

IMPROVING THE RATE OF CONVERGENCE OF ‘HIGH ORDER FINITE ELEMENTS’ ON POLYHEDRA II: MESH REFINEMENTS AND INTERPOLATION

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ABSTRACT. We construct a sequence of meshes \mathcal{T}'_k that provides *quasi-optimal rates of convergence* for the solution of the Poisson equation on a bounded polyhedral domain with right hand side in H^{m-1} , $m \geq 2$. More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain and let $u \in H^1(\Omega)$ be the solution of the Poisson problem $-\Delta u = f \in H^{m-1}(\Omega)$, $m \geq 2$, $u = 0$ on $\partial\Omega$. Also, let S_k be the Finite Element space of continuous, piecewise polynomials of degree $m \geq 2$ on \mathcal{T}'_k and let $u_k \in S_k$ be the finite element approximation of u , then $\|u - u_k\|_{H^1(\Omega)} \leq C \dim(S_k)^{-m/3} \|f\|_{H^{m-1}(\Omega)}$, with C independent of k and f . Our method relies on the a priori estimate $\|u\|_{\mathcal{D}} \leq C \|f\|_{H^{m-1}(\Omega)}$ in certain anisotropic weighted Sobolev spaces $\mathcal{D} = \mathcal{D}_{a+1}^{m+1}(\Omega)$, with $a > 0$ small, determined only by Ω . The weight is the distance to the set of singular boundary points (*i.e.*, edges). The main feature of our mesh refinement is that a segment AB in \mathcal{T}'_k will be divided into two segments AC and CB in \mathcal{T}'_{k+1} as follows: $|AC| = |CB|$ if A and B are equally singular and $|AC| = \kappa|AB|$ if A is more singular than B . We can choose $\kappa \leq 2^{-m/a}$. This allows us to use a uniform refinement of the tetrahedra that are away from the edges to construct \mathcal{T}'_k .

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INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain. Consider the Poisson problem

$$(1) \quad -\Delta u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0 \text{ on } \partial\Omega,$$

where Δ is the Laplace operator. Let $S_k \subset H_0^1(\Omega)$ be a sequence of finite dimensional spaces of dimension $\dim(S_k)$ and let $u_k \in S_k$ be the Finite Element approximation of the solution u of Equation (1) for $f \in H^{m-1}(\Omega)$ arbitrary. We shall say that the sequence S_k achieves *quasi-optimal rates of convergence with respect to the dimension of the space S_k* (or, simply, *quasi-optimal rates of convergence*) if there exists a constant $C > 0$, independent of k and f , such that

$$(2) \quad \|u - u_k\|_{H^1(\Omega)} \leq C \dim(S_k)^{-m/3} \|f\|_{H^{m-1}(\Omega)}, \quad u_k \in S_k.$$

If Ω is bounded with *smooth* boundary, then it is well known [13, 23, 26] that we can choose our sequence of tetrahedralization \mathcal{T}'_k to be quasi-uniform, provided that the Dirichlet boundary conditions are suitably treated. On the other hand, if Ω is not smooth, an important result of Wahlbin [58] states that a quasi-uniform sequence of triangulations will *not* lead to optimal rates of convergence for the sequence u_k . (See also [12, 39, 59].) Nevertheless, if Ω is a polygonal domain in the plane, it was shown by Babuška in the ground breaking paper [11] that there exist sequences S_k that will achieve optimal rates of convergence. See also Raugel [52], Apel, Sändig, and Whiteman [8], Babuška, Kellogg, and Pitkäranta [14], or the book of Oganessian and Rukhovets [51] for alternative proofs of this result. Yet another proof of this result was given in a previous paper of ours for polygonal domains, [20].

Let us fix $m \geq 2$. It is the purpose of this paper to construct an explicit sequence S_k of finite element spaces providing quasi-optimal rates of convergence with respect to the dimension of the approximation space. Our method is based on the refining methods of [14, 20], on a regularity result in the spirit of [24], and on interpolation results in the spirit of [5] (see also [6, 7]). The approximation spaces S_k are associated to a sequence of conforming meshes \mathcal{T}'_k and consists of continuous, piecewise polynomials of degree m . The condition $m \geq 2$ is probably just an artifact of our proof (better interpolants will probably help eliminate this condition). The sequence of meshes \mathcal{T}'_k is obtained by combining a uniform division of the tetrahedra that do not intersect the edges of Ω with a graded refinement toward the edges. Namely, we divide a vertex AB in \mathcal{T}'_k to yield AC and CB in \mathcal{T}'_{k+1} such that, if A is more singular than B , then $|AC|/|AB| = \kappa$, for a parameter $\kappa = 2^{-m/a}$, as in [8, 20, 52]. (The parameter a must be small enough so that the regularity Theorem 2.2 is satisfied.) On the other hand, the uniform refinement procedure is such that AB is divided into two equal parts if A and B are equally singular.

Our results, based on *a priori* estimates in weighted Sobolev spaces, should be compared to the results using *a posteriori* estimates and adaptive mesh refinements. See for example [22, 45, 48] and the references therein.

We now describe the contents of our paper. In Section 1 we recall the definitions and properties of weighted Sobolev spaces and we introduce some anisotropically weighted Sobolev spaces. We also introduce and fix an initial decomposition \mathcal{T}_0 of Ω into tetrahedra (close to the vertices), triangular prisms (close to the edges) and an interior region Λ_0 . In Section 2, we prove a well posedness of the Poisson

problem in anisotropic weighted Sobolev spaces (Theorem 2.2), generalizing the results of [21, 24, 36, 37, 38, 41, 42, 43, 49] and others. In Section 3, we explain the general principles for constructing our tetrahedralizations (*i.e.*, meshes), and formulate some conditions on our tetrahedralizations that ensure quasi-optimal rates of convergence (with respect to the dimension of the approximation space). In Section 4, we explain in detail the construction of our tetrahedralizations \mathcal{T}'_n . Fix n , then the tetrahedralization \mathcal{T}'_n is obtained in turn from a decomposition (or partition) \mathcal{T}_n of Ω into tetrahedra and straight triangular prisms (except \mathcal{T}_0 , which is allowed also to contain an interior polyhedral region that is not a tetrahedron or a prism). Each prism in our decompositions will have a fixed diagonal (“mark”) that will determine a decomposition of it into three tetrahedra. The meshes \mathcal{T}'_k , $k \geq 1$, are obtained from the decompositions \mathcal{T}_k by dividing each prism into three tetrahedra, as explained. To obtain \mathcal{T}'_0 , we also have to tetrahedralize the interior region Λ_0 without introducing additional edges on the boundary of Λ_0 except for the marks (we allow however additional vertices and edges interior to Λ_0). So, in order to describe the meshes \mathcal{T}'_n , it is enough to describe the decompositions \mathcal{T}_n . In Section 5, we obtain some refinements of the usual interpolation results on standard simplices (provided that one edge is perpendicular to a face). The interpolation result of this section separates the variable z (parametrizing an edge) from the variables x and y parametrizing the plane perpendicular to the edge, to account for different behaviors of the solution u with respect to these variables close to the edge. These interpolation results are used then in Section 6 to prove interpolation results on thin tetrahedra which finally show that our sequence of meshes \mathcal{T}'_k leads to quasi-optimal rates of convergence (with respect to the dimension of the approximation space) for $m \geq 2$. Our approach is summarized in Section 7, where we also state our main and final result that the sequence S_k of spaces of continuous, order $m \geq 2$ piecewise polynomials on \mathcal{T}'_k provides quasi-optimal rates of convergence with respect to the dimension of the Finite Element space.

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1. ISOTROPIC AND ANISOTROPIC SOBOLEV SPACES WITH WEIGHTS

In this section, we shall describe the various function spaces used in what follows. We begin with an initial partition of our domain, which will be used throughout the paper. We shall use the notation from our main references, [5, 6, 20, 21, 24] and [43], as much as possible.

1.1. Polyhedral domains. *Throughout this paper, Ω will be a fixed bounded polyhedral domain.* The polygonal and polyhedral domains used in this paper are a subclass of the domains considered in [31].

Definition 1.1. *A bounded polygonal domain is a bounded, connected, open subset $D \subset \mathbb{R}^2$, $\partial D = \partial \bar{D}$, together with a choice of finitely many points $\{A_k\} \subset \partial D$ such that the boundary of D is the finite union of the straight, closed segments $[A_k, A'_k]$,*

called *the closed sides of D* , where $\{A_k\} = \{A'_k\}$, each A_k belongs to exactly two closed sides of D , no two distinct open sides (A_k, A'_k) have a point in common, and no point A_k can be eliminated from this definition.

The points A_k are called the *vertices* of D . We shall sometimes replace \mathbb{R}^2 with an affine space in the definition of polygonal domain.

If the boundary of D is connected, then we agree that $A'_k = A_{k+1}$ or $A'_k = A_1$. However, we do not require the boundary of D_j to be connected, in general. The condition $\partial D = \partial \bar{D}$ means that we do not allow D to be on both sides of its boundary (as it is the case in domains with cracks). This condition is included for simplicity and is not essential in what follows.

We are now ready to introduce bounded polyhedral domains.

Definition 1.2. A *bounded polyhedral domain* $\Omega \subset \mathbb{R}^3$ is a bounded, connected open satisfying:

- (1) $\partial \Omega = \partial \bar{\Omega}$;
- (2) there exist finitely many disjoint polygonal domains $D_j \subset \partial \Omega = \partial \bar{\Omega}$ whose closures cover the boundary: $\partial \Omega = \cup \bar{D}_j$;
- (3) each side of any of the domains D_j belongs to exactly one other domain D_k ; and
- (4) no adjacent faces are coplanar.

The sets D_j are called the *open faces* of Ω , the sides of D_j are the *open edges* of Ω , and the vertices of D_j are also the *vertices* of Ω .

A note on terminology: First, the faces, edges, or domains are considered as open sets of the minimal affine space they belong to. Nevertheless, we may say “the vertex P belongs to the edge $e = (AB)$ ” in the sense that $P = A$ or $P = B$ (of course $A, B \in \bar{e} = [AB]$). However, $P \notin e$. Sometimes, to avoid confusion, we shall say “the open edge e ” instead of saying simply “the edge e .” Similarly, when we shall say “the closed edge e ,” we shall mean \bar{e} . The use of “open face” and “closed face” is similar.

1.2. Initial decomposition. We begin with an initial decomposition $\mathcal{T}_0 = \{\Lambda_0, \Lambda_1, \dots, \Lambda_N\}$, $\bar{\Omega} = \cup_{j=0}^N \bar{\Lambda}_j$ with the following properties:

- (1) Each Λ_j is a polyhedral domain;
- (2) If $\bar{\Lambda}_j$ contains a vertex of Ω , then it is a tetrahedron, it contains no other vertex of Ω , and it intersects at most one open edge of Ω ;
(A region of this type will be called a *type I* region if it does not intersect any open edge of Ω and it will be called a *type II* region if it intersects exactly one open edge of Ω , see Figure 1.1.)
- (3) If $\bar{\Lambda}_j$ contains no vertex of Ω , but intersects an open edge e of Ω , then Λ_j is a triangular prism with basis triangles with acute angles (*i.e.*, $< \pi/2$) and with three edges parallel to e ;
(A region of this type will be called a *type III* region.)
- (4) The three parallel edges of the prisms are perpendicular to the bases.
- (5) $\Lambda_0 = \Omega \setminus (\cup_{j \geq 1} \bar{\Lambda}_j)$ and is the only region whose closure does not intersect any closed edge of Ω ;
- (6) We assume that our partition is *conforming*, in the usual sense that if $\bar{\Lambda}_j$ and $\bar{\Lambda}_k$ have more than a point in common, then their intersection is a common edge or a common face of both regions.

Let us make now some comments on the initial decomposition \mathcal{T}_0 of Ω satisfying the conditions that we have just explained. There are four types of regions Λ_j in $\mathcal{T}_0 = \{\Lambda_0, \Lambda_1, \dots\}$: type I, II, III, and Λ_0 . The region Λ_0 is the only one that is not a tetrahedron or a triangular prism. The regions of type I and II consist only of tetrahedra, and the regions of type III consist only of prisms. Condition 4 is introduced only for convenience. One could easily adapt the proofs below to the case when the bases are parallel, but not necessarily perpendicular to the three parallel edges. With some additional work, one could consider even more general prisms, which is convenient in implementation. However, we shall leave this for later work. Other conditions can probably be relaxed as well (like Condition 2, for example, see also below).

In Section 4, we shall construct a sequence of decompositions \mathcal{T}_k of Ω defined in part by the initial decomposition \mathcal{T}_0 and a parameter $\kappa \in (0, 1/2]$. The decomposition \mathcal{T}_k , $k \geq 1$, will consist only of prisms and tetrahedra (unlike \mathcal{T}_0 , which also includes the region Λ_0 that is neither a tetrahedron nor a prism). Our sequence of meshes \mathcal{T}'_n , $n \geq 1$, will be obtained from the decompositions \mathcal{T}_n by further dividing the prisms into tetrahedra in a precise way that will be explained.

Assume Ω is the regular tetrahedron $A_1A_2A_3A_4$ represented in Figure 1.1. A picture of a possible choice of an initial decomposition of $\Omega = A_1A_2A_3A_4$ is obtained from the decomposition of Figure 1.1 by deforming the points on the edge of Ω to obtain segments perpendicular to the edge (so that Condition 4 is satisfied). This can be achieved for the regular tetrahedron $\Omega = A_1A_2A_3A_4$ as follows. We first continuously move the point D_1 on $[A_1A_2]$ such that the plane determined by $[D_1D_{13}D_{14}]$ is perpendicular on $[A_1A_2]$. We then similarly reposition D_3 and D_4 and the other points close to A_1, A_3 , and A_4 . These changes preserve the topology of our tetrahedralization.

Let us also notice that every vertex of Ω is contained in at least four small tetrahedra of our initial decomposition, see Figure 1.1. This is needed because no tetrahedron of our initial decomposition is allowed to intersect two open edges, as in [5, 6]. In implementation, this condition may turn out not to be necessary in general, although it will be necessary at the reentrant corners.

The union of the closed tetrahedra $\bar{\Lambda}_j$ containing a given vertex P of Ω will form a neighborhood \mathcal{V}_P of P in $\bar{\Omega}$. Similarly, the union of the closed tetrahedra and the closed prisms $\bar{\Lambda}_j$ containing part of a given edge e of Ω will form a neighborhood \mathcal{V}_e of e in $\bar{\Omega}$. These neighborhoods are analogues of the regions of influence in [24, 41], for example. In the following, by

$$(3) \quad \mathcal{V}_P := \cup_{P \in \bar{\Lambda}_j} \bar{\Lambda}_j \quad \text{and} \quad \mathcal{V}_e := \cup_{e \cap \bar{\Lambda}_j \neq \emptyset} \bar{\Lambda}_j$$

we shall always denote the neighborhoods introduced above.

