IMPROVING THE RATE OF CONVERGENCE OF 'HIGH ORDER FINITE ELEMENTS' ON POLYHEDRA I: A PRIORI ESTIMATES

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Abstract. Let $\mathcal{T}_k$ be a sequence of triangulations of a polyhedron $\Omega \subset \mathbb{R}^n$ and let S_k be the associated finite element space of continuous, piecewise polynomials of degree m. Let $u_k \in S_k$ be the finite element approximation of the solution u of a second order, strongly elliptic system $Pu = f$ with zero Dirichlet boundary conditions. We show that a weak approximation property of the sequence S_k ensures optimal rates of convergence for the sequence u_k. The method relies on certain a priori estimates in weighted Sobolev spaces for the system $Pu = 0$ that we establish. The weight is the distance to the set of singular boundary points. We obtain similar results for the Poisson problem with mixed Dirichlet–Neumann boundary conditions on a polygon.

Contents

Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider the boundary value problem

$$-\Delta u = f, \quad u|_{\partial\Omega} = g, \tag{1}$$

where Δ is the Laplace operator. Let $S_k \subset H^1_0(\Omega)$ be a sequence of finite dimensional spaces such that $\cup S_k$ is dense in $H^1_0(\Omega)$. Denote by N_k the dimension of S_k and let $u_k \in S_k$ be the Finite Element (*i.e.,* Galerkin) approximation of the solution u of Equation (1) with $g = 0$ and $f \in H^{m-1}(\Omega)$. See [6, 8] for the basics on the Finite Element Method.

Following the standard terminology, we shall say that *the sequence S_k achieves optimal rates of convergence for the sequence u_k* if there exists a constant $C > 0$,

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independent of k and f, such that

$$\|u - u_k\|_{H^1(\Omega)} \leq C N_k^{-m/n} \|f\|_{H^{m-1}(\Omega)}. \tag{2}$$

If $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain and $m = 1$, then it is well known [8] that we can chose $\mathcal{T}_k$ to be a quasi-uniform sequence of triangulations. On the other hand, if Ω is not convex, an important result of Wahlbin [39] (see also [40]) states that a quasi-uniform sequence of triangulations will *not* lead to optimal rates of convergence for the sequence u_k. Nevertheless, if Ω is a polygonal domain in the plane, it was shown by Babuška in the ground breaking paper [5] that there exist sequences S_k that will achieve optimal rates of convergence. See also Raugel [36] for another proof of this important result, a proof that is closer to our methods. It is the purpose of this paper to initiate a study of the existence of a sequence S_k with these properties when Ω is a polyhedral domain in three dimensions using the method of [12], where the case of a polygon was considered. Our method is also similar to the method of [28], where similar, but different, weighted Sobolev spaces regularity results were used.

Let us recall the definition of the *weighted Sobolev spaces*

$$\mathcal{K}_a^\mu(\Omega) = \{u \in L^2_{\text{loc}}(\Omega),\ \vartheta^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \text{for all } |\alpha| \leq \mu\}, \tag{3}$$

where $\mu \in \mathbb{Z}_+ = \{0, 1, \ldots, \}$ and $\vartheta(x)$ is the distance from a point $x \in \Omega$ to the set $\partial_{\text{sing}}\Omega \subset \partial\Omega$ of non-smooth boundary points of Ω. We endow this space with the induced Hilbert space norm. In this paper we establish the a priori estimate

$$\|u\|_{\mathcal{K}^{m+1}_{1+\epsilon}(\Omega)} \leq C\|f\|_{\mathcal{K}^{m-1}_{-1+\epsilon}(\Omega)}, \quad \text{if } |\epsilon| \text{ is small}, \tag{4}$$

which, as we shall see in the next section, is crucial for our approach to obtaining optimal rates of convergence for the Finite Element Method on polyhedral domains in three dimensions.

To explain our approach, let us first recall that, for a smooth and bounded domain Ω, the choice of a quasi-uniform sequence of triangulations will achieve optimal rates of convergence due to the following basic well posedness result [18, 37].

Theorem 0.1. *If Ω is smooth and bounded, the boundary value problem* (1) *has a unique solution $u \in H^{s+2}(\Omega)$, which depends continuously on $f \in H^s(\Omega)$ and $g \in H^{s+3/2}(\partial\Omega)$, $s \geq -1$.*

It is well known that Theorem 0.1 is not valid for non-smooth domains. An analysis of the difficulties that arise for general Lipschitz is contained in the papers of Jerison and Kenig [21], Mitrea and Taylor [34], for instance. The paper [33] deals with mixed boundary value problems on Lipschitz domains.

In this paper, we consider the boundary value problem (1) for Ω a *bounded polyhedral domain* in $\mathbb{R}^n$, $n = 2, 3$. Also, we replace Poisson's equation $\Delta u = f$ with a strongly elliptic system. Let us denote by $\partial_{\text{sing}}\Omega$ the set of singular points of the boundary of Ω (that is, the set of points $p \in \partial\Omega$ such $\partial\Omega$ is not smooth in a neighborhood of p). More precisely, when $n = 2$, $\partial_{\text{sing}}\Omega$ is the set of vertices of Ω, and, when $n = 3$, $\partial_{\text{sing}}\Omega$ is the union of all edges of Ω. Results specific to polyhedral domains are contained in the papers of Babuška and Guo [20], Bacuta, Bramble, and Xu [9], Costabel [13], Dauge [14, 15], Elschner [16], Kondratiev [23], Lubuma and Nicaise [28, 29], Mazya and Rossman [30], van Petersdorff and Stefan [38], and others. See also [8], Section 5.5, for a leisurely presentation of the regularity issue that is more in the spirit of our paper.

One of our main results, Theorem 4.2, is a well posedness for a generalization of the boundary value (1) to a system in $n = 2$ or $n = 3$ dimensions that satisfies the *strong Legendre conditions*. In two dimensions, we also consider mixed Dirichlet–Neumann boundary conditions for the Laplace operator, as long as on each edge we have a definite type of condition (Dirichlet or Neumann), and no two adjacent edges have Neumann boundary conditions. For simplicity, we now formulate one of our main results, Theorem 4.2, for the Laplace equation on a polyhedral domain and Dirichlet boundary conditions. (The case $n = 2$ of the next theorem can be found in [25], Theorem 6.6.1, except maybe for the determination of the Sobolev spaces on the boundary.)

Theorem 0.2. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded, polyhedral domain and $\mu \in \mathbb{Z}_+$. Then there exists $\eta > 0$ such that the boundary value problem (1) has a unique solution $u \in \mathcal{K}^{\mu+1}_{a+1}(\Omega)$ for any $f \in \mathcal{K}^{\mu-1}_{a-1}(\Omega)$, any $g \in \mathcal{K}^{\mu+1/2}_{a+1/2}(\partial\Omega)$, and any $|a| < \eta$. This solution depends continuously on f and g. If $\mu = a = 0$ and $g = 0$, this solutions is the solution of the associated variational problem.*

The continuity part of this theorem gives right away the a priori estimate (4) (see also Remark 4.3). Our proof can be extended to the higher dimensions. However, in higher dimensions, one has to deal with tremendous topological complications, see [10] for instance.

We now describe the contents of the sections of the paper in more detail. In the first section, we explain our approach to constructing sequences of triangulations $\mathcal{T}_k$ achieving optimal rates of convergence for the sequence u_k of finite element approximations of the Poisson problem with zero Dirichlet boundary conditions using weighted Sobolev spaces. Let us stress that the result is in terms of the usual Sobolev spaces, and the weighted Sobolev spaces appear only as an intermediate step. This leaves room for improving our results, for instance by considering data f that belongs to a space large than $H^{m-1}(\Omega)$. In the second section, we introduce the weighted Sobolev spaces that are used in the a priori estimate (4). We also review in this section some of the properties of these function spaces, including a regularity theorem and a trace theorem. In the third section, we prove a Hardy–Poincaré type inequality. In section 4 we recall the strong Legendre condition and state our two well-posedness results. The last section, Section 5 contains the proofs of these two well-posedness results.

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1. Optimal rates of convergence

Let $\Omega \subset \mathbb{R}^n$ be an open polyhedral domain. Thus Ω is also bounded and connected. Let u be the solution of the Poisson problem with zero Dirichlet boundary condtitions, that is, of the boundary value problem

$$-\Delta u = f, \quad u|_{\partial\Omega} = 0. \tag{5}$$

In this section, we present our approach to constructing sequences of triangulations $\mathcal{T}_k$ of Ω achieving optimal rates of convergence for the sequence u_k of Finite Element approximations of u. Our results apply also to other types of operators and to other boundary conditions, but for simplicity we formulate them only for Equation (5). Our method is a generalization of the method presented in [12], where we dealt with the case of a polygon in the plane.

Let $H^m(\Omega) = \{u, \partial^\alpha u \in L^2(\Omega), |\alpha| \leq m\}$, as usual. We shall say that the boundary value problem (5) on (the bounded polyhedral domain) $\Omega \subset \mathbb{R}^n$ has *the shift property in weighted Sobolev spaces* if it satisfies the a priori estimate estimate

$$\|u\|_{\mathcal{K}^{m+1}_{1+\epsilon}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)}. \tag{6}$$

for $\epsilon > 0$ small and a constant $C > 0$ independent of f. Note that the continuous inclusions

$$H^{m-1}(\Omega) \subset \mathcal{K}^{m-1}_0(\Omega) \subset \mathcal{K}^{m-1}_{-1+\epsilon}(\Omega), \quad 0 \leq \epsilon \leq 1, \tag{7}$$

show that the estimate (4) implies the estimate (6). If $\mathcal{O} \subset \mathbb{R}^n$ is a smooth bounded domain, then we can take $\vartheta = 1$, and hence $\|u\|_{\mathcal{K}^{m+1}_{1+\epsilon}(\mathcal{O})} = \|u\|_{H^{m+1}(\mathcal{O})} \leq C\|f\|_{H^{m-1}(\mathcal{O})}$. Therefore $\mathcal{O}$ has the shift property in weighted Sobolev spaces. If Ω is not smooth, we will not have $u \in H^{m+1}(\Omega)$ in general, so the above argument does not apply. Nevertheless, in the following sections, we shall show that the estimate (4) is satisfied by polyhedral domains $\Omega \subset \mathbb{R}^3$. We stress that it is crucial to obtain $\epsilon > 0$.

