $B_o(t,\omega)\ B_f(t,\omega)$ emerges as an overall measure of the spectral characteristics of a process, and constitutes an effective criterion to assess the feasibility and accuracy of the estimation procedure in a given situation. Of the various possible interpretations of the optimal design relations, the interpretation based on a minimization of the maximum possible mean-square error over both frequency and time is the simplest and the most amenable to the practical computation of spectra. A particularly interesting point about these time- and frequency independent design relations is that, given an arbitrarily long interval, they provide the filters which require the smallest widths, and therefore, the shortest record length to construct spectral estimates which are, in sampling quality, at least equal to or better than the estimate characterized by the worst minimal error M_o.

An examination of Figs. 1 and 2 indicates that the general character of the estimates here in all cases compares very favorably with the theoretical forms (in contrast to the results obtained by Priestley, 1965). It is evident that the optimal design relations eliminate the inadequacies and subjectivity associated with the available forms of spectral design relations and illustrate very effectively the salient features and validity of the concepts involved in the evolutionary spectral theory.

REFERENCES

ABDRABBO, N. A. and PRIESTLEY, M. B. (1969). Filtering non-stationary signals. J. R. Statist. Soc. B, 31, 151-159.

BROWN, L. J. (1967). Methods for the analysis of non-stationary time series with applications to oceanography. Tech. Rep. HEL 16-3, University of California, Berkeley, California.

DUBMAN, M. R. (1965). The spectral characterization and comparison of non-stationary processes. (Unpublished Rocketdyne Research Report.)

DUFFIN, R. J., PETERSON, E. L., and ZENER, C. (1967). Geometric Programming Theory and Applications. John Wiley and Sons, Inc., New York.

FANO, R. M. (1950). Short-time autocorrelation functions and power spectra. J. Acoust. Soc. Amer., 22, 546-550.

HAMMOND, J. K. (1968). On the response of single and multidegree of freedom systems to non-stationary random excitations. J. Sound Vib. 7 (3), 393-416.

MARKS, W. D. (1970. Spectral analysis of the convolution and filtering of non-stationary stochastic processes. J. Sound Vib. 11 (19), 19-63.

PAGE, C. H. (1952). Instantaneous power spectra. J. Appl. Phys., 23, 103-106.

PARZEN, E. (1961). Mathematical considerations in the estimation of spectra. Technometrics, 3, 167-190.

PRIESTLEY, M. B. (1965). Evolutionary spectra and non-stationary processes. J. R. Statist. Soc. B, 27, 204-207.

____________ (1966). Design relations for non-stationary processes. J. R. Statist. Soc. B, 28, 228-240.

____________ (1967). Power spectral analysis of non-stationary random processes. J. Sound Vib., 6, 86-97.

____________ (1971). Some notes on the physical interpretation of spectra of non-stationary stochastic processes. J. Sound Vib., 17 (1), 51-54.

PRIESTLEY, M. B., and SUBBA RAO, T. (1969). A test for non-stationarity of time series. J. R. Statist. Soc. B, 31, 140-149.

SHINOZUKA, M. (1970). Random processes with evolutionary power. J. Engineer. Mech. ASCE, 96 (EM4), 543-545.

SHINOZUKA, M., and JAN, C.-M. (1971). Simulation of multivariate and multi-dimensional processes II. Tech. Rep. 12, Civil Engineering and Engineering Mech., Columbia University, New York.

TAYFUN, M. A., and YANG, C. Y. (1972). Discussion on an approach to time-varying spectral analysis. J. Engineer. Mech. ASCE, 98 (EM6), 1606-1608.