In what follows, occasionally it will be convenient to use also some unbounded domains. We are especially interested in dihedral angles and polyhedral cones. Let D_α be defined, using cylindrical coordinates (r, θ, z) , by $D_\alpha := \{0 < \theta < \alpha\} \subset \mathbb{R}^3$. In general, a *dihedral angle* $D \subset \mathbb{R}^3$ is a set obtained from D_α by orthogonal transformations (rotations and translations). Let $\omega \subset S^2$ be a domain of the unit sphere $S^2 \subset \mathbb{R}^3$. Then we define $C_\omega := \{tx', x' \in \omega, t > 0\} \subset \mathbb{R}^3$. We shall say that C_ω is a *polyhedral cone* if it coincides with a bounded polyhedral domain in a neighborhood of the origin and is *not* a dihedral angle. In general, a *polyhedral cone* is a set obtained from a polyhedral cone of the form C_ω by

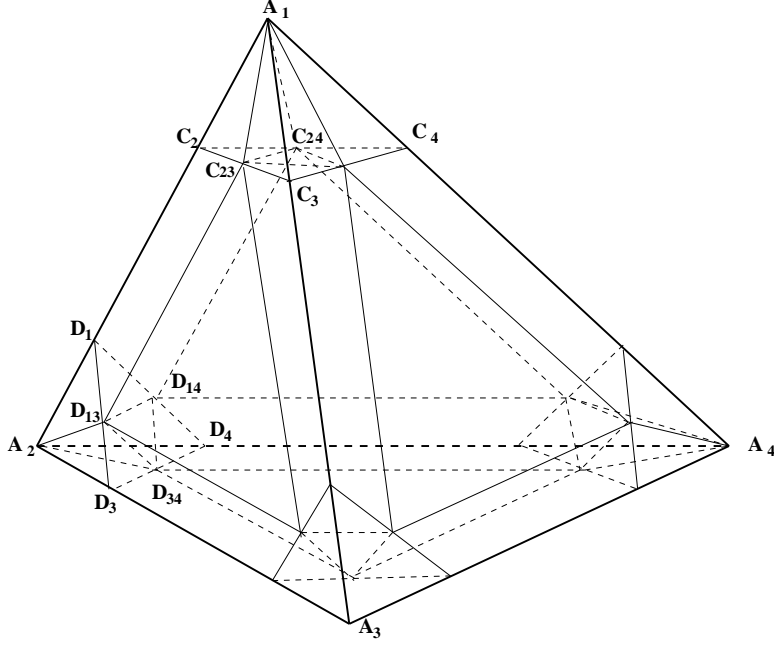


FIGURE 1.1. Initial decomposition.

orthogonal transformations. (So, in our terminology, a dihedral angle is *not* a polyhedral cone.)

1.3. Isotropic Sobolev spaces. We now introduce three classes of Sobolev spaces on our bounded polyhedral domain Ω . Although we are interested mainly in the definition of these spaces for *bounded polyhedral domains*, for technical reasons, it is convenient to define them in a slightly greater generality on *any open* set $V \subset \mathbb{R}^3$.

We shall use the standard notation for partial derivatives, namely $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. If $n = 3$, we shall also write $\partial^\alpha = \partial_x^{\alpha_x} \partial_y^{\alpha_y} \partial_z^{\alpha_z}$. The usual Sobolev spaces on our open set V are then

$$H^m(V) = \{u : V \rightarrow \mathbb{C}, \partial^\alpha u \in L^2(V), |\alpha| \leq m\}.$$

(If m is small, we need to consider equivalence classes of such functions.)

We now introduce two local Sobolev spaces. The space $H_{\text{loc}}^m(V)$ is the space of (equivalence classes of) functions $f : V \rightarrow \mathbb{C}$ such that $f \in H^m(\omega)$ for every open subset ω with compact closure $\bar{\omega} \subset V$. Similarly, for a bounded polyhedral domain, $H_{\text{sl}}^m(\Omega)$ is the space of (equivalence classes of) functions $f : \Omega \rightarrow \mathbb{C}$ such that $f \in H^m(\omega)$ for every open subset ω with the property that $\bar{\omega}$ does not intersect any edge of Ω . (The index “sl” means “semi-local.”)

Let us denote by $\vartheta(x)$ the distance from $x \in V$ to the set of singular points of the boundary of V . Then the *Babuška–Kondratiev* spaces $\mathcal{K}_a^m(V)$ are defined by

$$(4) \quad \mathcal{K}_a^m(V) := \{u, \vartheta^{|\alpha|-a} \partial^\alpha u \in L^2(V), |\alpha| \leq m\}, \quad m \in \mathbb{Z}_+, a \in \mathbb{R}.$$

For a bounded polyhedral domain, $\vartheta(x)$ is the distance from x to the union of the closed edges of Ω . Then the Babuška–Kondratiev spaces satisfy $\mathcal{K}_a^m(\Omega) \subset H_{\text{sl}}^m(\Omega) \subset H_{\text{loc}}^m(\Omega)$.

We now introduce some functions that will play a crucial role in what follows. Let $\text{dist}(A, B)$ be the Euclidean distance between two sets A and B . Then we shall denote

$$(5) \quad \rho_P(p) := \text{dist}(p, P) \quad \text{and} \quad r_e(p) := \text{dist}(p, e).$$

(Note that r_e is the distance to the closure of e and *not* the distance to the line containing e , as it was the case in [4], for example.)

With this notation, we have the following simple alternative description of the Babuška–Kondratiev spaces. This description will not be used, however, in what follows.

Remark 1.3. Recall from the previous subsection (Equation 3) the neighborhood \mathcal{V}_P of the generic vertex P (defined as the union of the closed tetrahedra $\bar{\Lambda}_j$ containing P) and the neighborhood \mathcal{V}_e of the generic edge e (defined as the union of the closed tetrahedra or prisms $\bar{\Lambda}_j$ adjacent to e). Then

$$(6) \quad u \in \mathcal{K}_a^m(\Omega) \Leftrightarrow \begin{cases} u \in H_{\text{sl}}^m(\Omega) \\ \rho_P^{|\alpha|-a} \partial^\alpha u \in L^2(\mathcal{V}_P), \quad \text{and} \\ r_e^{|\alpha|-a} \partial^\alpha u \in L^2(\mathcal{V}_e), \quad |\alpha| \leq m \end{cases}$$

In particular, $u \in \mathcal{K}_a^m(\Omega)$ implies that $u \in H^m(\Lambda_0)$, where $\Lambda_0 = \Omega \setminus (\cup_{j \geq 1} \bar{\Lambda}_j)$, as before, because the closure of Λ_0 does not intersect any edge of Ω .

Our Babuška–Kondratiev spaces depend on only one real parameter (excluding the smoothness index m). It is possible to define similar spaces depending on as many parameters as the total number of vertices and edges, as in [24] or in [43]. This may lead to improved regularity and well-posedness results, and hence to improved numerical methods. The theoretical treatment of the general case is, however, very similar to that considered in this paper, so we shall not consider separately this more general case.

Let $C_\omega := \{tx', x' \in \omega, t > 0\} \subset \mathbb{R}^3$ be a polyhedral cone. (Recall that this means that C_ω coincides with a bounded polyhedral domain in a neighborhood of the origin and is *not* a dihedral angle.) Then $\vartheta(x)$ is the distance from x to the edges of Ω and it satisfies $\vartheta(tx) = t\vartheta(x)$. Let $\alpha_t : C_\omega \rightarrow C_\omega$, $\alpha_t(x) = tx$, be the dilation by t . Also, we shall denote by $\alpha_t(f) = f \circ \alpha_t$ for any function defined on C_ω . This definition is changed accordingly for polyhedral cones with the vertex not necessarily at the origin.

Lemma 1.4. *Let $\Omega = C_\omega$. Then $\|\alpha_t(f)\|_{\mathcal{K}_a^m(\Omega)} = t^{a-3/2} \|f\|_{\mathcal{K}_a^m(\Omega)}$.*

Proof. The proof is by direct calculation, as in [20]. □

Let $D_\alpha := \{0 < \theta < \alpha\}$ be the dihedral angle considered above. Then $\vartheta(p) = r = \sqrt{x^2 + y^2}$ is the distance from $p = (x, y, z)$ to the edge $x = y = 0$ of D . The above lemma remains valid, but it will be more important for us to consider the translation invariance of the space $\mathcal{K}_a^m(D_\alpha)$ and of its norm with respect to translations along the edge of D_α .

1.4. Weights and differential operators. It is elementary (and easy, see [4] for details) to check that if P is an order m differential operator P , then

$$(7) \quad P : \mathcal{K}_a^s(\Omega) \rightarrow \mathcal{K}_{a-m}^{s-m}(\Omega)$$

is continuous for $s \geq m$. In particular, any first order derivative of a function $u \in \mathcal{K}_a^s(\Omega)$ will be in $\mathcal{K}_{a-1}^{s-1}(\Omega)$, a simple fact that will be used many times without further comment. Also, we shall need the easy fact that

$$(8) \quad \mathcal{K}_a^s(\Omega) \subset \mathcal{K}_{a'}^{s'}(\Omega), \quad \text{for } s \geq s' \text{ and } a \geq a',$$

with a continuous inclusion map. The above two equations show that, at a formal level, the Babuška–Kondratiev spaces behave in a somewhat similar way with respect to the two indices m and a .

We now recall the definition of $r_\Omega : \bar{\Omega} \rightarrow [0, \infty)$, the smoothed distance to the edges considered in [3, 4, 21, 24, 29], for example. Let $\tilde{r}_e : \bar{\Omega} \rightarrow [0, \infty)$ be a continuous function satisfying the following properties:

- (i) $r_e(x) \leq \tilde{r}_e(x) \leq 2r_e(x)$, where $r_e(x)$ is the distance from x to the edge e , as in Equation (5);
- (ii) \tilde{r}_e is smooth outside the edge e (a property not satisfied in general by r_e);
- (iii) $r_e(x) = \tilde{r}_e(x)$ if the foot P of the perpendicular from x to the line containing e is inside e ;
- (iv) If the foot P of the perpendicular from x to the line containing e is outside e , let A be the vertex of e closest to P and α_t the dilation of ratio t and center A . Then we also require that $\tilde{r}_e(\alpha_t(x)) = t\tilde{r}_e(x)$.

Then we define:

$$(9) \quad r_\Omega := \left(\prod_P \rho_P \right) \left(\prod_{e=[AB]} \frac{\tilde{r}_e(x)}{\rho_A(x)\rho_B(x)} \right),$$

the first product being over all vertices P of Ω and the second product being over all edges $e = [AB]$ of Ω . Some of the main properties of the function r_Ω are that it is smooth outside the edges and that

$$(10) \quad \vartheta(x)/K \leq r_\Omega(x) \leq K\vartheta(x)$$

for some large constant K . Close to an edge but away from the vertices, the Equation (10) follows from $r_e(x) \leq \tilde{r}_e(x) \leq 2r_e(x)$ (the relation (i) above). Close to a vertex P , the Equation (10) follows from the dilation invariance of $\rho_P \prod(\tilde{r}_e/\rho_P)$, where the product is taken over all edges e containing P . This leads to the following alternative description of the Babuška–Kondratiev spaces:

$$\mathcal{K}_a^m(\Omega) := \{u, r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), \quad |\alpha| \leq m\}, \quad m \in \mathbb{Z}_+, \quad a \in \mathbb{R}.$$

Let us choose next $\rho_\Omega : \bar{\Omega} \rightarrow [0, \infty)$ to be smooth and > 0 except at the vertices and to satisfy

$$(11) \quad \rho_\Omega(x) = \rho_P(x) \quad \text{if } x \in \mathcal{V}_P$$

(i.e., if x is close enough to P , then $\rho_\Omega(x)$ is the distance from x to P). We have $K^{-1}\rho_\Omega \leq \prod_P \rho_P(x) \leq K\rho_\Omega$, for some constant $K > 0$. We shall call ρ_Ω be a “smoothed distance to the vertices of Ω .” (The reason we do not chose $\prod_P \rho_P(x)$ for ρ_Ω is that we want ρ_Ω to be homogeneous of degree one close to the vertices.)

Then we have the following lemma that is similar to the results in [20] and is proved as in that paper.

Lemma 1.5. *Let $a, b \in \mathbb{R}$ and let P be a differential operator of order k with smooth coefficients.*

- (i) The multiplication by r_Ω^b gives rise to an isomorphism $\mathcal{K}_a^m(\Omega) \rightarrow \mathcal{K}_{a+b}^m(\Omega)$. In particular, we have that $r_\Omega^b \mathcal{K}_a^m(\Omega) = \mathcal{K}_{a+b}^m(\Omega)$.
- (ii) The map $P : \mathcal{K}_a^m(\Omega) \rightarrow \mathcal{K}_{a-k}^{m-k}(\Omega)$ is well defined and continuous.
- (iii) Let $\Omega = C_\omega$ be a cone centered at the origin and $L_b(u) = \rho^{-b}(P\rho^b u) - Pu$. Then $L_b : \mathcal{K}_a^m(\Omega) \rightarrow \mathcal{K}_{a-k}^{m-k}(\Omega)$ is a family of bounded maps depending continuously on b in the norm of bounded operators between these spaces.

Proof. The proofs of (i) and (ii) are by a simple, direct calculation, see [4, 20] for details.

To prove (iii), assume first that $P = \partial_x$. We notice then that $L_b(u) = bx\rho^{-2}u$ (a multiplication operator). The inequality $\partial^\beta(x\rho^{-2}) \leq C\rho^{-|\beta|-1}$ shows that the function $r_\Omega^{|\beta|+1}\partial^\beta(x\rho^{-2})$ is bounded for any multiindex β . The identity

$$(12) \quad \partial^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta u \partial^{\alpha-\beta} v$$

then gives that multiplication with $x\rho^{-2}$ defines a bounded operator $\mathcal{K}_a^m(\Omega) \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega)$. In general, assume that $P = \partial^\gamma$, with $\gamma = (\gamma_x, \gamma_y, \gamma_z)$ arbitrary. It is enough to show that $L_b + P = \rho^{-b}P\rho^b$ depends continuously on b . We then proceed by induction, keeping track of the orders of operators, using also (ii) and the identity

$$\rho^{-b}\partial^\gamma\rho^b = (\rho^{-b}\partial_x\rho^b)^{\gamma_x}(\rho^{-b}\partial_y\rho^b)^{\gamma_y}(\rho^{-b}\partial_z\rho^b)^{\gamma_z}.$$

The result follows since the composition of two continuous families of bounded operators is again continuous. \square

1.5. Anisotropic Sobolev spaces. As in [5, 6, 24], we shall also need to consider certain *anisotropic Sobolev spaces* $\mathcal{D}_a^m(\Omega)$ that we define in this section. Our spaces seem to be slightly different from the ones defined before. We define the spaces $\mathcal{D}_a^m(\Omega)$, $m \in \mathbb{Z}$, $m \geq 1$, $a \in \mathbb{R}$, by induction. We also define first these spaces for a dihedral angle D_α , then for a polyhedral cone C_ω , and then in general for a polyhedral domain Ω .