Let $\mathcal{T}_k$ be a sequence of triangulations of Ω and let S_k be the associated finite element space of continuous, piecewise polynomials of degree m. If u is a smooth function on $\overline{\Omega}$, we shall denote by $u_I \in S_k$ the interpolant of u associated to the family of equidistributed nodes (we are considering only Lagrange finite element spaces).

Let $B(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$, as usual, so that $B(u,v) = (f,v)$ for all $v \in H^1_0(\Omega)$, (where $H^1_0(\Omega)$ is the closure of $\mathcal{C}^\infty_{\rm c}(\Omega)$ in $H^1(\Omega)$, as customary). Let $u_k \in S_k$ be the Finite Element approximation of the solution u of Equation (5), that is, u_k is the unique element of S_k such that $B(u_k, v) = (f,v)$ for all $v \in S_k$.

Let N_k be the dimension of the space S_k and $|u|^2_{H^1} = B(u,u)$. We shall say that S_k satisfies *the weak degree m-approximation property in weighted Sobolev spaces* if there exists $\epsilon > 0$ and $C > 0$ independent of $u \in \mathcal{K}^{m+1}_{1+\epsilon}(\Omega)$ such that

$$|u - u_I|_{H^1} \leq C N_k^{-m/n} \|u\|_{\mathcal{K}^{m+1}_{1+\epsilon}(\Omega)}. \tag{8}$$

The factor $N_k^{-m/n}$ behaves asymptotically like h^m if $\mathcal{T}_k$ is a sequence of quasi-uniform meshes of (typical) size $h = h(k) \approx N_k^{-1/n}$. In our case, however, $\mathcal{T}_k$ cannot be a quasi-uniform sequence of triangulations. Moreover, no estimate of the form (8) is possible for $\epsilon \leq 0$, so it is crucial to assume $\epsilon > 0$.

Recall that we have denoted by S_k the Finite Element space of continuous, piecewise polynomials of degree m associated to the triangulation $\mathcal{T}_k$ of Ω. Our method for obtaining a sequence $\mathcal{T}_k$ of triangulations such that the sequence S_k achieves optimal rates of convergence is based on the following result.

Theorem 1.1. *Let $m \geq 1$ and $\epsilon > 0$ be fixed. Assume that the sequence of triangulations $\mathcal{T}_k$ of the polyhedral domain Ω satisfies the weak degree m-approximation property in weighted Sobolev spaces and that the Poisson equation with zero Dirichlet boundary conditions has the shift property in weighted Sobolev spaces. Then* *there exists a constant $C > 0$, independent of k and f, such that*

$$\|u - u_k\|_{H^1(\Omega)} \leq C N_k^{-m/n} \|f\|_{H^{m-1}(\Omega)}.$$

That is, the sequence S_k achieves optimal rates of convergence for the sequence u_k.

Proof. Combining Equations (6) with (8), we obtain

$$|u - u_I|_{H^1} \leq C N_k^{-m/n} \|f\|_{H^{m-1}(\Omega)}. \tag{9}$$

Cea's lemma, $|u - u_k|_{H^1} \leq |u - u_I|_{H^1}$, and the equivalence of the $\|\cdot\|_{H^1}$ and $|\cdot|_{H^1}$ norms on $H^1_0(\Omega)$ complete the result. □

The rest of this paper is devoted to establishing the estimate (4) on polyhedral domains in $\mathbb{R}^3$ (see Theorem 4.2 and the Remark 4.3 following it). Our results are slightly more general, in view of further applications. See [30, 35] for results on polyhedral domains in $\mathbb{R}^3$. The equations and the weighted Sobolev spaces considered in those references are more general than the ones considered in our paper. However, those references only establish the Fredholm condition on bounded domains, whereas for numerical methods we need full solvability (and, in fact, even well posedness). Even more general weighted Sobolev spaces in arbitrary dimensions are considered in [10].

In [11], the second paper of this series, we shall establish the weak approximation property, Equation (8). More precisely, we shall establish for Ω a suitable domain in three dimension that we have

$$\|u - u_I\|_{\mathcal{K}^1_1(\Omega)} \leq C N_k^{-m/n} \|u\|_{\mathcal{K}^{m+1}_{1+\epsilon}(\Omega)}, \tag{10}$$

for a suitable sequence of triangulations $\mathcal{T}_k$ that depends on m and ϵ. Since the norms $\|\cdot\|_{\mathcal{K}^1_1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $H^1_0(\Omega)$ (this follows from the results of this paper), Equation (10) implies Equation (8). We again stress that there is no hope of obtaining estimates of the form (10) for $\epsilon \leq 0$, so it is crucial to be able to assume $\epsilon > 0$.

2. Sobolev spaces

We recall now the weighted Sobolev spaces $\mathcal{K}^\mu_a(\Omega)$, $\mathcal{K}^s_a(\partial\Omega)$, $\mu \in \mathbb{Z}$ and $a, s \in \mathbb{R}$, for the cases when Ω is a *curvilinear polygonal domain* in two dimensions or a *straight polyhedral domain* in three dimensions. (The definitions of curvilinear polygonal domains in two dimensions and of straight polyhedra in three dimensions are recalled below.) We also review the properties of these Sobolev spaces. This section is based on [1], owing also to [2, 32]. The weighted Sobolev consider here were considered before by many researchers, most notably Kondratiev in [23], but see also the paper by Babuška and Guo [20].

Throughout this paper Ω will be an open set satisfying $\partial\Omega = \partial\overline{\Omega}$. The conditions $\partial\Omega = \partial\overline{\Omega}$ is designed to rule out the case when Ω is on both sides of parts of its boundary. The case when Ω is on both sides of the boundary can also be treated by our method, but that would lead to some complications that we prefer to avoid.

2.1. Definition of weighted Sobolev spaces.

Let f be a continuous function on Ω, $f > 0$ on the interior of Ω. We define the *μth Sobolev space with weight f* (and index a) by

$$\mathcal{K}^\mu_{a,f}(\Omega) = \{u : \Omega \to \mathbb{C},\ f^{|\alpha| - a} \partial^\alpha u \in L^2(\Omega),\ \text{for all } |\alpha| \leq \mu\}, \quad \mu \in \mathbb{Z}_+, \tag{11}$$

($\mathbb{Z}_+ = \{0, 1, \dots\} \subset \mathbb{Z}$). The norm on $\mathcal{K}^\mu_{a,f}(\Omega)$ is

$$\|u\|^2_{\mathcal{K}^\mu_{a,f}(\Omega)} := \sum_{|\alpha|\leq\mu} \|f^{|\alpha|-a}\partial^\alpha u\|^2_{L^2(\Omega)}.$$

Thus, $\mathcal{K}^0_{0,f}(\Omega) = L^2(\Omega)$. We could restrict in some problems to real valued functions. The theories for real valued and complex valued functions are the same.

Similarly, we let

$$(12)\quad \mathcal{K}^\mu_{a,f}(\partial\Omega) = \{u : \partial\Omega \to \mathbb{C},\ f^{|\alpha|-a}\partial^\alpha u \in L^2(\partial\Omega),\ \text{for all } |\alpha| \leq \mu\}, \quad \mu \in \mathbb{Z}_+.$$

In the above definition, only derivatives *tangent* to the faces are to be considered.

Definition 2.1. For $\mu \in \mathbb{Z}_+$, we let $\mathcal{K}^\mu_a(\Omega) = \mathcal{K}^\mu_{a,\vartheta}(\Omega)$ and $\mathcal{K}^\mu_a(\partial\Omega) = \mathcal{K}^\mu_{a,\vartheta}(\partial\Omega)$, where $\vartheta(x)$ is the distance from $x \in \Omega$ to $\partial_{\text{sing}}\Omega$, the set of singular points of the boundary of Ω.

The spaces $\mathcal{K}^\mu_a(\Omega)$ introduced in the above definition are the same as the spaces considered in Equation (3) of the Introduction. These weighted Sobolev spaces are closely related to weighted Sobolev spaces on non-compact manifolds considered, for instance, in [17, 19, 26]. We now introduce the negative order Sobolev spaces with weight by duality with pivot $\mathcal{K}^0_0(\Omega)$, as usual. Let $\mathcal{C}^\infty_{\text{c}}(\Omega_1)$ denote the space of smooth functions with compact support in the open set Ω_1.

Definition 2.2. Let $\mu \in \mathbb{N} = \mathbb{Z}_+ \smallsetminus \{0\}$. We define $\overset{\circ}{\mathcal{K}}{}^\mu_a(\Omega)$ to be the closure of $\mathcal{C}^\infty_{\text{c}}(\Omega)$ in $\mathcal{K}^\mu_a(\Omega)$. The space $\mathcal{K}^{-\mu}_{-a}(\Omega)$ is defined to be the dual of $\overset{\circ}{\mathcal{K}}{}^\mu_a(\Omega)$.