APPENDIX

Derivation of Optimal Design Relations

The geometric programming technique proceeds by identifying the non-negative weights μ_j defined by

$$\mu_j = \left\{ k_j \; y_j(h_o, T_o') \right\} \Big/ M_o, \; (j = 1,2,3,4), \tag{A.1}$$

where the subscript zero denotes the optima, and k_j, y_j $(j = 1,2,3,4)$ are defined in (3.12) through (3.14), with

$$M_o = M_o \; (\bar{C}, t, \omega) = \min_{h,T'} \; M(h,T'; \; \bar{C}, t, \omega). \tag{A.2}$$

The weights $\mu_j (j = 1,2,3,4)$ describe the fraction of the total minimal error M_o that should be assigned to the corresponding terms of M_o to achieve this minimum. Clearly, then $(\mu_1 + \mu_2 + \mu_3)$ and μ_4 correspond, respectively, to the contributions of the bias and variability errors to M_o. The weights satisfy the normality and orthogonality conditions defined, respectively, as

$$\sum_{j=1}^{4} \mu_j = 1, \tag{A.3}$$

and

$$\sum_{j=1}^{4} a_j \mu_j = 0 \quad , \quad \sum_{j=1}^{4} b_j \mu_j = 0, \tag{A.4}$$

where a_j and b_j $(j=1,2,3,4)$ are defined in (3.13). These conditions constitute three simultaneous equations in four unknowns $\mu_j (j=1,2,3,4)$. The fourth equation is the dual function defined as

$$d(\mu_1, \mu_2, \mu_3, \mu_4) = \prod_{j=1}^{4} (k_j / \mu_j)^{\mu_j} \tag{A.5}$$

A basic result in the geometric programming technique is that the maximum of the dual function equals the minimum of the primal function M. Hence, we proceed by solving first (A.3) and (A.4) for, say, $\mu_j (j=2,3,4)$ in terms of μ_1, and obtain

$$\begin{aligned} \mu_2 &= (1/3) - 2\mu_1, \\ \mu_3 &= \mu_1 \\ \mu_4 &= 2/3. \end{aligned} \qquad (A.6)$$

At this point we note that $\mu_1 + \mu_2 + \mu_3 = 1/3$ and $\mu_4 = 2/3$ indicate that the minimal error M_o should consist of bias and variability errors in a one to two ratio. Therefore, the optimal design rule as to how bias and variability should be proportioned has already emerged before we have evaluated either M_o or, the corresponding parameters h_o and T'_o. Using now (A.6), we may rewrite (A.5) as

$$d(\mu_1) = \left\{(k_1 k_3)/\ \mu_1^2\right\}^{\mu_1} \left\{ k_2/(\tfrac{1}{3} - 2\mu_1)\right\}^{\frac{1}{3} - 2\mu_1} \left\{ k_4/(2/3) \right\}^{2/3}.$$

Maximizing the dual function $d(\mu_1)$, subject to the condition $0 \leq \mu_1 \leq (1/6)$ imposed by the non-negativity requirement on μ_j's and results in (A.6), yields the optimal value

$$\mu_1 = \left[3\left\{2 + \frac{k_2}{\sqrt{(k_1 k_3)}}\right\} \right]^{-1} = 1/12,$$

by noting that $k_2/\sqrt{(k_1 k_3)} = 2$. Therefore,

$$M_o = d(1/12) = 3(k_1 k_2^2 k_3 k_4^8/4)^{1/12},$$

or, on account of the definitions (3.14),

$$M_o = 3 \left\{ \frac{(1 + \delta_{o,\omega}) \; C_{w1} \; C_{w2} \; C_{g1} \; C_{g2}}{2 \; B_o(t,\omega) \; B_f(t,\omega)} \right\}^{2/3}.$$

We may now, using (A.1), determine the optimal parameters, h_o and T'_o, as

$$h_o = (3/M_o)^{\frac{1}{4}} \left\{ C_{g1}/B_f(t,\omega) \right\},$$

and

$$T'_o = (M_o/3)^{\frac{1}{4}} \left\{ B_o(t,\omega)/C_{w1} \right\},$$

so that

$$(h_o/T'_o) = \left\{ \frac{C_{w1}^2 \; C_{g1}^2}{B_o^2(t,\omega) \; B_f^2(t,\omega) \; C_{w2} \; C_{g2}} \right\}^{1/3}.$$

The resolution of spectral estimates $\hat{f}(t,\omega)$ in frequency- and time-domains corresponding to the optima is given by

$$\gamma_o = \lambda_o = (M_o/3)^{\frac{1}{4}} .$$

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