Let $D_\alpha := \{0 < \theta < \alpha\}$ be the dihedral angle considered also earlier. Then $\vartheta(p) = r_{D_\alpha}(p) = r(p) = \sqrt{x^2 + y^2}$ is the distance from $p = (x, y, z)$ to the edge $x = y = 0$ of D . We let

$$(13) \quad \mathcal{D}_a^m(D_\alpha) := \{u \in \mathcal{K}_a^m(D_\alpha), \partial_z u \in \mathcal{D}_a^{m-1}(D_\alpha)\}, \quad \mathcal{D}_a^1(D_\alpha) := \mathcal{K}_a^1(D_\alpha).$$

Thus the spaces $\mathcal{D}_a^1(D_\alpha)$ are, in fact, independent of a . We endow the space $\mathcal{D}_a^m(D_\alpha)$, $m \geq 2$, with the norm

$$(14) \quad \|u\|_{\mathcal{D}_a^m(D_\alpha)}^2 := \|u\|_{\mathcal{K}_a^m(D_\alpha)}^2 + \|\partial_z u\|_{\mathcal{D}_a^{m-1}(D_\alpha)}^2.$$

Note that if $u \in \mathcal{K}_a^m(D_\alpha)$, then we can only say that $\partial_z u \in \mathcal{K}_{a-1}^{m-1}(D_\alpha)$, so the condition $\partial_z u \in \mathcal{D}_a^{m-1}(D_\alpha)$ in the definition of the space D_α is non-trivial. The space $\mathcal{D}_a^m(D_\alpha)$ and its norm are invariant with respect to translations parallel to the z -axis. This definition is changed accordingly for dihedral angles not passing through the origin.

The definition of the norm on $\mathcal{D}_a^m(D_\alpha)$, Equation (14) gives

$$(15) \quad \|u\|_{\mathcal{D}_a^m(D_\alpha)}^2 := \|u\|_{\mathcal{K}_a^m(D_\alpha)}^2 + \|\partial_z u\|_{\mathcal{K}_{a-1}^{m-1}(D_\alpha)}^2 + \dots \\ + \|\partial_z^{m-2} u\|_{\mathcal{K}_a^2(D_\alpha)}^2 + \|\partial_z^{m-1} u\|_{\mathcal{K}_a^1(D_\alpha)}^2.$$

(Note that the last term is slightly different from the others.)

We shall denote by $\alpha_\perp = (\alpha_x, \alpha_y) \in \mathbb{Z}_+^2$ a multi-index in the first variable, and by $\partial^{\alpha_\perp} = \partial_x^{\alpha_x} \partial_y^{\alpha_y}$. We denote $|\alpha_\perp| = \alpha_x + \alpha_y$, as before. Also, we shall denote by \mathcal{F}_K the space of functions that vanish outside the set K . The following lemma provides an alternative definition of the norm on the spaces \mathcal{D} close to the edges.

We have the following.

Lemma 1.6. *Let $a \geq 0$ and $\chi : D_\alpha \rightarrow C$ be a smooth function. Define*

$$\|\chi\|_{m,\infty} = \sum_{k=0}^m \sum_{|\alpha_\perp| \leq m-k} \|r^{|\alpha_\perp|} \partial^{\alpha_\perp} \partial_z^k \chi\|_{L^\infty(D_\alpha)}.$$

Then, for any compact set $K \subset \mathbb{R}^3$, there exists a constant $C > 0$ such that $\|\chi u\|_{\mathcal{D}_{a+1}^m(D_\alpha)} \leq C \|\chi\|_{m,\infty} \|u\|_{\mathcal{D}_{a+1}^m(D_\alpha)}$ for all $u \in \mathcal{D}_{a+1}^m(D_\alpha) \cap \mathcal{F}_K$. The constant C depends only on K and m .

Proof. The definitions of the norms and Equation (12) give $\|\chi u\|_{\mathcal{K}_{a+1}^m(D_\alpha)} \leq C \|\chi\|_{m,\infty} \|u\|_{\mathcal{K}_{a+1}^m(D_\alpha)}$, if u has support in K . Then our relation follows from Equations (15) and (12) and the relation $a \geq 0$. (The condition that u has support in K is needed in order to bound $r^{|\alpha|}$ by a multiple of $r^{|\alpha_\perp|}$. The relation $a \geq 0$ is needed in order to compare the terms containing the \mathcal{K}_1^1 -norms with the terms containing the \mathcal{K}_{a+1}^{m+1} -norms.) \square

We shall need the following explicit description of the norm on the spaces $\mathcal{D}_a^m(D_\alpha)$. This description is similar to the one in [24], and will be used for our interpolation estimates on meshes with thin tetrahedra near the edges in Section 6.

Lemma 1.7. *Let $r(p) = \sqrt{x^2 + y^2}$ be the distance from $p = (x, y, z)$ to the edge $x = y = 0$ of the dihedral angle $D_\alpha = \{(r, \theta, z), 0 < \theta < \alpha\}$ and define*

$$\begin{aligned} \|u\|_{\mathcal{D}_a^m(D_\alpha)}^2 &= \sum_{k=0}^{m-2} \sum_{|\alpha_\perp| \leq m-k} \|r^{|\alpha_\perp|-a} \partial^{\alpha_\perp} \partial_z^k u\|_{L^2(D_\alpha)}^2 \\ &+ \|r^{-1} \partial_z^{m-1} u\|_{L^2(D_\alpha)}^2 + \|\partial_x \partial_z^{m-1} u\|_{L^2(D_\alpha)}^2 + \|\partial_y \partial_z^{m-1} u\|_{L^2(D_\alpha)}^2 + \|\partial_z^m u\|_{L^2(D_\alpha)}^2. \end{aligned}$$

Let K be a compact set and assume $a \leq 2$. Then $u \rightarrow \|u\|_{\mathcal{D}_a^m(D_\alpha)}$ defines a norm on $\mathcal{D}_a^m(D_\alpha) \cap \mathcal{F}_K$ that is equivalent to the original norm on $\mathcal{D}_a^m(D_\alpha) \cap \mathcal{F}_K$. More precisely, we have $\|u\|_{\mathcal{D}_a^m(D_\alpha)} \leq \|u\|_{\mathcal{D}_a^m(D_\alpha)} \leq C \|u\|_{\mathcal{D}_a^m(D_\alpha)}$ on $\mathcal{D}_a^m(D_\alpha) \cap \mathcal{F}_K$.

Proof. The result then follows by expanding the definitions of the norms on the spaces $\mathcal{K}_a^m(D_\alpha)$, $\mathcal{K}_a^{m-1}(D_\alpha)$, ..., $\mathcal{K}_a^2(D_\alpha)$, and $\mathcal{K}_1^1(D_\alpha)$ using Equation (4). To obtain the exact form given in the lemma, we also need to use also that $\|r^{l-a} \partial^{\alpha} u\|_{L^2(D_\alpha)} \leq C \|r^{k-a} \partial^{\alpha} u\|_{L^2(D_\alpha)}$, if $0 \leq k \leq l \leq m$, with a constant $C > 0$ that is independent of k and l .

The condition $a \leq 2$ is needed to compare the norms of the terms $r^{|\alpha|-a} \partial^{\alpha} \partial_z^k u$, $k \leq m-2$, with the norms of the corresponding terms in $\|\partial_z^{m-1} u\|_{\mathcal{K}_1^1}$. \square

Now let $C_\omega = \{tx', t > 0, x' \in \omega \subset S^2\}$ be as above. We assume that ω is such that C_ω is not a dihedral angle, that is, we assume that $\mathcal{C} := C_\omega$ is a polyhedral cone (ω and $\mathcal{C} := C_\omega$ will remain fixed throughout the following discussion). Let $\rho_0(x)$ denote the distance from x to the origin (=the vertex of \mathcal{C}). Then we let

$$(16) \quad \mathcal{D}_a^1(\mathcal{C}) := \rho_0^{a-1} \mathcal{K}_1^1(\mathcal{C}) = \{\rho_0^{a-1} v, v \in \mathcal{K}_1^1(\mathcal{C})\},$$

with norm $\|u\|_{\mathcal{D}_a^1(\mathcal{C})} := \|u/\rho_0^{a-1}\|_{\mathcal{K}_a^1(\mathcal{C})}$.

In general, for $m \geq 2$, let $\rho\partial_\rho = x\partial_x + y\partial_y + z\partial_z$ be the infinitesimal generator of dilations. That is $\rho\partial_\rho(f) = \lim_{t \rightarrow 0} t^{-1}(\alpha_t(f) - f)$. Then, for $m \geq 2$, we define by induction

$$(17) \quad \mathcal{D}_a^m(\mathcal{C}) := \{u \in \mathcal{K}_a^m(\mathcal{C}), \rho\partial_\rho(u) \in \mathcal{D}_a^{m-1}(\mathcal{C})\}, \quad \mathcal{C} = C_\omega,$$

with norm

$$(18) \quad \|u\|_{\mathcal{D}_a^m(\mathcal{C})}^2 := \|u\|_{\mathcal{K}_a^m(\mathcal{C})}^2 + \|\rho\partial_\rho(u)\|_{\mathcal{D}_a^{m-1}(\mathcal{C})}^2.$$

We again notice that for $v \in \mathcal{K}_a^m(\mathcal{C})$, we only have $\rho\partial_\rho v \in \mathcal{K}_a^{m-1}(\mathcal{C})$, in general, so the condition $\rho\partial_\rho u \in \mathcal{D}_a^{m-1}(\mathcal{C})$ defining the space $\mathcal{D}_a^m(\mathcal{C})$ is non-trivial. The spaces $\mathcal{D}_a^m(\mathcal{C})$ are defined similarly for any polyhedral cone \mathcal{C} .

Before we define the anisotropic Sobolev spaces $\mathcal{D}_a^m(\Omega)$ in general, it is useful to discuss some properties of the various Sobolev spaces introduced so far. We begin with the following analogue of the Lemma 1.4, which will also justify in part our definition of the spaces $\mathcal{D}_a^m(C_\omega)$.

Lemma 1.8. *Let $\Omega = C_\omega$. Then $\|\alpha_t(f)\|_{\mathcal{D}_a^m(\Omega)} = t^{a-3/2}\|f\|_{\mathcal{D}_a^m(\Omega)}$.*

Proof. For $m = 1$ this is a direct calculation based on Lemma 1.4 and the definitions. For the other values of m it follows by induction, using again Lemma 1.4. \square

Let \mathcal{C} be a polyhedral cone and e be an arbitrary edge of \mathcal{C} . Let D_e be the dihedral angle that has the edge e and two faces in common with \mathcal{C} .

Recall that \mathcal{F}_K denotes the space of functions with support in the set K .

Lemma 1.9. (1) *Let K be a compact set that does not intersect any edge of \mathcal{C} with the exception of e . Then $\mathcal{D}_a^m(\mathcal{C}) \cap \mathcal{F}_K = \mathcal{D}_a^m(D_e) \cap \mathcal{F}_K$ and the norms of the spaces $\mathcal{D}_a^m(\mathcal{C})$ and $\mathcal{D}_a^m(D_e)$ are equivalent on this common subspace.*

(2) *Let K be a compact set that does not intersect any edge of \mathcal{C} (no exceptions). Then $\mathcal{D}_a^m(\mathcal{C}) \cap \mathcal{F}_K = \mathcal{D}_a^m(D_e) \cap \mathcal{F}_K = H^m(\mathcal{C}) \cap \mathcal{F}_K$ and the norms of the spaces $\mathcal{D}_a^m(\mathcal{C})$, $\mathcal{D}_a^m(D_e)$, and $H^m(\mathcal{C})$ are equivalent on this common subspace.*

Proof. This follows from the definitions. \square

We are ready now to define the spaces $\mathcal{D}_a^m(\Omega)$ for any bounded polyhedral domain Ω by “gluing” the spaces \mathcal{D}_a^m already defined using a natural partition of unity ϕ_j , in which we associate one function ϕ_j to each vertex, one to each edge, and one to the interior. Recall the canonical neighborhoods \mathcal{V}_P and \mathcal{V}_e of a vertex P or of an edge e of Ω (Equation 3). For each vertex P , we shall denote by C_P the polyhedral cone spanned by the vertex P of Ω , that is $C_P = \{P + t(x' - P), t > 0, x' \in \Omega\}$. Similarly, for each edge e , we shall denote by D_e the dihedral angle spanned by e , that is, the dihedral angle that has e on the edge and contains two faces of Ω .

Recall that, throughout this paper, Ω denotes a bounded polyhedral domain in three dimensions and that $\mathcal{T}_0 = \{\Lambda_0, \Lambda_1, \dots\}$ is a finite decomposition of Ω into polyhedral domains with $\bar{\Lambda}_0$ not intersecting any edge of Ω and all the other regions Λ_j consisting of tetrahedra and straight triangular prisms. Let N_e be the number of edges of Ω and N_v be the number of vertices of Ω .

Let $N := N_e + N_v$ and ϕ_j , $j = 0, 1, \dots, N$, be a partition of unity consisting of smooth functions such that:

- (i) $\phi_0 = 0$ in a neighborhood of the edges;

- (ii) $\phi_0 = 1$ on $\Lambda_0 \in \mathcal{T}_0$;
- (iii) Assume the vertices of Ω have been denoted P_1, \dots, P_{N_v} , then $\phi_j = 1$ in a neighborhood of P_j and $\phi_j = 0$ outside \mathcal{V}_{P_j} , $j = 1, \dots, N_v$;
- (iv) each of the remaining functions ϕ_j (i.e., $j > N_v$) has support inside one of the sets \mathcal{V}_e , for some edge $e = e_j$ of Ω .

We shall write $C_j := C_{P_j}$ and $D_j = D_{e_j}$. Then $\text{supp } \phi_j \subset \bar{C}_j$ for $1 \leq j \leq N_v$ and $\text{supp } \phi_j \subset \bar{D}_j$ for $j > N_v$.

Definition 1.10. Let Ω be a bounded polyhedral domain. We define $\mathcal{D}_a^m(\Omega)$ as the space of functions $u \in \mathcal{K}_a^m(\Omega)$ such that $\phi_j u \in \mathcal{D}_a^m(C_j)$ for $1 \leq j \leq N_v$ and $\phi_j u \in \mathcal{D}_a^m(D_j)$ for $j > N_v$. We endow $\mathcal{D}_a^m(\Omega)$ with the norm

$$(19) \quad \|u\|_{\mathcal{D}_a^m(\Omega)}^2 := \|\phi_0 u\|_{H^m(\Omega)}^2 + \sum_{j=1}^{N_v} \|\phi_j u\|_{\mathcal{D}_a^m(C_j)}^2 + \sum_{j=N_v+1}^{N_v+N_e} \|\phi_j u\|_{\mathcal{D}_a^m(D_j)}^2.$$

Lemma 1.9 guarantees that the definition of the space $\mathcal{D}_a^m(\Omega)$ is independent of the choice of the partition of unity ϕ_j with the indicated properties (in particular, different choices of the partition of unity ϕ_j lead to equivalent norms).