The space $\overset{\circ}{\mathcal{K}}{}^\mu_a(\Omega)$ can be defined also as the space of functions whose normal derivative of order at most $\mu - 1$ vanish at the boundary, as in the classical case [1], but we shall not need this, except in the case $\mu = 1$. There is therefore a close parallel between the definitions of our weighted Sobolev spaces and the usual Sobolev spaces on a smooth bounded domain in $\mathbb{R}^n$. We shall not introduce the spaces $\mathcal{K}^s_{-a}(\Omega)$, $s \in \mathbb{R} \smallsetminus \mathbb{Z}$, in order to avoid some subtleties in the case $s \in 1/2 + \mathbb{Z}$, see [27].

These definitions will be made completely explicit below in two and three dimensions. We need however to introduce first curvilinear polygonal domains in two dimensions and straight polyhedra in three dimensions. These definitions are certainly not new; they just formalize familiar concepts.

2.2. **Two dimensions.** We now define curvilinear polygonal domains in two dimensions and make more explicit our definitions in this case. We shall denote by B^n the open unit ball in $\mathbb{R}^n$.

Definition 2.3. An open, connected subset $\Omega \subset M$ of a two dimensional manifold M will be called a *curvilinear polygonal domain* if $\overline{\Omega}$ is compact and, for every point $p \in \partial\Omega$, there exists a diffeomorphism $\phi_p : V_p \to B^2$, $\phi_p(p) = 0$, defined on a neighborhood $V_p \subset M$, such that

$$(13)\qquad \phi_j(V_p \cap \Omega) = \{(r\cos\theta, r\sin\theta),\ 0 < r < 1,\ 0 < \theta < \alpha_p\}, \quad \alpha_p \in (0, 2\pi).$$

For each curvilinear polygonal domain Ω, we fix a subset $\{A_1, A_2, \dots, A_k\} \subset \partial\Omega$ containing all points $p \in \partial\Omega$ for which $\alpha_p \neq \pi$. This set will be called the set of *vertices* of Ω. In particular, $\partial\Omega \smallsetminus \{A_1, A_2, \dots, A_k\}$ will be the union of finitely many smooth curves, called the *open edges* of Ω. Note that we can always enlarge the set of vertices.

For Ω a curvilinear polygonal domain in the plane, $\vartheta(x)$ is the distance from x to the vertices of Ω and the resulting spaces $\mathcal{K}_a^\mu(\Omega) = \mathcal{K}_{a,\vartheta}^\mu(\Omega)$ are the spaces introduced by Kondratiev [23]. Thus, if we enlarge the set of vertices, we obtain a different weighted Sobolev space. This is justified, for example, if the boundary conditions change. Then it is convenient to consider as vertices the points where the boundary conditions change.

An explicit description of the Sobolev spaces on the boundary is obtained as follows. Let Ω be a curvilinear polygonal domain in the plane and let ∂_t be the derivative with respect to the arc length. We define the weighted Sobolev spaces on the boundary by

$$\mathcal{K}_a^\mu(\partial\Omega) = \{u : \partial\Omega \to \mathbb{C},\ \vartheta^{k-a}\partial_t^k u \in L^2(\partial\Omega),\ \text{for all } k \leq \mu\}\,, \quad \mu \in \mathbb{Z}_+. \tag{14}$$

Then $\mathcal{K}_{-a}^{-\mu}(\partial\Omega)$ is defined to be the dual of $\mathcal{K}_a^\mu(\partial\Omega)$, for $\mu \in \mathbb{Z}_+$ and $a \in \mathbb{R}$. The spaces $\mathcal{K}_a^s(\partial\Omega)$, $s \in \mathbb{R}$, are defined by interpolation. Unlike the case of Ω, there is no subtlety in the definition of the spaces $\mathcal{K}_{-a}^{-\mu}(\partial\Omega)$, $\mu \notin \mathbb{Z}_+$.

2.3. **Three dimensions.** We now turn to three dimensions. By an *affine space* we shall denote the translation of a subspace of a vector space V. A *straight polygon* is a curvilinear polygonal domain $\mathbb{P} \subset V$ of a two dimensional affine space such that all the edges of $\mathbb{P}$ are straight (*i.e.,* on a straight line).

Definition 2.4. A *straight polyhedron* $\Omega \subset \mathbb{R}^3$ is a bounded, connected open set such that there exist finitely many disjoint straight polygons $D_j \subset \partial\Omega = \partial\overline{\Omega}$ such that $\partial\Omega = \cup\overline{D_j}$ and each edge of any of the polygons D_j belongs to exactly two of the polygons D_l. The *edges* of Ω are the edges of the faces D_j.

The weighted Sobolev spaces on the boundary in three dimensions are introduced, more or less, as in the two dimensions. We let Ω be a straight polyhedron in space, and take $\vartheta(x)$ to be the distance from x to the edges of Ω. Let $\mu \in \mathbb{Z}_+$ and $D_j \subset \partial\Omega$ be as in Definition 2.4, then we define

$$\mathcal{K}_a^\mu(\partial\Omega) = \{u : \partial\Omega \to \mathbb{C},\ \vartheta^{|\alpha|-a}\partial^\alpha u \in L^2(D_j),\ \text{for all } j \text{ and } |\alpha| \leq \mu\}. \tag{15}$$

The derivatives $\partial^\alpha u$ on the face D_j are computed using any orthogonal coordinate system on (the affine plane containing) D_j. The spaces $\mathcal{K}_a^s(\partial\Omega)$, $-s \in \mathbb{Z}_+$ are defined by duality and then, for $s \in \mathbb{R} \smallsetminus \mathbb{Z}$, by interpolation, as in the two dimensional case.

As we have already mentioned above, the condition $\partial\Omega = \partial\overline{\Omega}$ is designed to rule out the case when Ω lies on *both* sides of its boundary. To deal with this case, as well as with more general domains, we need the concept of a "curvilinear polyhedral domain," which, together with the higher dimensional cases, is discussed in [10]. The definition of the weighted Sobolev spaces on the boundary becomes then significantly more complicated.

2.4. **Properties of weighted Sobolev spaces.** *From now on and throughout the paper, Ω will be either a curvilinear polygonal domain in the plane or a straight polyhedron in space.*

The following result on weighted Sobolev spaces generalizes the well known results on Sobolev spaces on domains with smooth boundary. The proof in [1] is to reduce to the case of a half-space using a suitable partition of unity. Recall that $\partial_{\text{sing}}\Omega$ denotes the set of singular points of the boundary of Ω (*i.e.,* the set of vertices, if $n = 2$, or the set of edges, if $n = 3$).

Theorem 2.5 (**Existence of traces).** *The restriction map*

$$\mathcal{C}_c^\infty(\overline{\Omega} \smallsetminus \partial_{sing}\Omega) \ni u \to u|_{\partial\Omega} \in \mathcal{C}_c^\infty(\partial\Omega \smallsetminus \partial_{sing}\Omega)$$

extends to a continuous, surjective map $\mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_{a-1/2}^{\mu-1/2}(\partial\Omega)$, $\mu \geq 1$. *Moreover,* $\mathcal{C}_c^\infty(\Omega)$ *is dense in the kernel of this map if* $\mu = 1$.

This theorem leads to the following equivalent definition of the space $\mathcal{K}_{-a}^{-\mu}(\Omega)$, $-\mu \in \mathbb{N}$. First let, for any $u \in \mathcal{C}^\infty(\Omega)$,

$$\|u\|_{\mathcal{K}_{-a}^{-\mu}(\Omega)} = \sup \frac{|(u, v)|}{\|v\|_{\mathcal{K}_a^\mu(\Omega)}}, \quad 0 \neq v \in \mathcal{C}_\text{c}^\infty(\Omega), \tag{16}$$

where $(\cdot,\cdot)$ denotes the standard L^2 inner product. Then we define $\mathcal{K}_{-a}^{-\mu}(\Omega)$ to be the completion of the space of smooth functions u on Ω that are such that $\|u\|_{\mathcal{K}_{-a}^{-\mu}(\Omega)} < \infty$.

The following proposition can be proved in small dimensions directly using spherical or polar coordinates. A more general result is proved in [1].

Proposition 2.6. *Let Ω be a curvilinear polygonal domain in the plane or a straight polyhedron. Let P be a differential operator of order m on Ω with smooth coefficients. Then P maps $\mathcal{K}_a^\mu(\Omega)$ to $\mathcal{K}_{a-m}^{\mu-m}(\Omega)$ continuously, for any $\mu \in \mathbb{Z}$.*

We also have the following regularity result. Let $P = (P_{kl})$, $k, l = 1, \ldots, q$, be a matrix of second order differential operators with smooth coefficients,

$$P_{kl}u = -\sum_{ij=1}^{n} a_{kl}^{ij} \partial_i \partial_j u + \ldots, \tag{17}$$

where the dots stand for lower order differentials and n is the dimension of the space, thus $n = 2$ or $n = 3$ in our case. Then $Pu := \big(\sum P_{kl}u_l\big)$ defines a continuous map $\mathcal{K}_{a+1}^{\mu+1}(\Omega)^q \to \mathcal{K}_{a-1}^{\mu-1}(\Omega)^q$. We shall denote the norm on $\mathcal{K}_a^\mu(\Omega)^q$ by $\|\cdot\|_{\mathcal{K}_a^\mu(\Omega)}$, without explicitly mentioning q in the notation.

As in [27], we shall say that P is *strongly elliptic on* Ω if there exists $\theta > 0$ such that, for any $\eta = (\eta_1, \ldots, \eta_q) \in \mathbb{C}^q$, any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, and any $x \in \Omega$, we have

$$\sum_{i,j,k,l} \text{Re}\,\big(a_{kl}^{ij}(x)\eta_k\overline{\eta}_l\big)\xi_i\xi_j \geq \theta|\xi|^2|\eta|^2, \tag{18}$$

with $|\cdot|$ denoting the norm on the corresponding finite dimensional space. The Laplacian Δ is an example of a strongly elliptic operator (with $q = 1$). The following elliptic regularity theorem for strongly elliptic matrices (or systems) is proved as its analogue for differential operators (*i.e.,* $q = 1$) in [1].