If $S \subset \Omega$ is any open set, then we shall denote by $\|u\|_{\mathcal{D}_a^m(S)}$ the quantity that we obtain if, in the integrals defining $\|u\|_{\mathcal{D}_a^m(\Omega)}$, we replace \int_{Ω} with \int_S everywhere. In particular, for any edge e and any measurable function $u : \mathcal{V}_e \rightarrow \mathbb{C}$, we define $\|u\|_{\mathcal{D}_a^m(\mathcal{V}_e)}$ by restricting the integrals defining $\|u\|_{\mathcal{D}_a^m(D_e)}$ to \mathcal{V}_e , where D_e is the dihedral angle generated by e . Similarly, for any vertex P of Ω , we define $\|u\|_{\mathcal{D}_a^m(\mathcal{V}_P)}$ by restricting the integrals defining $\|u\|_{\mathcal{D}_a^m(C_P)}$ to \mathcal{V}_P , where C_P is the polyhedral cone with vertex at P generated by Ω . In particular, since $|u|_{H^1(S)} \leq \|u\|_{\mathcal{K}_{a+1}^m(S)}$, we obtain

$$(20) \quad |u|_{H^1(S)} \leq C_{\Omega} \|u\|_{\mathcal{D}_{a+1}^m(S)},$$

where $a \geq 0$, $m \geq 1$, and C_{Ω} depends only on Ω and not on the subset $S \subset \Omega$.

2. ESTIMATES FOR POISSON'S PROBLEM

In this section we derive estimates on our Poisson problem (1) ($-\Delta u = f$ in Ω , $u = 0$ on the boundary of Ω) in the anisotropic weighted Sobolev spaces $\mathcal{D}_{a+1}^{m+1}(\Omega)$ introduced in the previous section. Our results extend the previous (isotropic) estimates from [21]. Similar results can be found in [5, 6, 24]. In the following, we will replace Ω with a domain \mathbb{P} that will be either a dihedral angle, a polyhedral cone, or a bounded polyhedral domain.

2.1. Preliminary results. Let us first recall the following result ($\mathbb{Z}_+ = \{0, 1, \dots\}$).

Theorem 2.1. *Let $m \in \mathbb{Z}_+$ and let $\mathbb{P} \subset \mathbb{R}^3$ be a dihedral angle, a polyhedral cone, or a bounded polyhedral domain. Then there exists $\eta > 0$ such that the boundary value problem (1) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\mathbb{P})$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\mathbb{P})$ and this solution depends continuously on f .*

For $\mathbb{P} = \Omega$ a bounded polyhedron and $m \geq 1$, the above result was announced in [24]. See [19] or [21] for a proof (including the case $m = 0$, which is crucial in our applications). The case of the dihedral angle $D_{\alpha} := \{0 < \theta < \alpha\}$ was not treated explicitly in these papers, but can be dealt with in exactly the same way. Related well-posedness results were obtained in [9, 10, 16, 37, 38, 42, 43] and

many other papers. The singular functions approach is an alternative to the well-posedness results, and sometimes provide additional information, see [17, 18, 31, 28, 30, 41, 50] for example. There are many papers dealing with numerical issues related to polyhedral domains. In addition to the ones mentioned earlier, we would like to mention also [25, 32, 35, 15, 53, 54, 55, 60]. See [44] for results on mixed boundary value problems and an explicit treatment of the case of dihedral angles and polyhedral cones. See also [3, 4, 46, 47] for some recent extensions and some additional details of the proof. Weighted norm estimates were also used in [40] to obtain improved *a posteriori* estimates and hence improved adaptive procedures.

Since Δ maps $\mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \rightarrow \mathcal{K}_{a-1}^{m-1}(\mathbb{P})$ continuously, another way of formulating the above theorem is that the map

$$\Delta : \mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \rightarrow \mathcal{K}_{a-1}^{m-1}(\mathbb{P}), \quad m \in \mathbb{Z}_+, |a| < \eta,$$

is an isomorphism. Let $\rho_{\mathbb{P}}$ denoted the “smoothed distance to the vertices of \mathbb{P} ,” as in Equation (11). (So $\rho_{\mathbb{P}}(p)$ is the distance to the vertices of \mathbb{P} close to these vertices, and otherwise is a smooth positive function outside the vertices.) Using Lemma 1.5, we notice next that the family $\rho_{\mathbb{P}}^{-b} \Delta \rho_{\mathbb{P}}^b$ depends continuously on b in the topology of the space of continuous, linear maps $\mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \rightarrow \mathcal{K}_{a-1}^{m-1}(\mathbb{P})$. Hence we obtain that

$$(21) \quad \Delta : \rho_{\mathbb{P}}^b \mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \xrightarrow{\simeq} \rho_{\mathbb{P}}^b \mathcal{K}_{a-1}^{m-1}(\mathbb{P}), \quad m \in \mathbb{Z}_+,$$

is an isomorphism for $|a|$ and $|b|$ small enough, with inverse denoted $\Delta_{m,a,b}^{-1}$.

For $|a|$ and $|b|$ small enough, we also have that $u := \Delta_{m,a,b}^{-1} f$ is also the solution of the variational problem

$$(22) \quad B(u, v) := \int_{\mathbb{P}} \nabla u \cdot \nabla v dx = \int_{\mathbb{P}} f v dx,$$

for any $v \in \rho_{\mathbb{P}}^{-b} \mathcal{K}_{a+1}^1(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\}$. This follows from the fact that $B_{\rho}(u, v) = B(\rho_{\mathbb{P}}^b u, \rho_{\mathbb{P}}^{-b} v)$ also depends continuously on a and b and the fact that for $a = b = 0$ we indeed obtain that u is the solution of the variational problem (a well known, classical fact [23, 33]).

In the following, we shall use only the case $a \geq 0, b \geq 0$. Then $\rho_{\mathbb{P}}^b \mathcal{K}_{a-1}^{m-1}(\mathbb{P}) \subset \mathcal{K}_{-1}^{-1}(\mathbb{P})$ and $\rho_{\mathbb{P}}^b \mathcal{K}_{a+1}^{m+1}(\mathbb{P}) \subset \mathcal{K}_1^1(\mathbb{P})$. Let

$$\Delta_D : \mathcal{K}_1^1(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \xrightarrow{\simeq} \mathcal{K}_{-1}^{-1}(\mathbb{P}),$$

be the isomorphism obtained by setting $m = a = b = 0$ in Equation (21). Then $\Delta_{m,a,b}^{-1} f = \Delta_D^{-1} f$ if $f \in \rho_{\mathbb{P}}^b \mathcal{K}_{a-1}^{m-1}(\mathbb{P})$, and $m \in \mathbb{Z}_+$, and $a, b \geq 0$ are small. The point of this discussion is that in what follows, we can consider a single realization of Δ , namely Δ_D above, in the sense that whenever we write $\Delta v, v$ will be in the space $\mathcal{K}_1^1(\mathbb{P})$. Similarly, whenever we shall write $\Delta_D^{-1} f$, the function f will be in $\mathcal{K}_{-1}^{-1}(\mathbb{P})$.

2.2. Anisotropic regularity. From the discussion and the results above, we shall now derive the following theorem inspired by the results in [5, 6], and [24] (note however that we obtain slightly more regularity than in the previous results).

Theorem 2.2. *Let $m \in \mathbb{Z}_+$ and let $\mathbb{P} \subset \mathbb{R}^3$ be a dihedral angle, a polyhedral cone, or a bounded polyhedral domain. Let $f \in H^{m-1}(\mathbb{P})$ if $m \geq 1$ and $f \in \rho_{\mathbb{P}}^a H^{-1}(\mathbb{P})$ if $m = 0$. We assume that f has support in a fixed compact set $K \subset \mathbb{P}$. Then there exists $\eta \in (0, 1]$ such that the boundary value problem (1) has a unique solution*

$u \in \mathcal{D}_{a+1}^{m+1}(\mathbb{P})$. This solution depends continuously on f , for any $0 \leq a < \eta$ and any $m \in \mathbb{Z}_+$ and coincides with the solution of the variational problem (22).

Note that for $m \geq 1$, our assumptions on the data f are independent of a . This is the case that is needed in applications. For the proof, however, we shall also need the case $m = 0$. Also, it is crucial for the applications that we have in mind that $u \in \mathcal{D}_{a+1}^{m+1}(\mathbb{P})$, $a > 0$, rather than just $u \in \mathcal{D}_1^{m+1}(\mathbb{P})$. The values of a for which this can be achieved ($a \in [0, \eta)$, for some $\eta > 0$), will depend, however, on the geometry of the domain \mathbb{P} . The results stated in [24] can give some estimates on η .

Proof. The proof of this theorem is rather long, so we shall divide it into three parts: the case of a dihedral angle, the case of polyhedral cone, and the general case. The first two cases will be proved by induction using Theorem 2.1. We shall use Equation (7) and (8) repeatedly without further comment. Also, we shall use that

$$(23) \quad H_0^1(\Omega) := H^1(\Omega) \cap \{u|_{\partial\Omega} = 0\} = \mathcal{K}_1^1(\Omega) \cap \{u|_{\partial\Omega} = 0\}$$

for any *bounded* polyhedral domain Ω , by the weighted Friedrichs-Poincaré inequality results of [21].

Step 1. $\mathbb{P} = D_\alpha$, a dihedral angle. We shall proceed by induction on m . If $m \geq 1$, we shall prove our statement under the more general assumption that $\partial_z^j f \in \mathcal{K}_{a-1}^{m-1-j}(D_\alpha)$ for $j \leq m-1$ and $\partial_z^m f \in \mathcal{K}_{-1}^{-1}(D_\alpha)$. Indeed, these assumptions are satisfied if f is as in the statement of our theorem (because $a \leq 1$ and f has support in a compact set.)

Let first $m = 0$. Then our assumption is that $f \in H^{-1}(D_\alpha)$. We have

$$H^{-1}(D_\alpha) := H_0^1(D_\alpha)^* = (\mathcal{K}_1^1(D_\alpha) \cap \{u|_{\partial D_\alpha} = 0\})^* =: \mathcal{K}_{-1}^{-1}(D_\alpha).$$

We need to prove that the equation $-\Delta u = f$, $u|_{\partial D_\alpha} = 0$ has a unique solution¹ $u \in \mathcal{D}_{a+1}^1(D_\alpha) = \mathcal{K}_1^1(D_\alpha)$ and that this solution depends continuously on f . Indeed, in view of $H^{-1}(D_\alpha) = \mathcal{K}_{-1}^{-1}(D_\alpha)$, this follows from Theorem 2.1. This completes the proof of the initial case ($m = 0$) of our induction.

Let us consider now the induction step. Let $m \geq 1$. Then $f \in \mathcal{K}_{a-1}^{m-1}(D_\alpha)$. We need to show that the equation $-\Delta u = f$, $u|_{\partial D_\alpha} = 0$ (Equation (1)) has a unique solution $u \in \mathcal{D}_{a+1}^{m+1}(D_\alpha)$, which depends continuously on f . By the definition of the spaces $\mathcal{D}_{a+1}^{m+1}(D_\alpha)$, this means that we need to show that

- (1) Equation (1) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(D_\alpha)$ and that this solution satisfies $\partial_z u \in \mathcal{D}_{a+1}^m(D_\alpha)$.
- (2) u and $\partial_z u$ depend continuously on f in the topology of the indicated spaces.

Theorem 2.1 gives that we can take $u := \Delta_D^{-1}(f)$ and that

$$\mathcal{K}_{a-1}^{m-1}(D_\alpha) \ni f \rightarrow u := \Delta_D^{-1}(f) \in \mathcal{K}_{a+1}^{m+1}(D_\alpha)$$

is continuous. We then need to show that

$$H^{m-1}(D_\alpha) \ni f \rightarrow \partial_z u := \partial_z(\Delta_D^{-1}(f)) \in \mathcal{D}_{a+1}^m(D_\alpha)$$

is well defined (*i.e.*, $\partial_z u \in \mathcal{D}_{a+1}^m(D_\alpha)$) and is continuous. This follows from

$$(24) \quad \partial_z \Delta_D^{-1}(f) = \Delta_D^{-1}(\partial_z f),$$

¹Note that we are not claiming that there are no other solutions u in other spaces, in fact, this equation has infinitely many solutions in $\mathcal{K}_b^1(D_\alpha)$, provided that b is small enough ($b \ll 0$).

which we shall prove in a moment. Assume therefore Equation (24). Then we have $\partial_z f \in H^{m-2}(D_\alpha) \subset \mathcal{K}_{a-1}^{m-2}(D_\alpha)$. By the induction hypothesis and the claimed relation (Equation 24), $\partial_z \Delta_D^{-1}(f) = \Delta_D^{-1}(\partial_z f) \in \mathcal{D}_{a+1}^m(D_\alpha)$ and depends continuously on f .

The remaining of this first step is to prove Equation (24). To this end, we shall use the standard techniques for proving the regularity results for boundary value problems on smooth, bounded domains [1, 2, 33, 34, 57] (in fact, what we need to prove is easier in our case). Let $\beta_t(x, y, z) = (x, y, z + t)$ and $\beta_t(u) := u \circ \beta_t$. Then $\beta_t(\vartheta) = \vartheta$ and hence the translation β_t maps $\mathcal{K}_a^m(D_\alpha)$ to itself isometrically. Similarly, β_t maps $H^{m-1}(D_\alpha)$ to itself continuously and we have

$$t^{-1}(\beta_t(f) - f) \rightarrow \partial_z f \text{ in } H^{m-2}(D_\alpha) \text{ as } t \rightarrow 0.$$

Similarly, $u_t := t^{-1}(\beta_t(u) - u) \in \mathcal{K}_{a+1}^{m+1}(D_\alpha) \subset \mathcal{K}_1^1(D_\alpha)$ (recall that $a \geq 0$). Moreover, $u_t \rightarrow \partial_z u$ in $\mathcal{K}_a^m(D_\alpha)$ as $t \rightarrow 0$.

$$\Delta(\partial_z u) = \lim_{t \rightarrow 0} \Delta(u_t) = \lim_{t \rightarrow 0} f_t = \partial_z f \in H^{m-2}(D_\alpha) \subset \mathcal{K}_{a-1}^{m-2}(D_\alpha) \subset \mathcal{K}_{-1}^{-1}(D_\alpha),$$

by the assumption that $m \geq 1$ and $a \leq 1$. Since $u = 0$ on the boundary of D_α (this makes sense because $u \in \mathcal{K}_{a+1}^{m+1}(D_\alpha)$ is regular enough to have a trace on the boundary). Since $\partial_z u \in \mathcal{K}_a^m(D_\alpha)$ we obtain that $\partial_z u$ is also smooth enough to have a trace on the boundary and $\partial_z u = 0$ on the boundary of D_α . Hence $\partial_z u = \Delta_D(\partial_z f)$. This completes the proof of the Equation (24).

Step 2. $\mathbb{P} = \mathcal{C}$, a polyhedral cone. We proceed similarly. Let ρ be the distance to the vertex of \mathcal{C} . So $\rho = \rho_{\mathcal{C}}$. We shall prove our statement under the more general assumption that or that $(\rho \partial_\rho)^j f \in \mathcal{K}_{a-1}^{m-1-j}(\mathcal{C})$ for $j \leq m-1$ and $(\rho \partial_\rho)^m f \in \rho^a \mathcal{K}_{-1}^{-1}(\mathcal{C})$. These assumptions are satisfied if f is as in the statement of our theorem because $(\rho \partial_\rho)^j f \in H^{m-1-j}(\mathcal{C})$ and $(\rho \partial_\rho)^j f$ has compact support, for $j \leq m-1$. For $j = m$, we also use $(\rho \partial_\rho)^m f = \rho[\partial_\rho(\rho \partial_\rho)^{m-1} f] \in \rho \mathcal{K}_{-1}^{-1}(\mathcal{C})$.

Let $m = 0$. Then our assumption is that $f \in \rho^a \mathcal{K}_{-1}^{-1}(\mathcal{C})$. Then, for $a \geq 0$ small enough, the equation $-\Delta u = f$, $u|_{\partial \mathcal{C}} = 0$ has a unique solution $u \in \rho^a \mathcal{K}_1^1(\mathcal{C}) =: \mathcal{D}_{1+a}^1(\mathcal{C})$, and that this solution depends continuously on f , by Equation (21) (which is a slight extension of Theorem 2.1). This takes care of the case $m = 0$.

Let now $m \geq 1$. As in the first step, we need to show that

- (1) Equation (1) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\mathcal{C})$ and $\rho \partial_\rho u \in \mathcal{D}_{a+1}^m(\mathcal{C})$.
- (2) u and $\rho \partial_\rho u$ depend continuously on f in the topology of the corresponding spaces.

We get $u \in \mathcal{K}_{a+1}^{m+1}(\mathcal{C})$ as in the first step by using Theorem 2.1. That theorem also gives that u depends continuously on f . We then only need to show that

$$\rho \partial_\rho u := \rho \partial_\rho(\Delta_D^{-1}(f)) \in \mathcal{D}_{a+1}^m(\mathcal{C})$$

is well defined and depends continuously on f satisfying the assumptions above (i.e., $(\rho \partial_\rho)^j f \in \mathcal{K}_{a-1}^{m-1-j}(\mathcal{C})$ for $j \leq m-1$ and $(\rho \partial_\rho)^m f \in \rho^a \mathcal{K}_{-1}^{-1}(\mathcal{C})$).

Let Δ' be the Laplace operator on the unit sphere. We notice that the formula $\Delta = \rho^{-2}((\rho \partial_\rho)^2 + \rho \partial_\rho + \Delta')$ gives

$$(25) \quad \Delta[\rho \partial_\rho(u)] = \rho \partial_\rho(\Delta u) + 2\Delta u = \rho \partial_\rho(f) + 2f.$$

(This formula also follows from the behavior of Δ with respect to dilations $\alpha_t(x) = tx$, $x \in \mathbb{R}^3$, as in the first step. We omit the similar details.) For $m \geq 1$ (our case),

it makes sense to restrict u and $\rho\partial_\rho u$ to the boundary, so they are both zero at the boundary. This gives

$$\rho\partial_\rho(u) = \Delta_D^{-1}(\rho\partial_\rho f + 2f).$$

We next show that

$$\Delta_D^{-1}(\rho\partial_\rho f + 2f) = \Delta_D^{-1}(\rho\partial_\rho f + 2f) \in \mathcal{D}_{a+1}^m(\mathcal{C}).$$

Indeed, we have that $\mathcal{K}_{a-1}^0(\mathcal{C}) \subset \rho^a \mathcal{K}_{-1}^{-1}(\mathcal{C})$, and hence $\rho\partial_\rho f + 2f$ satisfies the same assumptions as f , but with m replaced with $m-1$. The induction hypothesis then gives $u \in \mathcal{D}_{a+1}^{m+1}(\mathcal{C})$, as desired.

Step 3. $\mathbb{P} = \Omega$ a bounded polyhedral domain. Let $f \in H^{m-1}(\Omega)$, if $m \geq 1$, or $f \in \rho^a H^{-1}(\Omega)$ if $m = 0$. In any case, we have $f \in H^{-1}(\Omega)$ (because $a \geq 0$), and hence $u := \Delta_D^{-1} f \in H_0^1(\Omega)$ is defined.

We want to show that $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$. For $m = 0$, as in the second step, this follows from the discussion preceding the statement of this theorem (which is turned based on Theorem 2.1), which shows that $u \in \rho^a \mathcal{K}_1^1(\Omega) = \mathcal{D}_{a+1}^1(\Omega)$. (Recall that $\rho = \rho_\Omega$ is now the smoothed distance to the vertices of Ω .) This takes care of the case $m = 0$.

Let us consider now the case $m \geq 1$. We shall proceed by induction, using the notation of Definition 1.10. In particular, ϕ_j is the partition introduced right before that definition. In particular, $\nabla\phi_j = 0$ in a neighborhood of each vertex. Then $\Delta(\phi_j u) = f_1 := \phi_j f + 2\nabla\phi_j \cdot \nabla u + (\Delta\phi_j)u$.

Let us assume that ϕ_j is supported near an edge e . We can assume that in the neighborhood \mathcal{V}_e of the edge e , all the functions ϕ_j depend only on z (the coordinate along the edge). Then $\Delta(\phi_j u) = f_1 := \phi_j f + 2\partial_z \phi_j \partial_z u + (\partial_z^2 \phi_j)u$. We have $\phi_j f \in H^{m-1}(\Omega)$ and has compact support. The induction hypothesis shows that $2\partial_z \phi_j \partial_z u + (\partial_z^2 \phi_j)u \in \mathcal{D}_{a+1}^{m-1}(\Omega)$, and hence $f_1 := \phi_j f + 2\partial_z \phi_j \partial_z u + (\partial_z^2 \phi_j)u$ satisfies the assumptions of Step 1. This shows that $\phi_j u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$.

Let us assume now that ϕ_j is supported near a vertex P . Then, similarly, f_1 satisfies the assumptions of Step 2 (because $2\nabla\phi_j \cdot \nabla u + (\Delta\phi_j)u = 0$ in a neighborhood of the vertex), and hence $\phi_j u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$. (Where $2\nabla\phi_j \cdot \nabla u + (\Delta\phi_j)u = 0$ is non-zero, we use the argument of the previous paragraph.)

Finally, $\phi_0 u$, the only remaining term of the form $\phi_j u$ not already considered, is supported away from the edges. Since $\Delta(\phi_0 u) = f_1 := \phi_0 f + 2\nabla\phi_0 \cdot \nabla u + (\Delta\phi_0)u \in H^{m-1}(\Omega)$, by induction. Elliptic regularity for smooth, bounded domains then shows that $\phi_0 u \in H^{m+1}(\Omega) \subset \mathcal{D}_{a+1}^{m+1}(\Omega)$. Since $u = \sum \phi_j u$, the third step is complete and so is the proof of our theorem. \square

Remark 2.3. For our main result, we shall need mostly the following two ingredients: the estimates of the above theorem close to the edges and the dilation invariance of the norm on \mathcal{D}_a^m for functions supported close to a vertex. The case of an infinite edge is much easier to check.

3. INITIAL TETRAHEDRALIZATION AND THE GENERAL STRATEGY

We now explain the general ideas and properties of our tetrahedralizations. The details of these constructions will be completed in the following sections. We begin by introducing marked prisms (*i.e.*, triangular prisms with a choice of a diagonal on one of the faces) and explain how they are tetrahedralized. Then we explain

how we obtain our initial tetrahedralization \mathcal{T}'_0 of Ω . We assume that we have fixed an initial decomposition $\mathcal{T}_0, \bar{\Omega} = \cup_j \bar{\Lambda}_j$, of Ω as in Subsection 1.2.

In the last two subsections we explain the properties of our sequence \mathcal{T}'_n of tetrahedralizations and show that they lead to quasi-optimal rates of convergence with respect to the dimension of the Finite Element space. The full details of the construction of the sequence \mathcal{T}'_n of tetrahedralizations will be given in the following sections.

3.1. Marked prisms and the initial tetrahedralization. In this subsection we describe the division of the prisms. Let us fix $\mathbb{P} := ABCA'B'C'$ to be a triangular prism with $AA', BB',$ and CC' parallel. Let us fix a diagonal d of one of the quadrilateral faces of \mathbb{P} . Then we shall call (\mathbb{P}, d) or \mathbb{P} a *marked prism* and we shall call the diagonal d the *mark* of this prism. Any marked prism (\mathbb{P}, d) leads to a *canonical tetrahedralization* of \mathbb{P} after we divide the other two quadrilateral faces of \mathbb{P} into two triangles using the diagonals that have a vertex in common with d .

For example, assume we have fixed the diagonal BC' (of the face $BCC'B'$). Then we draw the diagonals $A'B$ and $A'C$ (of the other two quadrilateral faces), to obtain a partition of \mathbb{P} into the three tetrahedra $A'ABC, A'BCC',$ and $A'B'BC',$ as in Section 5.

We shall also assume that each prism Λ_j in our initial decomposition is a marked prism as follows. Let e be the unique edge of Ω that intersects $\bar{\Lambda}_j$. We assume that the mark d of Λ_j (*i.e.*, the fixed diagonal of one of the quadrilateral faces of Λ_j) belongs to the quadrilateral face of Λ_j that is *not* adjacent e . In other words, all prisms in our initial decomposition are marked prisms and the mark does not intersect any edge of Ω . This gives exactly two choices for the mark.

Two prisms in our initial decomposition Λ_j and Λ_k are called *adjacent* if they have a face in common (this implies that they correspond to the same edge e of Ω). To simplify our considerations, whenever possible, we shall choose our marks so that if Λ_j and Λ_k are adjacent, then the two corresponding marks (*i.e.*, fixed diagonals) have an end point in common. In this way, only one choice has to be made for each edge of Ω .

We also assume that we have fixed a tetrahedralization of Λ_0 such that no additional edges were introduced on the boundary of Λ_0 (except the marks of the prisms). We do allow however additional internal vertices, which will determine additional internal edges. Then we divide each prism into three tetrahedra as determined by the mark (this is as explained above, that is, using the fixed diagonals and such that the new diagonals of the faces adjacent to the edges have a point in common with the fixed diagonals). The resulting tetrahedralization \mathcal{T}'_0 of Ω (obtained by tetrahedralizing Λ_0 and the prisms of \mathcal{T}_0) will be called the *initial tetrahedralization* and our construction guarantees that this initial tetrahedralization defines a conforming mesh.

For instance, if $\Omega = A_1A_2A_3A_4$, to obtain the initial tetrahedralization, we proceed as follows. We first divide the small tetrahedra adjacent to the edges to obtain our initial decomposition, as explained in Subsection 1.2. See Figure 1.1. Then we fix a diagonal of the rectangle $D_{13}D_{14}C_{23}C_{24}$ (the point C_{23} is the middle of C_2C_3). Say we fix $D_{13}C_{24}$. This leads to a tetrahedralization of the prism $D_1D_{13}D_{14}C_{23}C_{24}C_2$ by introducing also C_2D_{13} and D_1C_{24} (these are the two diagonals of the faces adjacent to the edge A_1A_2 that have a point in common with the fixed diagonal $D_{13}C_{24}$). We proceed analogously with the other two prisms.

Then we introduce the barycenter of Ω as an additional vertex and join it with all the vertices of the innermost region Λ_0 to obtain a tetrahedralization of this region without additional edges on its boundary (except the fixed diagonals).

3.2. The sequence of tetrahedralization. We shall construct in Section 4, for any parameter $\kappa \in (0, 1/2]$, a sequence \mathcal{T}_n of decompositions of Ω . The decomposition \mathcal{T}_0 is our initial decomposition as in Subsection 1.2, which we assume to be fixed from now on. We also assume that the marks on the prisms of the initial decomposition \mathcal{T}_0 and the tetrahedralization \mathcal{T}'_0 are fixed from now on.

The decompositions \mathcal{T}_n , $n \geq 1$, are decompositions of Ω into finitely many tetrahedra and marked prisms, *i.e.*, if $T \in \mathcal{T}_n$, $n \geq 1$, then T is either a tetrahedron or a marked prism $T = (\mathbb{P}, d)$, with the mark never adjacent to any edge of Ω . Note that \mathcal{T}_0 will be slightly different from the other decompositions because it contains also a region Λ_0 that is not a prism or tetrahedron. The sequence of decompositions \mathcal{T}_n will have the following properties.

- (i) \mathcal{T}_0 is the initial decomposition and it satisfies the conditions of Subsection 1.2, \mathcal{T}'_0 is the initial tetrahedralization of Ω as above (that with Λ_0 divided into tetrahedra without introducing any additional edges on the boundary of Λ_0 , except the marks of the prisms, and the prisms divided into three tetrahedra using the mark).
- (ii) \mathcal{T}_n is a decomposition of Ω into disjoint tetrahedra and straight triangular marked prisms: $\bar{\Omega} = \cup_{T \in \mathcal{T}_n} \bar{T}$.
- (iii) \mathcal{T}_{n+1} is a refinement of \mathcal{T}_n , in the sense that each region $\Lambda \in \mathcal{T}_{n+1}$ is contained in exactly one region $\Lambda' \in \mathcal{T}_0$.
- (iv) If we canonically divide each prism of \mathcal{T}_n into three tetrahedra in the unique way specified by the mark of that prism, then the new decomposition of Ω , denoted \mathcal{T}'_n , is a conforming mesh (that is, if two tetrahedra in \mathcal{T}'_n have more than an edge in common, then they have a face in common).
- (v) Each prism $T \in \mathcal{T}_n$ has one of the three parallel edges contained in one of the edges of Ω , but does not contain any vertex of Ω .
- (vi) Let $\mathcal{V}_{P,n}$ be the union of the closed tetrahedra of \mathcal{T}_n that are adjacent to the vertex P of Ω . Then $\mathcal{V}_{P,n}$ is a neighborhood of P in $\bar{\Omega}$ such that every tetrahedron $T \in \mathcal{T}_k$, $k \geq n$ that intersects $\mathcal{V}_{P,n}$ is completely contained in $\mathcal{V}_{P,n}$.
- (vii) Let α_t be the dilation of ratio t and center P . There is a fixed parameter κ (usually $\kappa = 2^{-m/a}$) with the property that α_κ maps the restriction of \mathcal{T}_n to $\mathcal{V}_{P,k}$ to the restriction of \mathcal{T}_{n+1} to $\mathcal{V}_{P,k+1}$.
- (viii) The number $k_{n,j}$ of regions $T \in \mathcal{T}_n$ that are contained in the region Λ_j of the initial decomposition satisfies $C^{-1}2^{3n} \leq k_{n,j} \leq C2^{3n}$, with C independent of n and j .
- (ix) Let S_n be the space of continuous, piecewise polynomials of degree m on \mathcal{T}'_n . Let $u \rightarrow u_{I,n}$ be the Lagrange interpolant associated to the “ m -simplex” and the mesh \mathcal{T}'_n restricted. Let $X := \Omega \setminus \cup_P \mathcal{V}_{P,1}$. If $\kappa \leq 2^{-m/a}$, there exists a constant $C > 0$ such that

$$|u - u_{I,n}|_{H^1(X)} \leq C2^{-nm} \|u\|_{\mathcal{D}_{a+1}^{m+1}(X)}$$

for any $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$, $u = 0$ on the boundary. The constant C is independent of n and u .