Theorem 2.7 (**Elliptic regularity).** *Let $P = (P_{kl})$ be a strongly elliptic matrix of second order differential operators with smooth coefficients on Ω. Assume that $Pu \in \mathcal{K}_{a-1}^{\mu-1}(\Omega)$ and $u|_{\partial\Omega} \in \mathcal{K}_{a+1/2}^{\mu+1/2}(\partial\Omega)$, $\mu \in \mathbb{Z}_+$, $a \in \mathbb{R}$, for some $u \in \mathcal{K}_{a+1}^0(\Omega)$. Then $u \in \mathcal{K}_{a+1}^{\mu+1}(\Omega)$ and*

$$\|u\|_{\mathcal{K}_{a+1}^{\mu+1}(\Omega)} \leq C\Big(\|Pu\|_{\mathcal{K}_{a-1}^{\mu-1}(\Omega)} + \|u\|_{\mathcal{K}_{a+1}^0(\Omega)} + \|u|_{\partial\Omega}\|_{\mathcal{K}_{a+1/2}^{\mu+1/2}(\partial\Omega)}\Big). \tag{19}$$

Note that the inclusion $\mathcal{K}^{\mu+1}_{a+1}(\Omega) \to \mathcal{K}^{0}_{a+1}(\Omega)$ is *not* compact, so the above regularity result *does not* imply that P is a Fredholm operator, regardless of the boundary conditions. (The property that P is a Fredholm operator is called in some papers *normal solvability* of the corresponding boundary value problem.)

2.5. **A "smoothing" of ϑ.** The function ϑ is continuous, but is not smooth at $\partial_{\rm sing}\Omega$. It is also not smooth at other points inside the domain Ω. Inside Ω the singularities are not essential, and below we shall explain how we can construct an explicit continuous function r_Ω on $\overline{\Omega}$ such that r_Ω is equivalent to ϑ (*i.e.,* r_Ω/ϑ is bounded from above and bounded away from 0) and is smooth on Ω. In particular, the condition that r_Ω and ϑ are equivalent shows that $r_\Omega > 0$ on $\overline{\Omega} \smallsetminus \partial_{\rm sing}\Omega$, and, most importantly,

$$\mathcal{K}^{\mu}_{a}(\Omega) := \mathcal{K}^{\mu}_{a,\vartheta}(\Omega) = \mathcal{K}^{\mu}_{a,r_\Omega}(\Omega).$$

That is, we could use either of the functions ϑ or r_Ω to define the weighted Sobolev spaces $\mathcal{K}^{\mu}_{a}(\Omega)$. The same remark applies for the Sobolev spaces on the boundary.

We also obtain that

$$r_\Omega X_1 \ldots r_\Omega X_k r_\Omega \in L^\infty(\Omega), \tag{20}$$

for any smooth vector fields (or derivatives) $X_1, \ldots, X_k$, which implies that

$$r_\Omega^t \mathcal{K}^{\mu}_{a}(\Omega) = \mathcal{K}^{\mu}_{a+t}(\Omega). \tag{21}$$

(Above, by $r_\Omega X f$ we have denoted the product of r_Ω and the function $X(f)$ obtained by applying X, viewed as a derivative, to the function f.) Neither of the Equations (20) and (21) remains true if we replace r_Ω with ϑ, which justifies once more our interest in constructing a function r_Ω with these properties.

A construction of the function r_Ω with these properties is as follows. Let g_0 be the Euclidean metric in $\mathbb{R}^n$, $n = 2, 3$. Let ρ_0 be the distance to the vertices of $\partial\Omega$. Thus $\rho_0 = \vartheta$ if $n = 2$. Since ρ_0 is not a smooth function in general, we introduce a continuous function $\tilde\rho_0 : \overline{\Omega} \to [0, \infty)$ that is smooth except at the vertices, coincides with ρ_0 close to the vertices, and satisfies $\rho_0/2 \le \tilde\rho_0 \le \rho_0$ on Ω.

If $n = 2$, we let $r_\Omega = \tilde\rho_0$.

If $n = 3$, we let $g_1 = \tilde\rho_0^{-2} g_0$ and define ρ_1 to be the distance to the edges of $\overline{\Omega}$ *in the metric* g_1. We then consider the function $\tilde\rho_1$ that is smooth outside the edges, coincides with ρ_1 close to the edges, and satisfies $\rho_1/2 \le \tilde\rho_1 \le \rho_1$ everywhere. Then we define $r_\Omega = \tilde\rho_0\tilde\rho_1$. This completes the definition of r_Ω in three dimensions.

A simple calculation then shows [1]

Lemma 2.8. *For any order m differential operator P with smooth coefficients, the family $P_\lambda := r_\Omega^{-\lambda} P r_\Omega^{\lambda}$ is a family of continuous linear maps $P_\lambda : \mathcal{K}^{\mu}_{a}(\Omega) \to \mathcal{K}^{\mu-m}_{a-m}(\Omega)$ depending continuously on λ.*

We shall need also a result on the normal derivative at the boundary. Let $\partial_N\Omega \subset \partial\Omega$ be a union of open faces of Ω (*i.e.,* a union of edges if $n = 2$). Let ν be the outer unit normal to these faces. Note that ν is defined everywhere on $\partial_N\Omega$, because an open face contains no singular boundary points. We shall denote by ∂_ν the directional derivative in the direction of ν.

Proposition 2.9. *The normal derivative map at the boundary*

$$\mathcal{C}^\infty_c(\overline{\Omega} \smallsetminus \partial_{sing}\Omega) \ni u \to \partial_\nu u|_{\partial\Omega} \in \mathcal{C}^\infty_c(\partial\Omega \smallsetminus \partial_{sing}\Omega)$$

extends to a continuous, surjective map $\partial_\nu : \mathcal{K}^{\mu}_{a}(\Omega) \to \mathcal{K}^{\mu-3/2}_{a-3/2}(\partial\Omega)$, $\mu \ge 2$.

Proof. Let $\nu_1 = r_\Omega \nu$. Let g_0 be the Euclidean metric. Then ν_1 is the normal unit vector with respect to the metric $g = r_\Omega^{-2} g_0$. Moreover, ν_1 extends to a smooth vector field X on Ω that is in the structural Lie algebra of vector fields on Ω [1]. In particular, we obtain a continuous map $X : \mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_a^{\mu-1}(\Omega)$. The composition of X with the trace map is ∂_{ν_1}, the normal derivative at the boundary in the direction of ν_1, so we obtain a continuous map $\partial_{\nu_1} : \mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_{a-1/2}^{\mu-3/2}(\partial\Omega)$.

The result then follows from $\partial_\nu = r_\Omega^{-1}\partial_{\nu_1}$ and Equation (21). □

3. A Poincaré Inequality

Since for $n = 2$ we consider also mixed boundary conditions, we need to introduce some additional notation. Thus, for $n = 2$ we shall decompose the boundary of Ω into two disjoint sets

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \tag{22}$$

with the following properties. First, $\partial_D\Omega$ is a union of closed (with respect to $\partial\Omega$) edges. Second, $\partial_N\Omega = \partial\Omega \smallsetminus \partial_D\Omega$ is a union of *disjoint* open edges.

The set $\partial_D\Omega$ is the subset of $\partial\Omega$ that will be assigned Dirichlet boundary conditions and $\partial_N\Omega$ is the subset that will be assigned Neumann boundary conditions. This implies, in particular, that every edge is assigned exactly one of the Dirichlet or Neumann boundary conditions and no two adjoint edges are assigned Neumann boundary conditions. Thus, on a triangle, at most one of the edges can be assigned Neumann boundary conditions. Recall, however, that if the type of the boundary condition changes inside one of the edges, then we can introduce new vertices so that these boundary conditions fit into our framework.

If $n = 3$, we agree that $\partial_D = \partial\Omega$ (so $\partial_N\Omega = \emptyset$), in order to have uniform statements for $n = 2$ and $n = 3$. The main goal of this section is to prove Theorem 3.3, which amounts to a Poincare inequality in weighted Sobolev spaces for functions vanishing on $\partial_D\Omega$. The following two lemmas settle the case $n = 2$.

With a little abuse of notation, we shall write $u(r,\theta) := u(r\cos\theta, r\sin\theta)$ for a function $u(x_1, x_2)$ expressed in polar coordinates. Below, $dx = dx_1 dx_2 \dots dx_n$.

Lemma 3.1. *Let $\mathcal{C} = \mathcal{C}_R(\alpha) := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2, 0 < r < R, 0 < \theta < \alpha\}$, $0 < \alpha < 2\pi$. Then*

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2}\, dx \le \left(\frac{2\alpha}{\pi}\right)^2 \int_{\mathcal{C}} |\nabla u|^2 dx$$

for any u, $\nabla u \in L^2(\mathcal{C})$ satisfying $u(r,\theta) = 0$ if $\theta = 0$, in the trace sense.

Proof. We include here the elementary proof from [35][Subsection 2.3.1]. See also [4, 22]. Let $\partial_\theta u(r,\theta)$ be the weak derivative of u, which exists and is a locally integrable function since $u \in H^1_{\mathrm{loc}}(\mathcal{C})$.

Fubini's theorem gives that the function $u(r,\theta)$ is a Lebesgue measurable function of θ, except maybe for r in a set of measure zero (*i.e.,* almost everywhere in r). Since $\mathcal{C}_c^\infty(U)$ has a countable dense subspace, for any open set $U \subset \mathbb{R}^n$, we obtain that, for r fixed outside a set of measure zero, the functions $u(r,\theta)$ and $\partial_\theta u(r,\theta)$ are measurable in θ, are locally square integrable, and the later is the weak derivative in θ of the first one.

The one dimensional Poincaré inequality then applies and gives

$$\int_0^\alpha |u|^2\, d\theta \le \left(\frac{2\alpha}{\pi}\right)^2 \int_0^\alpha |\partial_\theta u|^2\, d\theta,$$

almost everywhere in r, because $u(r,0)=0$. Moreover, both sides of the above inequality define measurable functions of r (we extend them with zero where they are not defined). Using again Fubini's theorem, we obtain

$$\int_{\mathcal{C}} \frac{|u|^2}{r^2}\,dx = \int_0^R \left(\int_0^\alpha |u|^2 d\theta\right) r^{-1}dr \leq \alpha^2 \int_0^R \left(\int_0^\alpha |\partial_\theta u|^2 r^{-1} d\theta\right) dr$$
$$\leq \alpha^2 \int_0^R \int_0^\alpha \left(\frac{|\partial_\theta u|^2}{r^2} + |\partial_r u|^2\right) r d\theta dr = \alpha^2 \int_{\mathcal{C}} |\nabla u|^2 dx.$$

This completes the proof. □

Using Lemma 3.1, we now prove the Poincaré inequality for a curvilinear polygon.

Lemma 3.2. *Let Ω be a curvilinear polygonal domain in a two dimensional manifold M. Fix an arbitrary metric g on M and let $\vartheta(z)$ be the distance from z to the vertices of Ω. Then there exists a constant $\kappa_\Omega > 0$ such that*

$$\|u\|^2_{\mathcal{K}^0_1(\Omega)} := \int_\Omega \frac{|u(z)|^2}{\vartheta(z)^2}\,dz \leq \kappa_\Omega \int_\Omega |\nabla u(z)|^2 dz$$

for any $u \in H^1(\Omega)$ satisfying $u = 0$ on $\partial_D \Omega$.

Proof. Let us fix, for any vertex $p \in \overline{\Omega}$, a diffeomorphism $\phi_p : V_p \to B^2$ such that $\phi_p(V_p \cap \Omega) = \mathcal{C}$, as in Definition (2.3)(a). By decreasing V_p, if necessary, we may assume that ϕ_p extends to a diffeomorphism defined in a neighborhood of $\overline{V}_p$ in M. Note that $\mathcal{C}$ satisfies the assumptions of Lemma 3.1.

Let $r_p(z) = r(\phi_p(z))$ be the distance from $\phi_p(z)$ to $0 = \phi_p(p) \in \mathbb{R}^2$. Then $\vartheta(z)/r_p(z)$ is continuous on $V_p \smallsetminus \{p\}$ and

$$\vartheta(z)/r_p(z) \ \text{ is bounded as } \ z \to p.$$

If follows that $\vartheta(z)/r_p(z)$ is bounded on V_p. Similarly, the norms of the differentials $D\phi_p$ and $D\phi_p^{-1}$ are bounded on V_p, by continuity. Hence the Jacobians $|\det(D\phi_p)| = |\frac{\partial x}{\partial z}|$ and $|\det(D\phi_p^{-1})| = |\frac{\partial z}{\partial x}|$ are bounded on V_p.

Let $u \in H^1_{\text{loc}}(\Omega)$, $u = 0$ on $\partial\Omega$. Then Lemma 3.1 gives

$$(23) \quad \int_{V_p\cap\Omega} \frac{|u(z)|^2}{\vartheta(z)^2}\,dz = \int_{V_p\cap\Omega} \frac{|u(z)|^2}{r_p(z)^2}\frac{r_p(z)^2}{\vartheta(z)^2}\,dz \leq C\int_{V_p\cap\Omega} \frac{|u(z)|^2}{r_p(z)^2}\,dz$$
$$= C\int_{\mathcal{C}} \frac{|u(\phi_p^{-1}(x))|^2}{r^2} \left|\frac{\partial z}{\partial x}\right| dx \leq C\int_{\mathcal{C}} \frac{|u(\phi_p^{-1}(x))|^2}{r^2}\,dx \leq C\int_{\mathcal{C}} |\nabla_x u(\phi_p^{-1}(x))|^2\,dx$$
$$\leq C\int_{V_p\cap\Omega} |\nabla_z u(z)|^2\,dz.$$

In the last inequality we have used the fact that the differential $D\phi_p$ is bounded in norm. (We have used above the usual convention that C denotes a generic constant that may be different in each inequality.)

We shall also need the usual Poincaré inequality

$$(24) \qquad \int_\Omega |u(z)|^2 dz \leq C \int_\Omega |\nabla u(z)|^2 dz.$$

By decreasing the neighborhoods V_p, we can assume that they are disjoint. Adding up then the inequality (24) and the inequalities (23), for all vertices p of Ω, we obtain

$$\int_\Omega f(z)|u(z)|^2 dz \le C \int_\Omega |\nabla u(z)|^2 dz,$$

where $f(z) = 1$ if $z \notin V_p$ and $f(z) = 1 + r_p^{-2}$ if $z \in V_p$. We claim that the function $\vartheta(z)^2 f(z)$ is bounded. Indeed, the function $\vartheta^2(z)$ is bounded on Ω. Outside all V_p, $f(z)$ is bounded as well. Finally, on V_p, we have already noticed that the product $\vartheta(z)/r_p(z)$ is bounded.

We therefore obtain $\vartheta^{-2}(z) \le C f(z)$, and hence

$$\int_\Omega |u(z)|^2 \vartheta(z)^{-2} dz \le C \int_\Omega |\nabla u(z)|^2 dz.$$

This completes the proof. □

We now prove the desired Poincaré inequality. Recall that Ω is either a curvilinear polygonal domain in $\mathbb{R}^2$ or a straight polyhedron in $\mathbb{R}^3$. Also, recall that $\partial_N \Omega = \emptyset$ if $n = 3$ and that $\partial_N \Omega$ cannot contain two adjacent edges if $n = 2$. A generalization of this theorem in higher dimensions is possible using induction [10], but requires a significantly more complicated topological machinery.

Theorem 3.3. *There exists a constant $\kappa_\Omega > 0$, depending only on Ω, such that*

$$\|u\|^2_{\mathcal{K}^0_1(\Omega)} := \int_\Omega \frac{|u(z)|^2}{\vartheta(z)^2}\, dz \le \kappa_\Omega \int_\Omega |\nabla u(x)|^2\, dx\,, \quad dx = dx_1 dx_2 \dots dx_n,\ n = 2, 3,$$

for any measurable function u such that $\nabla u \in L^2(\Omega)$ and $u|_{\partial_D \Omega} = 0$.

Proof. The result was proved for $n = 2$, so let us assume that $n = 3$. The idea of the proof is to cover the domain Ω with open sets $\mathcal{C}$ on which the integration simplifies and we can use the usual Poincaré inequality. The result follows by adding the corresponding inequalities.

We shall write $dV = dxdydz$ for the volume element. Also, recall that $\vartheta(x)$ denotes the distance from $x \in \overline{\Omega}$ to the edges of Ω.

Let us consider the inequality

$$\|u\|^2_{\mathcal{K}^0_1(D)} := \int_D \frac{|u(x)|^2}{\vartheta(x)^2}\, dx \le C \int_D |\nabla u(x)|^2\, dx, \quad u = 0 \text{ on } \overline{D} \cap \partial\Omega \tag{25}$$

for suitable subdomains $D \subset \Omega$. The statement is exactly the inequality (25) for $D = \Omega$. The proof of our inequality for $D = \Omega$ will be obtained by adding the inequality (25) for suitable subdomains D.

We shall consider three types of subdomains $D \subset \Omega$, denoted Ω_e, $\Omega_{v,e}$, and Ω_v, where e is an edge of Ω and v is a vertex of Ω, $v \in e$. These domains are defined in terms of two small enough positive numbers $\epsilon > \delta > 0$, depending only on Ω, such that the following three properties are satisfied:

1. For any edge e of Ω, let Cyl_e be the right cylindrical domain of radius δ whose axis of symmetry is the line containing the edge e and whose bases intersect e at distance ϵ from its two vertices. (These two basis are orthogonal to e.) We assume that ϵ and δ were chosen small enough so that the domain $\Omega_e := \Omega \cap Cyl_e$ can be characterized, in suitable cylindrical coordinates, by

$$\Omega_e = \{(r, \theta, z),\ 0 < r < \delta,\ 0 < \theta < \theta_e,\ 0 < z < z_e := |e| - 2\epsilon\},$$

where $|e|$ is the length of the edge e, and, moreover, $\vartheta = r$ on Ω_e. In these cylindrical coordinates, the edge e is on the z-axis (in particular, $r = 0$ on e).

2. For any vertex v and any edge e containing v, we consider the right conical domain $Con_{v,e}$ with vertex v and base the same with one of the bases of Cyl_e. The cone $Con_{v,e}$ is therefore such that its symmetry axis is the line containing the edge e. We assume that ϵ and δ were chosen small enough so that domain $\Omega_{v,e} := \Omega \cap Con_{v,e}$ can be characterized in cylindrical coordinates by

$$\Omega_{v,e} = \{(r, \theta, z),\ 0 < r < \frac{\delta}{\epsilon} z,\ 0 < \theta < \theta_e,\ 0 < z < \epsilon\}$$

and, moreover, $\vartheta = r$ on $\Omega_{v,e}$.