From now on, S_n denotes the Finite Element space of continuous, piecewise polynomial of order m on the mesh \mathcal{T}'_n with Dirichlet boundary conditions.

Let us now make the simple, but important, remark that Condition (viii) guarantees that

$$(26) \quad C^{-1}2^{3n} \leq \dim(S_n) \leq C2^{3n},$$

where C depends on m and Ω , but not on n .

Remark 3.1. Condition (ix) will be shown to be true for $m \geq 2$. The results of [5] (see also the references therein) indicate that this condition is *not true* for $m = 1$. For $m = 1$, it is necessary to use an ‘‘averaged interpolant,’’ as in [5], but we shall not address this issue in this paper.

3.3. Quasi-optimal rates of convergence. Let us denote by \mathcal{T}'_n the associated tetrahedralization of Ω , as before. Let S_n be the Finite Element space of continuous, piecewise polynomials on \mathcal{T}'_n , as we have agreed. Also, let us denote by $u_{I,n}$ the Lagrange interpolant of u associated to the m -simplex (*i.e.*, to uniformly distributed nodes). If $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$, then u has enough regularity to be defined at all nodes of \mathcal{T}'_k , except the ones on the edges. In order to define the interpolant in this case, we set u to be equal to zero at those edge nodes. This is justified in view of the fact that u satisfies Dirichlet boundary conditions.

Assume that we have constructed a sequence \mathcal{T}'_n of decompositions of Ω satisfying the Conditions (i–ix) of the previous subsection. (So the results below will be proved in this paper to be true only under the additional assumption $m \geq 2$; they will probably remain true for $m = 1$ with a different choice of the interpolant, see [5] and Remark 3.1.)

Theorem 3.2. *Assume that Conditions (i–ix) are satisfied. Let $a \in (0, 1/2]$ and $0 < \kappa \leq 2^{-m/a}$. Then there exists $C > 0$ such that*

$$|u - u_{I,k}|_{H^1(\Omega)} \leq C2^{-km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\Omega)},$$

for any $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$, $u = 0$ on the boundary, and any $k \in \mathbb{Z}_+$.

Proof. Let $\mathcal{V}_{P,n}$ be as in Condition (vi). Recall that $X := \Omega \setminus \cup \mathcal{V}_{P,1}$. Let us fix P and let $Y_n := \mathcal{V}_{P,n} \setminus \mathcal{V}_{P,n+1}$. Then $Y_0 := \mathcal{V}_{P,0} \setminus \mathcal{V}_{P,1} \subset X := \Omega \setminus \cup_Q \mathcal{V}_{Q,1}$. By Condition (vi), \mathcal{T}'_k , $k > n$, provides a decomposition of Y_n (that is, Y_n will be contained in the union of the closures of the regions of \mathcal{T}'_k that intersect Y_n). In particular, Y_n is tetrahedralized with tetrahedra in \mathcal{T}'_k , for $k > n$.

Let $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$. We claim that we can find a constant C independent of n and k such that

$$(27) \quad \begin{aligned} |u - u_{I,k}|_{H^1(Y_n)} &\leq C2^{-km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(Y_n)}, \quad 1 \leq n < k, \quad \text{and} \\ |u - u_{I,k}|_{H^1(\mathcal{V}_{P,k})} &\leq C2^{-km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\mathcal{V}_{P,k})}. \end{aligned}$$

This equation is enough to prove our result, since the decomposition of Ω into the regions $\mathcal{V}_{P,k}$, Y_n , $1 \leq n \leq k-1$, and $X := \Omega \setminus \cup_P \mathcal{V}_{P,1}$, together with Equation

(27) and Condition (ix) (with n replaced with k) give

$$\begin{aligned} |u - u_{I,k}|_{H^1(\Omega)}^2 &= \sum_P \left(|u - u_{I,k}|_{H^1(\mathcal{V}_{P,k})}^2 + \sum_{n=1}^{k-1} |u - u_{I,k}|_{H^1(Y_n)}^2 \right) + |u - u_{I,k}|_{H^1(X)}^2 \\ &\leq C2^{-2km} \sum_P \left(\|u\|_{\mathcal{D}_{a+1}^{m+1}(\mathcal{V}_{P,k})}^2 + \sum_{n=1}^{k-1} \|u\|_{\mathcal{D}_{a+1}^{m+1}(Y_n)}^2 \right) + C2^{-2km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(X)}^2 \\ &= C2^{-2km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\Omega)}^2, \end{aligned}$$

with a constant $C > 0$ independent of k and n .

To prove our Claim, Equation (27), let $v \in \mathcal{D}_{a+1}^{m+1}(Y_n)$. Let $t = \kappa^n \leq 2^{-nm/a}$ and $\alpha_t(x) := tx$. Then α_t maps Y_0 to Y_n . Let $v = u \circ \alpha_t$, which will then be a function defined on Y_0 . Condition (vii) and the behavior of the Lagrange interpolant under change of coordinates give $u_{I,k} \circ \alpha_t = v_{I,k-n}$ on Y_0 , $k > n$. In turn, Lemma 1.8 and Condition (ix) give, for $t = \kappa^n \leq 2^{-nm/a}$,

$$\begin{aligned} |u - u_{I,k}|_{H^1(Y_n)} &= t^{1/2} |u \circ \alpha_t - u_{I,k} \circ \alpha_t|_{H^1(Y_0)} = t^{1/2} |v - v_{I,k-n}|_{H^1(Y_0)} \\ &\leq C_0 t^{1/2} 2^{-(k-n)m} \|v\|_{\mathcal{D}_{a+1}^{m+1}(Y_0)} \leq C_0 2^{-km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(Y_n)}, \end{aligned}$$

because $a \leq 1/2$. This proves the first part of Equation (27).

To prove the second part of Equation (27), let $u \in H^1(\mathcal{V}_{P,k})$. Also, let $t = \kappa^k \leq 2^{-km/a}$, $\alpha_t(x) = tx$, as before, and $v = u \circ \alpha_t$. Then α_t maps $\mathcal{V}_{P,0}$ to $\mathcal{V}_{P,k}$ and hence v will be defined on $\mathcal{V}_{P,0}$. We shall again use the dilation α_t to define $v = u \circ \alpha_t$, which will then be a function defined on $\mathcal{V}_{P,0}$. Condition (vii) and the behavior of the Lagrange interpolant under change of coordinates give $u_{I,k} \circ \alpha_t = v_{I,0}$. Let $\chi : \Omega \rightarrow [0, \infty)$ be a smooth function that is equal to 0 in a neighborhood of the edges, but is equal to 1 at all other nodal points of \mathcal{T}'_0 . Then $v_{I,0} = (\chi v)_{I,0}$ on $\mathcal{V}_P = \mathcal{V}_{P,0}$, because $u = 0$ at the boundary. In turn, Lemma 1.8 and a standard interpolation estimate give, for $t = \kappa^k \leq 2^{-km/a}$,

$$\begin{aligned} (28) \quad t^{-1/2} |u - u_{I,k}|_{H^1(\mathcal{V}_{P,k})} &= |u \circ \alpha_t - u_{I,k} \circ \alpha_t|_{H^1(\mathcal{V}_{P,0})} = |v - v_{I,0}|_{H^1(\mathcal{V}_{P,0})} \\ &\leq (|v - \chi v|_{H^1(\mathcal{V}_{P,0})} + |\chi v - (\chi v)_{I,0}|_{H^1(\mathcal{V}_{P,0})}) \leq C(|v|_{H^1(\mathcal{V}_{P,0})} + |\chi v|_{H^{m+1}(\mathcal{V}_{P,0})}) \\ &\leq C \|u \circ \alpha_t\|_{\mathcal{D}_{a+1}^{m+1}(\mathcal{V}_{P,0})} = C t^{a-1/2} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\mathcal{V}_{P,k})} \leq C t^{-1/2} 2^{-km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\mathcal{V}_{P,k})}. \end{aligned}$$

This completes the proof of our claim (Equation 27) and hence the proof of our theorem. \square

Let $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ be the weak solution of

$$(29) \quad -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Let $u_n \in S_n$ be the discrete solution of this equation, that is, the solution of the equation $B(u_n, v_n) = \int_{\Omega} f v_n(x) dx$, for all $v_n \in S_n$. Then $|u - u_n|_{H^1(\Omega)}$ will be called the *finite element error*. A standard application of Cea's Lemma [23, 26] and of Theorems 2.2 and 3.2 give then the following theorem.

Theorem 3.3. *Let S_n be the Finite Element spaces of continuous piecewise polynomials of degree m associated to a tetrahedralization \mathcal{T}'_n of Ω . Assume as above that \mathcal{T}'_n is associated to a sequence of decompositions \mathcal{T}_n satisfying the conditions*

(i-ix) of Subsection 3.2. Let $a \in (0, 1/2]$ be as in Theorem 2.2 and $\kappa \leq 2^{-m/a}$. Then there exists $C > 0$ such that

$$\|u - u_n\|_{H^1(\Omega)} \leq C 2^{-nm} \|u\|_{\mathcal{D}_{a+1}^{m+1}} \leq C \dim(S_n)^{-m/3} \|f\|_{H^{m-1}(\Omega)},$$

for a constant C independent of n and $f \in H^{m-1}(\Omega)$.

Note that the assumptions of the above theorem imply that $\dim(S_n) \leq C 2^{3n}$ and $\|u\|_{\mathcal{D}_{a+1}^{m+1}} \leq C_a \|f\|_{H^{m-1}(\Omega)}$. Also, let us notice that we make no additional assumption on the regularity of u (other than $\Delta u \in H^{m-1}(\Omega)$).

Since for $f \in H^{m-1}(\Omega)$ we cannot expect anything better than $u \in H_{\text{loc}}^{m+1}(\Omega)$ and $\dim(S_n) \sim 2^{3n}$, it follows that the order of the error $\|u - u_n\|_{H^1(\Omega)}$ is of the same order of magnitude as the interpolation error $\inf_{\chi \in S_n} \|u - \chi\|_{H^1(\Omega)}$. This is the sense in which the rate of convergence provided by Theorem 3.3 is quasi-optimal.

The existence of a sequence \mathcal{T}'_n satisfying the conditions of Theorem 3.3 for $m \geq 2$ will be proved in the following sections. Our procedure will likely have to be slightly changed for $m = 1$, by replacing the Lagrange interpolant with an averaged interpolant, as in [5]. The details of this still need to be worked out. See also Remark 3.1.

4. PARTITIONING, TETRAHEDRALIZATION, AND REFINEMENT STRATEGIES

We now present a procedure that will associate to a bounded, polyhedral domain Ω , an initial decomposition \mathcal{T}_0 of Ω into tetrahedra and marked prism, an initial tetrahedralization as in Subsection 3.1 (of the preceding section), and a parameter $\kappa \in (0, 1/2]$, a sequence \mathcal{T}_n of decompositions of Ω into finitely many tetrahedra and marked prisms satisfying the conditions of Subsection 3.2. This construction will involve no further choices, so it is canonical (*i.e.*, algorithmic). The verification of Condition (ix), however, will only be completed in Section 6.

The tetrahedralization \mathcal{T}'_n of Ω , for any n , is obtained from the decomposition \mathcal{T}_n of Ω by dividing each marked prism into three tetrahedra, as determined by the mark and explained in Subsection 3.1.

4.1. The parameter κ and the refinement of edges and faces. Our meshes, tetrahedralizations, and partitions of Ω will depend on the initial decomposition, the initial tetrahedralization and a parameter $\kappa \in (0, 1/2]$. For $\kappa = 1/2$ we shall obtain a quasi-uniform sequence of meshes. To obtain quasi-optimal rates of convergence with respect to the dimension of the Finite Element Space of order m piecewise polynomials, we usually need to take κ small. For example, if $0 < a < \eta$, where η is as in Theorem 2.2, we shall prove that we recover quasi-optimal rates of convergence (with respect to the dimension of the FEM space of order m piecewise polynomials) for

$$(30) \quad \kappa = 2^{-m/a}.$$

See Theorem 3.3. This is the same choice of the parameter κ as in [20], where it was seen to be optimal for $m = 1$ and a re-entrant corner in numerical tests in [20]. (This remains to be established in three dimensions.)

Given a point $P \in \Omega$, we shall say that P is of *type V* if it is a vertex of Ω ; we shall say that P is of *type E* if it is on an open edge of Ω . Otherwise, we shall say that it is of *type S*. Note that the type of a point depends only on Ω and not on any partition or meshing. Then the initial tetrahedralization will consist of edges

of type **VE**, **VR**, **ER**, **EE**:=**E**², and **R**². (The way our initial decomposition and initial tetrahedralization was defined, no edge will be of type **VV**.) The points of type **V** will be regarded as more singular than the points of type **E**, and the points of type **E** will be regarded as more singular than the points of type **R**.

Let AB be an edge in one of our decompositions \mathcal{T}_n . Then in \mathcal{T}_{n+1} , this edge will be decomposed in two segments, AC and CB , such that $|AC| = \kappa|AB|$ if A is more singular than B (*i.e.*, if AB is of type **VE**, **VR**, or **ER**). In particular, C will be closer to the more singular point (except when $\kappa = 1/2$). If A and B are as singular (*i.e.*, if AB is of type **E**² or **R**²), then we take C to be the middle of AB . This procedure is as in [8, 20, 52]. See Figure 4.1.

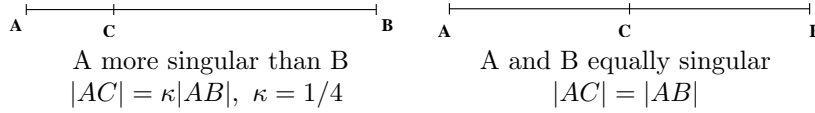


FIGURE 4.1. Edge decomposition

Let ABC be a triangle in the decomposition \mathcal{T}_n . Then in \mathcal{T}_{n+1} , this triangle will be divided into four other triangles, with the exception when ABC is of type **VER**. This will be achieved as follows. We divide each side of ABC into two segments, as explained above. Then we divide ABC into four triangles by joining the three new points. See Figure 4.2.