3. Let $B(v, t)$ be the open ball of radius t centered at v. We also assume that $\epsilon > 0$ is small enough so that for any vertex v of Ω, the domain $\Omega_v := \Omega \cap B(v, 2\epsilon)$ can be characterized in (suitable) spherical coordinates (ρ, ω) centered at v by

$$\Omega_v = \{(\rho, \omega), 0 < \rho < 2\epsilon,\ \omega \in \omega_v\},$$

where v corresponds to $\rho = 0$ and $\omega_v \subset S^2$ is a polygonal domain on the unit sphere $S^2 = \partial B^2 \subset \mathbb{R}^3$.

Assuming that $\epsilon, \delta > 0$ were chosen such that the conditions 1, 2, and 3 above are satisfied, we shall now prove (25) for D equal to one of the domains Ω_e, $\Omega_{v,e}$, or Ω_v. We need to stress at this point the crucial importance of the relation $\vartheta = r$ on the first two types of domains.

Let us prove first the inequality (25) for $D = \Omega_e = W_\delta \times (0, z_e)$, where $W_\delta = \{0 < r < \delta,\ 0 < \theta < \theta_e\}$ in suitable cylindrical coordinates (r, θ, z). It is enough to check that any $u \in C^\infty(D)$ such that $u(r, 0, z) = 0$ satisfies also

$$\int_D \frac{|u(x)|^2}{r^2} dV \leq C \int_D |\nabla u(x)|^2 dV \leq C \int_\Omega |\nabla u(x)|^2 dV. \tag{26}$$

Using Lemma (3.1) and the formula for $|\nabla u|$ in cylindrical coordinates, we obtain (using a similar reasoning to that of Lemma 3.1):

$$\begin{aligned} \int_D \frac{|u|^2}{r^2}\, dV &= \int_0^{z_e} \int_{W_\delta} \frac{|u|^2}{r}\, dr d\theta dz \leq C \int_0^{z_e} \int_{W_\delta} \left(\frac{|\partial_\theta u|^2}{r} \right) dr d\theta dz \\ &\leq C \int_0^{z_e} \int_{W_\delta} \left(\frac{|\partial_\theta u|^2}{r} + r|\partial_\rho u|^2 + r|\partial_z u|^2 \right) dr d\theta dz = C \int_D |\nabla u(x)|^2 dV. \end{aligned} \tag{27}$$

Let us consider next the case $D = \Omega_{v,e}$. Then the proof proceeds exactly in the same way, except that we replace W_δ in the integrals of the last equation with $W_{\delta z}$ the sector domain defined by $0 < \epsilon r < \delta z$ and $0 < \theta < \theta_e$.

Finally, if $D = \Omega_v$, we proceed as in Lemma (3.1), using also the formula

$$|\nabla u|^2 = u_\rho^2 + \frac{1}{\rho^2} u_\phi^2 + \frac{1}{\rho^2 \sin^2 \phi} u_\theta^2 \tag{28}$$

and the relation

$$\int_{\omega_v} |u|^2\, dS \leq C \int_{\omega_v} \left(u_\phi^2 + \frac{1}{\sin^2 \phi} u_\theta^2 \right) \sin \phi\, d\phi d\theta = C \int_{\omega_v} |\nabla_{(\phi,\theta)} u|^2\, dS,$$

which is nothing but the usual Poincaré's inequality on ω_v (dS is the volume element on ω_v). We then obtain

$$(29)\quad \int_{\Omega_v} \frac{|u|^2}{\rho^2} dV = \int_0^{2\epsilon} \int_{\omega_v} |u|^2 dS d\rho \leq C \int_0^{2\epsilon} \int_{\omega_v} \left(u_\phi^2 + \frac{1}{\sin^2 \phi} u_\theta^2 \right) dS d\rho$$
$$\leq C \int_0^{2\epsilon} \int_{\omega_v} \left(u_\rho^2 + \frac{1}{\rho^2} u_\phi^2 + \frac{1}{\rho^2 \sin^2 \phi} u_\theta^2 \right) \rho^2 dS d\rho = C \int_{\Omega_v} |\nabla u(x)|^2 dV.$$

Adding the inequalities (26) for all $D = \Omega_e$ and all $D = \Omega_{v,e}$, the inequalities (29) for all $D = \Omega_v$, and the usual Poincaré inequality, $\int_\Omega |u|^2 dV \leq C \int_\Omega |\nabla u(x)|^2 dV$, we obtain

$$\int_\Omega h|u|^2 dV \leq C \int_\Omega |\nabla u(x)|^2 dV,$$

where $h(x) = 1+$ terms of the form r^{-2} or ρ^{-2}.

There will be one term r^{-2} each time when $x \in \Omega_e$ or $x \in \Omega_{v,e}$ and one term ρ^{-2} each time when $x \in \Omega_v$. Therefore $h \geq r^{-2} = \vartheta^{-2}$ on Ω_e and on $\Omega_{v,e}$, $h \geq \rho^{-2} \geq C\vartheta^{-2}$ on Ω_v and *outside* all of $\Omega_e \cup \Omega_{v,e}$, and, finally, $h \geq 1 \geq C\vartheta^{-2}$ outside of $\Omega_e \cup \Omega_{v,e} \cup \Omega_v$. Therefore $h \geq C\vartheta^{-2}$ on *the whole of* Ω. This completes the proof of our inequality for u smooth. By a standard density argument, we obtain the desired result (25) for all functions in $H_0^1(\Omega)$. □

From the above Poincaré type inequalities we obtain the following corollary.

Corollary 3.4. *Let Ω be either a curvilinear polygonal domain or a straight polyhedral domain in three dimensions. Then the norms $\|\cdot\|_{H^1(\Omega)}$, $\|\cdot\|_{\mathcal{K}_1^1(\Omega)}$, and the seminorm $|\cdot|_{H^1(\Omega)}$ are equivalent on $H_0^1(\Omega)$. In particular, $H_0^1(\Omega) = \overset{\circ}{\mathcal{K}}{}_1^1(\Omega)$.*

Proof. We have

$$|u|^2_{H^1(\Omega)} \leq \|u\|^2_{H^1(\Omega)} \leq \|u\|^2_{\mathcal{K}_1^1(\Omega)} = \|u\|^2_{\mathcal{K}_1^0(\Omega)} + |u|^2_{H^1(\Omega)} \leq (\kappa_\Omega^2 + 1)|u|^2_{H^1(\Omega)}$$

for any $u \in \mathcal{C}_c^\infty(\Omega)$, where κ_Ω is the constant in Theorem 3.3. The result follows then from the density of $\mathcal{C}_c^\infty(\Omega)$ in both $H_0^1(\Omega)$ and $\overset{\circ}{\mathcal{K}}{}_1^1(\Omega)$. □

4. Two well-posedness results

We now state our two well-posedness results. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a curvilinear polygonal domain, if $n = 2$, or a straight polyhedron, if $n = 3$. The first result is a well posedness result on Ω for systems with smooth coefficients satisfying the strong Legendre condition (Equation (31)) and with Dirichlet boundary conditions. For $n = 2$ we also consider mixed Dirichlet–Neumann boundary conditions for the Poisson problem on a curvilinear polygonal domain (Theorem 4.4).

We consider on Ω a $q \times q$ system of second order differential equations given by a matrix $P = (P_{kl})$,

$$Pu = \sum_{ij} \partial_i\big(A^{ij} \partial_j u\big) + \sum_j B^j \partial_j u + Cu,$$

$u = (u_1, u_2, \ldots, u_q)$, where $A^{ij} = A^{ji}$, B^j, and C are matrices in $\mathbb{C}^{q\times q}$ whose entries are smooth functions (a_{kl}^{ij}), (b_{kl}^j), and (c_{kl}), $k, l = 1, \ldots, q$, that is,

$$(30)\qquad P_{kl} u_l = -\sum_{ij=1}^n \partial_i\big(a_{kl}^{ij} \partial_j u_l\big) + \sum_{j=1}^n b_{kl}^j \partial_j u_l + c_{kl} u_l.$$

Recall the following well known definition.

Definition 4.1. We say that P *satisfies the strong Legendre condition* if there exists $\theta > 0$ such that

$$\sum_{kl=1}^{q} \sum_{ij=1}^{n} \mathrm{Re}\left(a_{kl}^{ij}(x)\zeta_{ki}\overline{\zeta}_{lj}\right) \geq \theta \sum_{k=1}^{q} \sum_{i=1}^{n} |\zeta_{ki}|^2, \quad \zeta = (\zeta_{ki}) \in \mathbb{C}^{nq},\ x \in \Omega. \tag{31}$$

Clearly, any system satisfying the strong Legendre condition is also strongly elliptic. Operators with scalar principal symbols and some of the ones arising in elasticity satisfy the strong Legendre condition.[1] See also [3, 7, 31].

In our first well-posedness result, Theorem 4.2, P will be a system satisfying the strong Legendre conditions and satisfying also the Conditions (32) and (33) stated next

$$b_{kl}^{j}(x) = \overline{b}_{lk}^{j}(x), \quad x \in \Omega, \tag{32}$$

and the matrix $\sum_j -\partial_j B^j + C$ has non-negative self-adjoint part, that is

$$\mathrm{Re}\Big(\sum_{kl=1}^{q} \sum_{j=1}^{n} (-\partial_j b_{kl}^{j}(x) + 2c_{kl}(x))\eta_k \overline{\eta}_l\Big) \geq 0, \quad \eta = (\eta_1, \ldots, \eta_q) \in \mathbb{C}^q,\ x \in \Omega. \tag{33}$$

On Ω we consider the boundary value problem

$$Pu = f \ \text{ on } \Omega, \qquad u = g \ \text{ on } \partial\Omega, \tag{34}$$

and its variational form

$$\begin{aligned} B(u,v) &:= \sum_{kl=1}^{q} \int_\Omega \Big(\sum_{ij=1}^{n} a_{kl}^{ij} \partial_j u_l \partial_i \overline{v}_k + \sum_{j=1}^{n} b_{kl}^{j} \partial_j u_l \overline{v}_k + c_{kl} u_l \overline{v}_k \Big) dx \\ &= \int_\Omega \Big(\sum_{k=1}^{q} f_k \overline{v}_k \Big) dx =: (f, v), \quad \text{for all } v \in \overset{\circ}{\mathcal{K}}{}_1^1 (\Omega)^q = H_0^1(\Omega)^q. \end{aligned} \tag{35}$$

We now state our first well-posedness result.