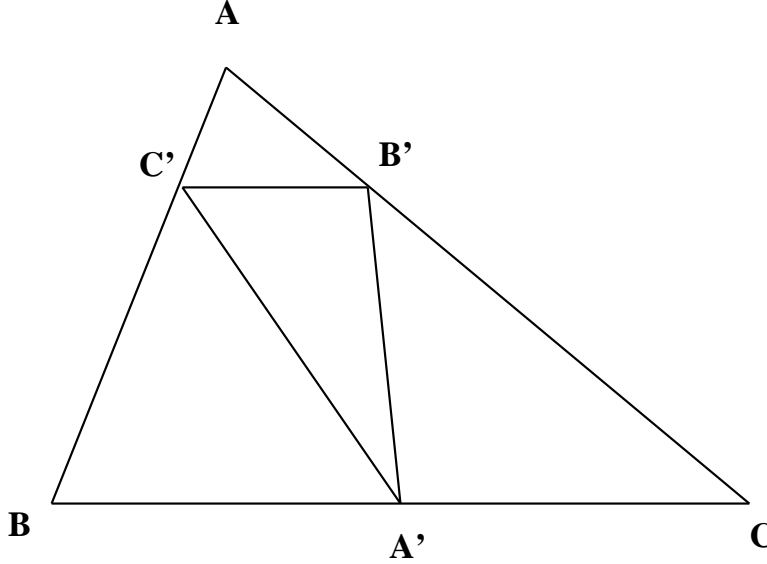


FIGURE 4.2. Face decomposition: A of type **V** or **E**, B and C of type **R**, $|AC'| = \kappa|AB|$, $|AB'| = \kappa|AC|$, $|A'B| = |A'C|$, $\kappa = 1/4$

On the other hand, if ABC is of type **VER** (with B of type **E**), then we remove the newly introduced segment that is opposite B . This will divide ABC into two triangles and a quadrilateral, with B belonging to this quadrilateral and not adjacent to any of the two triangles. The newly formed quadrilateral will belong to a prism

in \mathcal{T}_{n+1} . See Figure 4.3. (Note that we assume also that $\hat{B} = 90^\circ$, an assumption that is convenient for our proof, but is probably not necessary for implementation.)

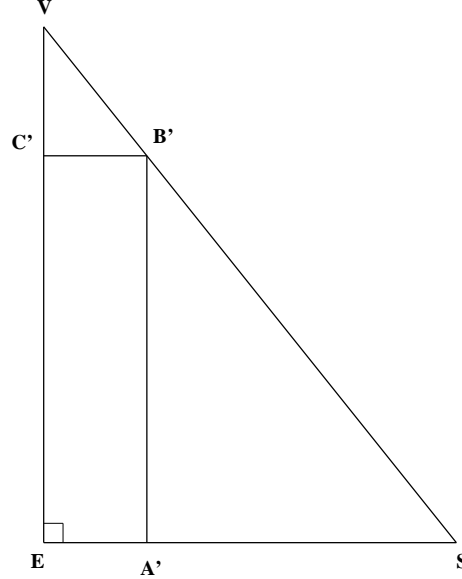


FIGURE 4.3. VER decomposition: $|VC'| = \kappa|VE|$, $|VB'| = \kappa|VR|$, $|EA'| = \kappa|ER|$, $A'C'$ was removed, $\angle E = 90^\circ$

Let $ABCD$ be a quadrilateral that appears in a prism of the decomposition \mathcal{T}_n . Then in \mathcal{T}_{n+1} , this quadrilateral is divided into four quadrilaterals by joining the two additional points on the opposite sides of $ABCD$ (these additional points were obtained as explained above). To obtain \mathcal{T}'_n , we must further divide each quadrilateral into two triangles using one of the diagonals. The choice of this diagonal is that it is either the mark of one of the prisms or that it has a point in common with a mark of one of the prisms. See Figure 4.8.

To summarize, each edge, triangle, or quadrilateral that appears in a tetrahedron or prism in the decomposition \mathcal{T}_n is divided in the decomposition \mathcal{T}_{n+1} in an intrinsic way, which depends only on the type of the vertices (of that edge, triangle, or quadrilateral). In particular, the way that a face in \mathcal{T}_n is divided to yield \mathcal{T}_{n+1} does not depend on the type of the other vertices of the tetrahedron or prism to which it belongs. This ensures that the tetrahedralization \mathcal{T}'_{n+1} , which is obtained from \mathcal{T}_{n+1} by dividing each prism in three tetrahedra, is a conforming mesh.

4.2. Division of tetrahedra and prisms. Unlike the edges and faces that are part of \mathcal{T}_{n+1} , the tetrahedra that are part of \mathcal{T}_{n+1} are usually *not* obtained by dividing a region in \mathcal{T}_n . In the following three sections we shall describe in detail how the regions of \mathcal{T}_n are obtained by using the following three refining procedures:

- (i) The *non-uniform refinement* is applied to a tetrahedron of type **VERR** or **VR³** in \mathcal{T}_n to yield regions of \mathcal{T}_{n+1} that will be either prisms or tetrahedra of various types.
- (ii) The *uniform refinement of level k* is applied to the tetrahedron of type **R⁴** that is either a tetrahedron in the initial tetrahedralization or is obtained by

- non-uniform refinement. When the uniform refinement of level k is applied to the tetrahedron of type \mathbf{R}^4 of \mathcal{T}'_n , it leads to tetrahedra of the decomposition \mathcal{T}_{n+k} (which are also tetrahedra of type \mathbf{R}^4 of the tetrahedralization \mathcal{T}'_{n+k}).
- (iii) The *semi-uniform refinement of level k* is applied to prisms that are either part of the initial decomposition \mathcal{T}_0 or are part of the decomposition \mathcal{T}_n as a result of a non-uniform division of a tetrahedron of \mathcal{T}_{n-1} . When the semi-uniform refinement of level k is applied to a marked prism in \mathcal{T}_n , it leads to 2^{3k} marked prisms of the decomposition \mathcal{T}_{n+k} .

It will follow by induction that every tetrahedron in the decompositions \mathcal{T}_k , $k \geq 0$, is of the type **VERR**, **VR³**, or **R⁴**, so we need not consider any other type of tetrahedron.

The general method for obtaining \mathcal{T}_n and \mathcal{T}'_n is as follows. First, recall that the tetrahedralization \mathcal{T}'_n is obtained from \mathcal{T}_n by dividing each prism in \mathcal{T}_n in three tetrahedra by using its mark. (So in order to define the meshes \mathcal{T}'_n , we need only define the decompositions \mathcal{T}_n and the marks on the prisms.) On the other hand, \mathcal{T}_n is *not* defined inductively from \mathcal{T}_{n-1} , but is rather obtained from the initial decomposition \mathcal{T}_0 and the initial tetrahedralization \mathcal{T}'_0 . Assume \mathcal{T}_k , $0 \leq k < n$, and \mathcal{T}'_0 were defined. We then define \mathcal{T}_n by dividing certain regions of \mathcal{T}_k , $0 \leq k < n$ as follows:

- (i) If Λ is a tetrahedron of type **R⁴** in the initial tetrahedralisation \mathcal{T}'_0 , then we apply to Λ the level n of uniform refinement (in this case $\Lambda \subset \Lambda_0$);
- (ii) If Λ is a marked prism of the initial decomposition \mathcal{T}_0 , then we apply to Λ the semi-uniform refinement of level n (in this case, $\Lambda \cap \Lambda_0 = \emptyset$);
- (iii) If Λ is a tetrahedron of type **R⁴** obtained by applying the non-uniform refinement procedure to some tetrahedron T in some \mathcal{T}_k , $0 \leq k < n$, then we apply to Λ the uniform refinement of level $n - k$ of (in this case, T must be of type **VERR** or **VR³**).
- (iv) If Λ is a marked prism obtained by applying the non-uniform refinement procedure to some tetrahedron T in some \mathcal{T}_k , $0 \leq k < n$, then we apply to Λ the level $n - k$ of semi-uniform refinement (again, T must be of type **VERR** or **VR³**).

By a performing the level 0 of a refinement to a region we mean that we do not change that region. Then \mathcal{T}_n consists of the regions defined above and the tetrahedra of type **VERR** or **VR³** obtained by the non-uniform refinement of a tetrahedron of the same type in \mathcal{T}_{n-1} .

All the regions considered in (i–iv) above and all the resulting regions that define \mathcal{T}_n are disjoint.

4.3. Uniform refinement. In this subsection, $T \in \mathcal{T}'_k$ will be a tetrahedron of type **R⁴** obtained in one of the following ways:

- (i) T is a tetrahedron $T \subset \Lambda_0$ of the initial tetrahedralization \mathcal{T}'_0 or
- (ii) T is a tetrahedron of type **R⁴** obtained by a non-uniform division of a tetrahedron of type **VR³** or **VER²**.

We now describe a *uniform refinement* strategy for dividing T . To obtain \mathcal{T}'_n , we need to apply to our tetrahedron $T \in \mathcal{T}'_k$ the level $n - k$ uniform refinement. We stress that this is not an inductive definition.

Let $T = A_1A_2A_3A_4$ be the given tetrahedron and let A_{ij} denote the midpoints of the edges A_iA_j , ($i < j$). The edges of the octahedron

$$\mathcal{O} := A_{12}A_{13}A_{14}A_{23}A_{24}A_{34}$$

form three parallelograms which intersect at the barycenter C of T and split the octahedron in eight tetrahedra. The *first level of uniform refinement of T* is defined as the splitting of T into 12 tetrahedra as shown in Figure 4.4. We note that $T \setminus \mathcal{O}$ consists of four tetrahedra similar with T and that \mathcal{O} is split in eight tetrahedra which belong to at most four different classes of similarity.

We introduce five parallelism-similarity classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ as follows. Let \mathcal{C}_0 be the class containing $T_0 := A_1A_2A_3A_4$, \mathcal{C}_1 be the class containing $T_1 := CA_{12}A_{13}A_{14}$, \mathcal{C}_2 be the class containing $T_2 := CA_{12}A_{23}A_{24}$, \mathcal{C}_3 be the class containing $T_3 := CA_{13}A_{23}A_{34}$ and \mathcal{C}_4 be the class containing $T_4 := CA_{14}A_{24}A_{34}$. We say that a tetrahedron τ_i belongs to \mathcal{C}_i if τ_i is similar with T_i and each edge of τ_i has the same direction with an edge of T_i . In other words, τ_i can be obtained from T_i by applying dilations, translations or point reflections but not rotations. For example $A_1A_{12}A_{13}A_{14} \in \mathcal{C}_0$ and $CA_{34}A_{24}A_{23} \in \mathcal{C}_1$ since the tetrahedron $CA_{34}A_{24}A_{23}$ is the point reflection of the tetrahedron $T_1 = CA_{12}A_{13}A_{14}$ with respect to C . Thus the first level of refinement produces 12 tetrahedra in five classes of parallelism-similarity.

To describe the next levels of refinement, we shall use affine coordinates relative to T . Namely, the point P is associated the affine coordinates $[x_1, x_2, x_3, x_4]$ if x_1, x_2, x_3, x_4 are the unique real numbers such that $x_1 + x_2 + x_3 + x_4 = 1$ and $OP = x_1OA_1 + x_2OA_2 + x_3OA_3 + x_4OA_4$. Then $T = \{x_j \geq 0\}$. We have that

$$\begin{aligned} A_1 &= [1, 0, 0, 0], & A_2 &= [0, 1, 0, 0], & A_3 &= [0, 0, 1, 0], & A_4 &= [0, 0, 0, 1], \\ A_{12} &= [1, 1, 0, 0]/2, & A_{13} &= [1, 0, 1, 0]/2, & A_{14} &= [1, 0, 0, 1]/2, \\ A_{23} &= [0, 1, 1, 0]/2, & A_{24} &= [0, 1, 0, 1]/2, & A_{34} &= [0, 0, 1, 1]/2, \\ & & \text{and } C &= [1, 1, 1, 1]/4. \end{aligned}$$

Thus, the nodal points associated with the first level of uniform of refinement are points of the form

$$[k_1, k_2, k_3, k_4]/4, \quad k_1 + k_2 + k_3 + k_4 = 4,$$

where k_1, k_2, k_3, k_4 are non-negative integers which are either all even or $k_1 = k_2 = k_3 = k_4 = 1$ (this last case corresponds to the point C)

We then define the *level n of uniform refinement of T* to consist of all the region in which T is divided by the planes given in affine coordinates by :

$$x_i = k2^{-n} \quad \text{and} \quad x_i + x_j = k2^{-n}, \quad i, j = 1, 2, 3, 4, \quad \text{and} \quad k \in \{0, 1, 2, \dots, 2^n - 1\}.$$

If we fix $x_i = k2^{-n}$, we see that the traces of the remaining planes on $x_i = k2^{-n}$ are given by $x_j = l2^{-n}$, so the triangle cut by $x_i = k2^{-n}$ from the tetrahedron T is divided into $(2^n - k)^2$ congruent triangles. If we ignore the planes $x_i + x_j = k2^{-n}$, then T is decomposed into $2^n(2^n + 1)(2^n + 2)/6$ tetrahedra similar and with the same orientation as the initial tetrahedron T , into $2^n(2^n - 1)(2^n - 2)/6$ tetrahedra similar but with the *opposite* orientation as the initial tetrahedron T (all tetrahedra are similar by a factor of 2^{-n} to T), and of $2^n(2^n - 1)(2^n + 1)/6$ regions similar to \mathcal{O} (by a factor of 2^{1-n}). This is seen by counting the regions between the planes $x_i = (2^n - k)2^{-n}$ and $x_i = (2^n - k - 1)2^{-n}$: then there are $k(k + 1)/2$ same

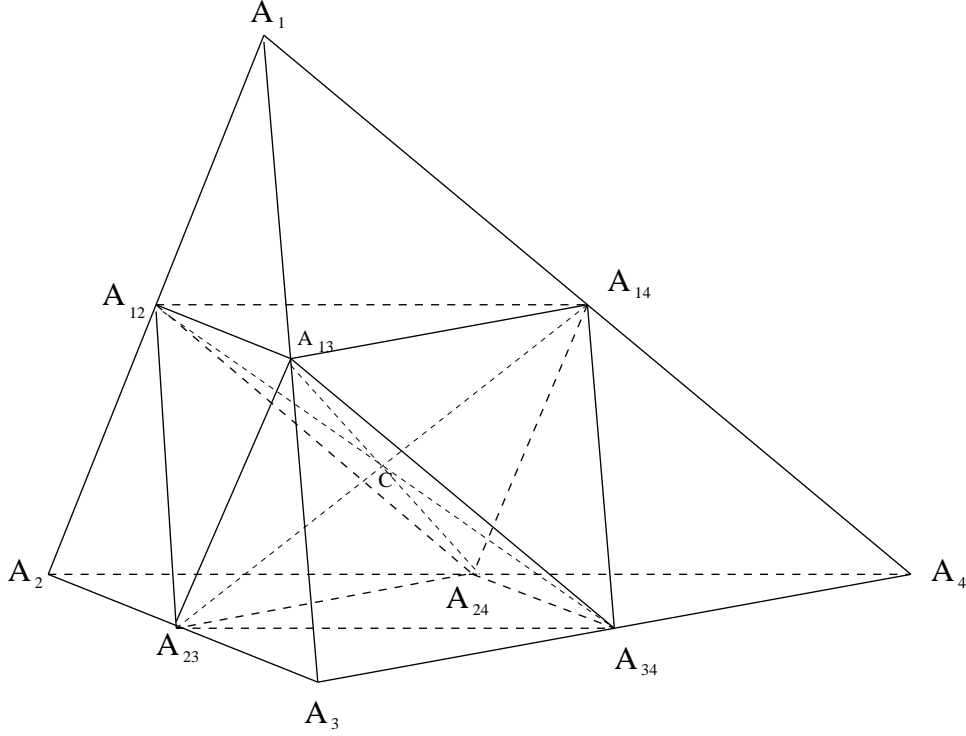


FIGURE 4.4. First level of uniform refinement

direction tetrahedra, $k(k-1)/2$ octahedra, and $(k-1)(k-2)/2$ opposite direction tetrahedra.