Theorem 4.2. *Let $\mu \in \mathbb{Z}_+$ and let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded curvilinear polygonal domain if $n = 2$ or a straight polyhedron if $n = 3$. Let P be a system with smooth coefficients on $\overline{\Omega}$ satisfying the strong Legendre condition, Equation* (31), *and Conditions* (32) *and* (33). *Then there exists $\eta > 0$ such that the boundary value problem* (34) *has a unique solution $u \in \mathcal{K}_{a+1}^{\mu+1}(\Omega)^q$ for any $f \in \mathcal{K}_{a-1}^{\mu-1}(\Omega)^q$, any $g \in \mathcal{K}_{a+1/2}^{\mu+1/2}(\partial\Omega)^q$, and any $|a| < \eta$. This solution depends continuously on f and g. If $\mu = a = 0$ and $g = 0$, this solutions is the solution of the associated variational problem, Equation* (35).

The proof of this theorem will be given in the following section.

Remark 4.3. The assumptions of the above theorem are satisfied if $P = -\Delta$, the Laplace operator. The continuous dependence of u on the data $f = -\Delta u$ amounts exactly to the estimate (4), because $g = u|_{\partial\Omega} = 0$ in this case.

[1] We owe the comment on the elasticity operators to Anna Mazzucato.

We now state our second well-posedness result on the Poisson problem with mixed Dirichlet–Neumann boundary value problem on curvilinear polygonal domains.

Let us assume for the rest of this section that Ω is a curvilinear polygonal domain in the plane. Also, we shall decompose the boundary $\partial\Omega$ into two subsets, $\partial_D\Omega$ and $\partial_N\Omega := \partial\Omega \smallsetminus \partial_D\Omega$ as in Equation (22). Thus, the disjoint decomposition of $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, is such that $\partial_D\Omega$ is a union of *closed* edges and that $\partial_N\Omega := \partial\Omega \smallsetminus \partial_D\Omega$ is a union of *open* edges.

Let $\nu = (\nu_1, \nu_2)$ be the outer unit normal to $\partial_N\Omega$ and ∂_ν be the directional derivative in the direction of ν, as before. This leads to the boundary value problem

$$-\Delta u = f, \qquad u = g_D \text{ on } \partial_D\Omega, \qquad \partial_\nu u = g_N \text{ on } \partial_N\Omega. \tag{36}$$

Let us denote by $\{A_1, A_2, \dots, A_k\}$ the vertices of Ω. Let α_j be the opening of the angle at the vertex A_j. Let $\eta_j := \pi/\alpha_j$, if Dirichlet boundary conditions are assigned to both sides containing A_j, $\eta_j = \pi/(2\alpha_j)$ otherwise. This second case corresponds to the case when on one of the sides containing A_j we assign Neumann boundary conditions and on the other side we assign Dirichlet boundary conditions. Let $\eta := \min\{\eta_j\}$.

For any subset $S \subset \partial\Omega$ that is a union of edges, we define the space $\mathcal{K}_a^\mu(S)$ to be the space of restrictions to S of distributions in $\mathcal{K}_a^\mu(\Omega)$.

Theorem 4.4. *Let $\mu \in \mathbb{Z}$, $\mu \geq 1$ and let $\Omega \subset \mathbb{R}^2$ be a bounded curvilinear polygonal domain in the plane. Let $\eta := \min\{\eta_j\} > 0$, as above. Then the boundary value problem (36) has a unique solution $u \in \mathcal{K}_{a+1}^{\mu+1}(\Omega)$ for any $f \in \mathcal{K}_{a-1}^{\mu-1}(\Omega)$, any $g_D \in \mathcal{K}_{a+1/2}^{\mu+1/2}(\partial_D\Omega)$, any $g_N \in \mathcal{K}_{a-1/2}^{\mu-1/2}(\partial_N\Omega)$, and any $|a| < \eta$. This solution depends continuously on f, g_D, and g_N.*

Let us notice that the condition that no two adjacent vertices are assigned Neumann boundary conditions is necessary in the above theorem. Indeed, if we assign Neumann boundary conditions to two adjacent vertices, then the resulting operator is not Fredholm for $a = 0$, by the results of Kondratiev [22] (the relevant results can also be found in [25]).

5. Proof of the well-posedness results

In this section, we prove our well-posedness results, Theorem 4.2 and 4.4. We follow the pattern of the proof for smooth domains [18] or [37]. First, we need the following lemmas. Recall that $H_0^1(\Omega) = \overset{\circ}{\mathcal{K}}{}_1^1(\Omega)$, by Corollary 3.4.

Lemma 5.1. *We have $(Pu, v) = B(u, v)$, for any $u, v \in H_0^1(\Omega)^q = \overset{\circ}{\mathcal{K}}{}_1^1(\Omega)^q$, where B is the form introduced in Equation (35).*

Proof. The proof is by integration by parts. □

Lemma 5.2. *We have*

$$\operatorname{Re} B(u, u) \geq \theta \|\nabla u\|_{L^2}^2 := \theta \sum_{k=1}^{q} \sum_{j=1}^{n} \|\partial_j u_k\|_{L^2}^2,$$

for any $u, v \in H_0^1(\Omega)^q = \overset{\circ}{\mathcal{K}}{}_1^1(\Omega)^q$, where B is the form introduced in Equation (35).

Proof. Equations (31), (32), and (33) give

$$\operatorname{Re} B(u,u) = \operatorname{Re}\left(\sum_{kl=1}^{q}\int_\Omega\Big(\sum_{ij=1}^{n} a_{kl}^{ij}\partial_j u_l \partial_i \overline{u}_k + \sum_{j=1}^{n} b_{kl}^{j}\partial_j u_l \overline{u}_k + c_{kl}u_l\overline{u}_k\Big)dx\right)$$

$$= \sum_{kl=1}^{q}\int_\Omega \operatorname{Re}\Big(\sum_{ij=1}^{n} a_{kl}^{ij}\partial_j u_l\partial_i\overline{u}_k\Big)dx + \sum_{kl=1}^{q}\int_\Omega \operatorname{Re}\Big((c_{kl} - \sum_{j=1}^{n}\partial_j b_{kl}^{j}/2)u_l\overline{u}_k\Big)dx$$

$$\geq \theta\sum_{l=1}^{q}\Big(\sum_{j=1}^{n}\int_\Omega |\partial_j u_l|^2 dx\Big) =: \theta\|\nabla u\|_{L^2}^2.$$

The lemma is proved. □

We now complete the proof of Theorem 4.2.

5.1. **Proof of Theorem 4.2.** First, let us notice that Theorem 2.5 allows us to reduce the proof to the case when $g = 0$.

We shall denote by $\big(u, v\big) := \sum_{k=1}^{q}\int_\Omega u_k(x)\overline{v}_k(x)dx$ the inner product on $L^2(\Omega)^q$. Let $\mathcal{H} \subset \mathcal{K}_1^1(\Omega)^q$ be the subspace consisting of the functions $u \in \mathcal{K}_1^1(\Omega)^q$ such that $u = 0$ on $\partial\Omega$. That is $\mathcal{H}$ is the kernel of the trace map $\mathcal{K}_1^1(\Omega)^q \to \mathcal{K}_{1/2}^{1/2}(\partial\Omega)^q$. In other words $\mathcal{H} = (\overset{\circ}{\mathcal{K}_1^1})^q$. Lemmas 5.1 and 5.2 and the Hardy–Poincaré inequality (Theorem 3.3 and Corollary 3.4) then give that there exists $\epsilon > 0$ such the following inequality

$$\operatorname{Re}\big(Pu, u\big) = \operatorname{Re} B(u,u) \geq \theta\|\nabla u\|_{L^2}^2 \geq \epsilon\|u\|_{\mathcal{K}_1^1(\Omega)}^2,$$

is satisfied for any $u \in \mathcal{H}$. In particular, B defines a continuous, coercive form on $\mathcal{H}$. The assumptions of the Lax-Milgram lemma are therefore satisfied, and hence $P : \mathcal{H} \to \mathcal{H}^* =: \mathcal{K}_{-1}^{-1}(\Omega)$ is an isomorphism This proves the result for $\mu = 0$ and $a = 0$.

Recall the function r_Ω introduced in Subsection 2.5. We know that $r_\Omega^\lambda \Delta r_\Omega^{-\lambda}$ depends continuously on λ and that $r_\Omega^t\mathcal{K}_a^\mu(\Omega) = \mathcal{K}_{a+t}^\mu(\Omega)$. As in [12], this allows us to conclude that there exists $\eta > 0$ such that P is an isomorphism for $\mu = 0$ and any $|a| < \eta$.

We now prove the result for $|a| < \eta$ and an arbitrary $\mu \in \mathbb{Z}_+$. Theorem 2.7 and the result we have just proved for $\mu = 0$ give that the map

$$P : \mathcal{K}_{a+1}^{\mu+2}(\Omega)^q \cap \{u|_{\partial\Omega} = 0\} \to \mathcal{K}_{a-1}^{\mu}(\Omega)^q$$

is surjective. Since this map is also continuous (Proposition 2.6) and injective (from the case $\mu = 0$), it is an isomorphism by the open mapping theorem. The proof of Theorem 4.2 is now complete.

Let $Pu = -\sum_{ij=1}^{n}\partial_i(a_{ij}\partial_j u)$ be a strongly elliptic second order differential operator on an open subset U of $\mathbb{R}^n$ (so $\sum a_{ij}(x)\eta_i\eta_j \geq \theta|\eta|^2$, for some $\theta > 0$ independent of $x \in U$ and $\eta \in \mathbb{R}^n$). Let us denote by

$$B(u,v) := \sum_{ij=1}^{n}\int_U a_{ij}(x)\partial_i u(x)\partial_j\overline{v(x)}dx \tag{37}$$

the associated form. Recall that $\nu = (\nu_1, \ldots, \nu_n)$ is unit outer normal to $\partial\Omega$. Denote $D_\nu^P u := \sum_{ij}\nu_i a_{ij}\partial_j u = 0$.

For the proof of Theorem 4.4, we shall need the following known regularity result for weak solutions of the Neumann problem. We keep the notation of Theorem 4.4.

Theorem 5.3. *Let $U \subset \mathbb{R}^n$ be a bounded domain with $\partial U = \partial_D U \cup \partial_N U$, the union of two disjoint, smooth manifolds, $\partial_D U \neq \emptyset$. Assume that $Pu = -\sum_{ij=1}^n \partial_i(a_{ij}\partial_j u)$ is a strongly elliptic differential operator with coefficients smooth on $\overline{U}$. Let us also assume that $f \in H^\mu(U)$, $\mu \in \mathbb{Z}_+$, $u \in H^1(U)$, $u = 0$ on $\partial_D U$, and*

$$B(u,v) := \sum_{ij=1}^n \int_U a_{ij}(x)\partial_i u(x)\partial_j \overline{v(x)}dx = \int_\Omega f(x)\overline{v(x)}dx \tag{38}$$

for all $v \in H^1(U)$ satisfying $v = 0$ on $\partial_D U$. Then $u \in H^{\mu+2}(U)$, $D_\nu^P u := \sum_{ij} \nu_i a_{ij}\partial_j u = 0$ on $\partial_N U$, and $Pu = f$. Moreover, there exists a constant $C > 0$, depending only on U, such that $\|u\|_{H^{\mu+2}(U)} \leq C\big(\|f\|_{H^\mu(U)} + \|u\|_{L^2(U)}\big)$.

Proof. We include the short proof for the benefit of the reader. Let

$$\mathcal{N} := \{w \in \mathcal{C}^\infty(\overline{U}),\ Pw = 0,\ w = 0 \text{ on } \partial_D U \text{ and } D_\nu^P w = 0 \text{ on } \partial_N U\}.$$

Since ∂U is smooth and $\mathcal{N} = \{0\}$, the boundary value problem

$$\begin{cases} Pw = f & \text{on } U \\ u = 0 & \text{on } \partial_D U \\ D_\nu^P w = 0 & \text{on } \partial_N U \end{cases} \tag{39}$$

has a solution $w \in H^{\mu+2}(U)$ whenever $f \perp \mathcal{N}$ in $L^2(U)$ and two solutions differ by an element of $\mathcal{N}$ (see, for example, [37]). We have also used here the fact that the boundary value problem (39) coincides with its adjoint.

An application of the Poincaré inequality shows that the form B is coercive on $\{u \in H^1(U),\ u = 0 \text{ on } \partial_D U\}$ and hence $\mathcal{N} = \{0\}$. Therefore a solution exists and is unique. By taking $v \in \mathcal{C}_c^\infty(U)$ in Equation (38), we obtain that $B(u,v) = B(w,v)$, and hence $u = w$ because B is coercive. In particular, $u \in H^{\mu+2}(U)$ and $D_\nu^P w = 0$ on $\partial_N U$. The last statement follows from the continuous dependence of the solutions of the boundary value (39) on its data. □

Remark 5.4. An analogue of Theorem 5.5 holds also when $\partial_N\Omega = \partial\Omega$, provided that the integrals of f and u vanish on all connected components of U.

This theorem gives the following version of Theorem 2.7. We use the same notation as in Theorem 5.3 for P and B, but with U replaced by a curvilinear polygonal domain $\Omega \subset \mathbb{R}^2$. The decomposition $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ is as in Equation (22).

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygonal domain, $a \in \mathbb{R}$ arbitrary. Assume the coefficients of the strongly elliptic differential operator $Pu = -\sum_{ij=1}^n \partial_i(a_{ij}\partial_j u)$ are smooth on $\overline{\Omega}$. Let us also assume that $f \in \mathcal{K}_{a-1}^{\mu-1}(\Omega)$, $\mu \geq 1$, and that $u \in \mathcal{K}_{a+1}^1(\Omega)$, $u = 0$ on $\partial_D\Omega$, satisfies*

$$B(u,v) = \int_\Omega f(x)\overline{v(x)}dx \tag{40}$$

for all $v \in \mathcal{K}_{1-a}^1(\Omega)$ such that $v = 0$ on $\partial_D\Omega$. Then $u \in \mathcal{K}_{a+1}^{\mu+1}(\Omega)$, $D_\nu^P u = 0$ on $\partial_N\Omega = \partial\Omega \smallsetminus \partial_D\Omega$, and $Pu = f$. Moreover, there exists a constant $C > 0$, depending only on Ω, such that $\|u\|_{\mathcal{K}_{a+1}^{\mu+1}(\Omega)} \leq C\big(\|f\|_{\mathcal{K}_{a-1}^{\mu-1}(\Omega)} + \|u\|_{\mathcal{K}_{a+1}^0(\Omega)}\big)$.

Proof. As in the proof in [1] of Theorem 2.7, the proof is reduced to the case of a smooth domain in $\mathbb{R}^2$ using a suitable partition of unity $\{\phi_k\}$. Then we use Theorem 5.3. □

Remark 5.6. For the above proof, the partition of unity $\{\phi_k\}$ takes a very simple form. Namely, fix a neighborhood V of a vertex of Ω. We can assume that $V = \{0 < \theta < \alpha\}$. Then we can choose $\phi_k(x) = \phi(2^k|x|)$ for k large and a suitable function ϕ.

We now complete the proof of Theorem 4.4.

5.2. **Proof of Theorem 4.4.** The proof is similar to that of Theorem 4.2. First, we show that we can assume that $g_D = 0$ and $g_N = 0$.

Indeed, using Theorem 2.5 and Proposition 2.9, we see that the map

$$\mathcal{K}^{\mu+1}_{a+1}(\Omega) \ni u \to (u|_{\partial_D\Omega}, \partial_\nu u|_{\partial_N\Omega}) \in \mathcal{K}^{\mu+1/2}_{a+1/2}(\partial_D\Omega) \oplus \mathcal{K}^{\mu-1/2}_{a-1/2}(\partial_N\Omega)$$

is continuous and surjective. Let $v \in \mathcal{K}^{\mu+1}_{a+1}(\Omega)$ such that $v = g_D$ on $\partial_D\Omega$, and $\partial_\nu v = g_N$ on $\partial_N\Omega$. By considering $w = u - v$, we see that the boundary value problem (36) is equivalent to $-\Delta w = f + \Delta v$, $w =$ on $\partial_D\Omega$, and $\partial_\nu w = 0$ on $\partial_N\Omega$.

Let $\mathcal{H} = \mathcal{K}^1_1(\Omega) \cap \{u = 0,\ \text{on}\ \partial_D\Omega\}$. We define the form $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ as before

$$B(u, v) := \int_\Omega \nabla u \cdot \nabla \overline{v} = \int_\Omega \big(\partial_x u \partial_x \overline{v} + \partial_y u \partial_y \overline{v}\big) dxdy$$

Then $|B(u, v)| \le \|u\|_{\mathcal{K}^1_1(\Omega)} \|v\|_{\mathcal{K}^1_1(\Omega)}$, so B is continuous. Also, $B(u, u) \ge \theta \|u\|^2_{\mathcal{K}^1_1(\Omega)}$, for some $\theta > 0$, by the Poincaré inequality, Theorem 3.3 (we do not have to use Lemma 5.2).

Let $f \in \mathcal{K}^{\mu-1}_{-1}(\Omega)$, $\mu \ge 1$. Then the L^2-inner product $(f, v) = \int_\Omega f(x)\overline{v(x)}dx$ is defined and continuous in v for $v \in \mathcal{K}^0_1(\Omega)$. Since $\mathcal{H} \subset \mathcal{K}^0_1(\Omega)$ continuously, $v \to (f, v)$ is a continuous linear functional on $\mathcal{H}$. The Lax-Milgram lemma now shows that the equation $B(u, v) = (f, v)$, for all $v \in \mathcal{H}$, has a solution $u \in \mathcal{H}$.

Theorem 5.5 then shows that $u \in \mathcal{K}^{\mu+1}_1(\Omega)$ and that $u = 0$, on $\partial_D\Omega$ and $\partial_\nu u = D^\Delta_\nu u = 0$ on $\partial_N\Omega$. The continuous dependence of u on f follows from the open mapping theorem again. We obtain that the map

$$\Delta : \mathcal{K}^{\mu+1}_{a+1}(\Omega) \cap \mathcal{H} \to \mathcal{K}^{\mu-1}_{a-1}(\Omega) \tag{41}$$

is an isomorphism for $a = 0$.

By the same argument as in [12], the map in Equation (41) will remain an isomorphism as long as it is Fredholm. It fails to be Fredholm when new singular functions appear. This gives the indicated value for η. Moreover, the largest η with this property will not depend on μ, since if we have an isomorphism for some value of a and $\mu = 2$, then we obtain an isomorphism for any μ, by the regularity theorem (Theorem 5.5) and the above argument.

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