Each region similar to \mathcal{O} will be then divided into eight tetrahedra (again similar to those \mathcal{O} has been divided into at the first level of refinement). This shows that all our tetrahedra belong to the similarity classes $\mathcal{C}_0, \dots, \mathcal{C}_4$. This shows that the level n of uniform refinement leads to a decomposition of our given tetrahedron T of type \mathbf{R}^4 into $(5 \cdot 2^{3n} - 2^{n+1})/3$ tetrahedra. For $n = 2$, we thus obtain 104 tetrahedra (for $n = 1$, this formula gives 12 tetrahedra, which is consistent with our previous observations and with the figure 4.4).

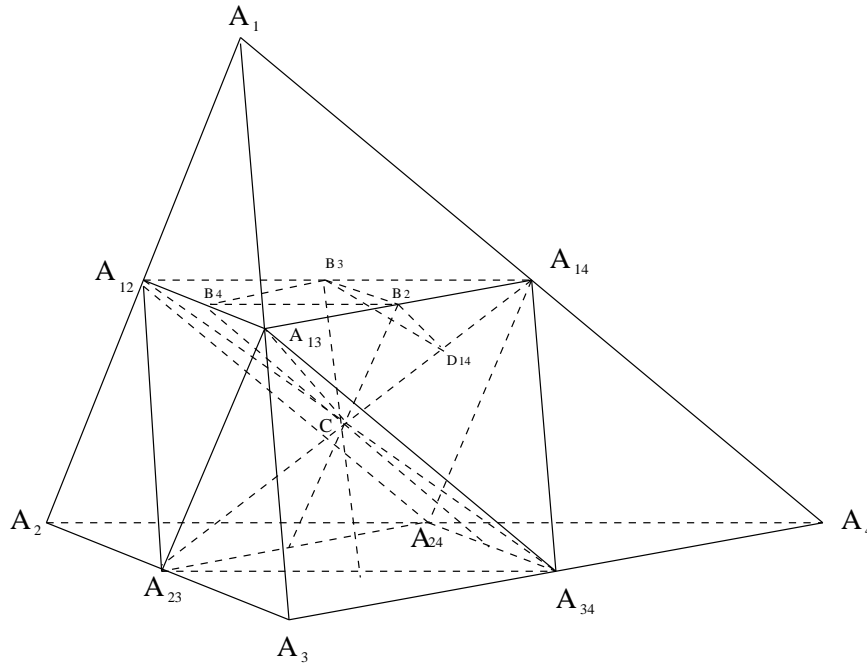
It is interesting to mention that the octahedron \mathcal{O} is divided into 56 tetrahedra in the second level of uniform refinement, so a complete picture of the second level of refinement would be useless. See, however, figure 4.5.

We define the distance between $P[x_1, x_2, x_3, x_4]$ and $Q[y_1, y_2, y_3, y_4]$ by

$$d(P, Q) := |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|.$$

The nodal points of the second level of uniform refinement are all the old points of the first level refinement together with all mid points of the edges of the first level refinement and can be characterized as the points of the form

$$\begin{aligned} & [k_1, k_2, k_3, k_4]/4, \quad k_1 + k_2 + k_3 + k_4 = 4, \quad \text{and} \\ & [k_1, k_2, k_3, k_4]/4 + [1, 1, 1, 1]/8, \quad k_1 + k_2 + k_3 + k_4 = 2, \end{aligned}$$

FIGURE 4.5. Refinement of one tetrahedron from \mathcal{C}_1 .

where k_1, k_2, k_3, k_4 are non-negative integers. A nodal point will be called even if it is of the first type and odd otherwise. At each level a nodal point will be named *even* if it is at the intersection of the planes $x_j = k2^{-n}$ and will be called *odd* otherwise. The odd points are the centers of the octohedra. For example, C is odd in the first level and even in the second level. The faces of the second level of refinement are of two types. The first type consists of triangular faces with all the vertices and mid points (of the second level refinement) being even points. To refine this type, just connect the mid points. The second type has all the vertices and one mid point as even points and the two remaining mid points are odd points. We split this triangular face in four triangles by connecting the even mid point with the opposite vertex and with the other two odd mid points. The refinement of faces induces the refinement of the tetrahedra. The tetrahedra of \mathcal{C}_0 are refined in the same manner we refined T . A tetrahedron from class \mathcal{C}_1 will be refined as shown in Figure 4.5. One can see from Figure 4.5 that a tetrahedron from class \mathcal{C}_1 is split in seven smaller tetrahedra: one in class \mathcal{C}_0 three in the same class \mathcal{C}_1 and one for each of the classes $\mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 . The splitting of the rest of the tetrahedra of the first level of refinement from $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ is similar.

Note that the sides of all the tetrahedra in the second level of refinement have length $1/2$ in the special metric and refinement preserves the number of parallelism-similarity classes. The refinement process can continue and it can be proved by induction that a new level of refinement will preserve the number of parallelism-similarity classes and the conformity of the mesh.

Since the strategy presented for refining a tetrahedron is symmetric with respect to the four vertices (or faces) of the tetrahedron the method extends naturally to

the case of polyhedral domains which can be split as union of tetrahedra such that any two tetrahedra are disjoint or share only a vertex or an edge or a face.

4.4. Non-uniform refinement. Let T be a tetrahedron of type **VERR** or **VR³** of some decomposition \mathcal{T}_n . It will follow by induction that every tetrahedron in the decompositions \mathcal{T}_k , $k \geq 0$, is of the type **VERR**, **VR³**, or **R⁴**. We begin by dividing first each of the edges and then each of the faces of T as described in Subsection 4.1.

If T is a tetrahedron of type **VR³**, we divide it in 12 tetrahedra like in the uniform strategy, but with the edges through the vertex of type **V** divided in the ration given by κ . These tetrahedra belong to \mathcal{T}_{n+1} . There will be a tetrahedron of type **VR³** and 11 tetrahedra of type **R⁴**. See Figure 4.6 Then, as explained in

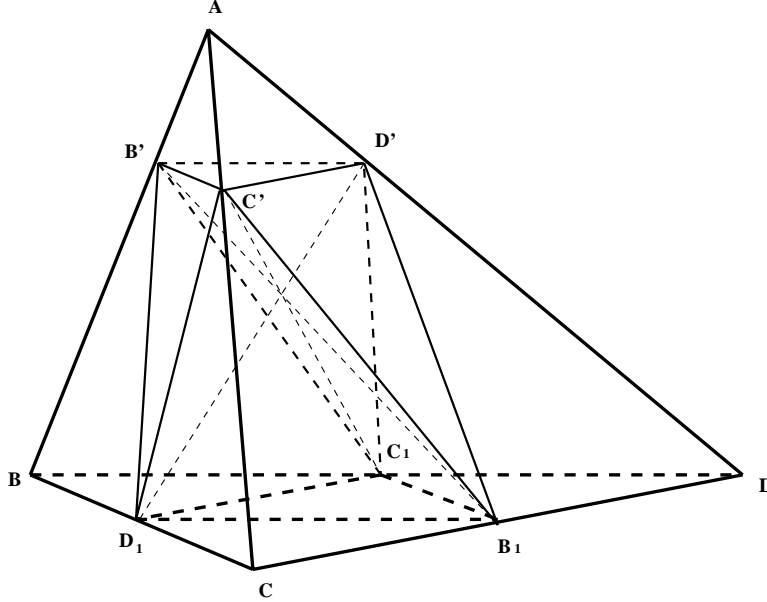


FIGURE 4.6. A of type **V**, B, C, D of type **R**

the previous subsection, we shall iterate this construction for the tetrahedron of type **VR³**, whereas the tetrahedra of type **R⁴** are divided according to the uniform strategy.

If, on the other hand, T is a tetrahedron of type **VERR**, with vertices denoted $ABCD$ in that order, we divide it into 4 tetrahedra of type **R⁴**, one tetrahedron of type **VR³**, and a prism. This is achieved by first dividing each face according to the previously explained strategies for dividing faces of type **VER**, **VR**, and **ERR**. The vertex of type **E** of T will belong only to the prism. Then we use the mark to divide one of the faces of the prism and this will be used to divide the pyramid into two tetrahedra. See Figure 4.7.

4.5. Semi-uniform refinements and conclusion. Let Λ be a prism in the decomposition \mathcal{T}_n . In particular, Λ is then a straight triangular prism with a distinguished diagonal (mark). To obtain \mathcal{T}_{n+1} , we divide each parallel edge of Λ in 2

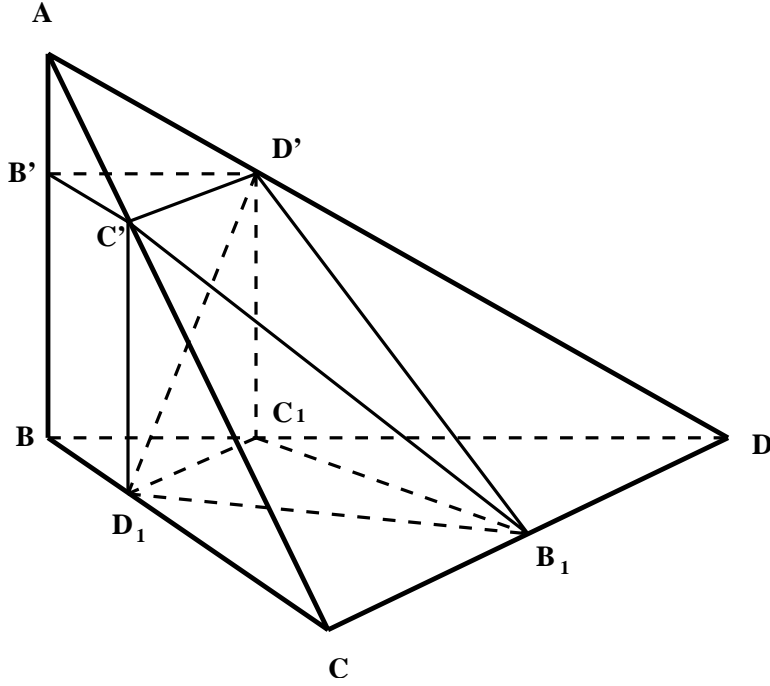


FIGURE 4.7. A of type **V**, B of type **E**, C, D of type **R** and $D_1D' =$ mark for the prism $BD_1C_1D'C_1B'$

equal sides. We also divide each base of Λ into four triangles, according to the division strategy of triangles of type **ERR**, as explained in Subsection 4.1. This yields a decomposition of Λ into eight prisms. Each face of Λ will be divided according to the prescription of the Subsection 4.1.

Let us now discuss the choice of the marks on the 8 smaller prisms. For $\kappa = 1/2$, we choose as marks the resulting diagonals parallel to the original mark. For general κ , we deform this choice from $1/2$ to our desired value for κ . This procedure can be iterated to define the level k of refinement of Λ , which will yield prisms of \mathcal{T}_{n+k} . To obtain the desired tetrahedralization \mathcal{T}'_{n+k} , we divide each of the resulting 2^{3k} marked prisms in three tetrahedra. See Figure 4.8 for the first level of semi-uniform refinement of a prism. We shall come back to these constructions in Subsection 6.1.

By examining our refinement procedure, we obtain the following (for all $m \geq 1$).

Theorem 4.1. *Let \mathcal{T}_n be the sequence of decompositions obtained by applying the procedure outlined in Subsection 4.2 using uniform, semi-uniform, and non-uniform refinements. Then \mathcal{T}_n satisfies the Conditions (i-viii) of Subsection 3.2. All tetrahedra of the resulting tetrahedralizations are of the type **VERR**, **VR³**, or **R⁴**. The region $X := \Omega \setminus \cup_P \mathcal{V}_{P,1}$ is the union of the closed prisms and tetrahedra of type **R⁴** of \mathcal{T}'_0 and \mathcal{T}'_1 .*

In particular, we have the following

Corollary 4.2. *The mesh \mathcal{T}'_n , when restricted to $\Omega \setminus \cup_P \mathcal{V}_{P,0}$ consists of the level n refinements (uniform and semi-uniform) of the tetrahedra of type **R⁴** of the initial tetrahedralization \mathcal{T}'_0 and of the prisms of the initial decomposition \mathcal{T}_0 . The mesh*

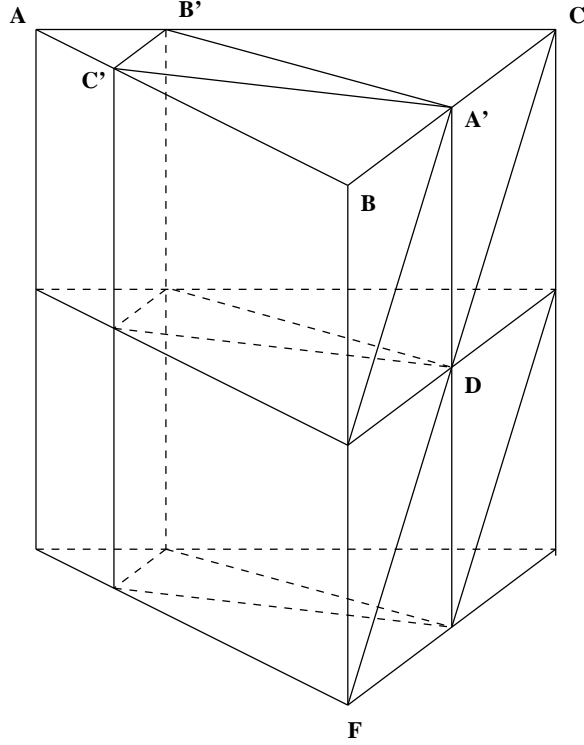


FIGURE 4.8. First level of semi-uniform refinement of a prism, $CD = \text{mark}$

\mathcal{T}'_n , when restricted to $\cup_P(\mathcal{V}_{P,0} \setminus \mathcal{V}_{P,1})$ consists of the level $n-1$ refinements (uniform and semi-uniform) of the tetrahedra of type \mathbf{R}^4 and of the prisms obtained by the non-uniform refinement of the tetrahedra in the initial decomposition \mathcal{T}_0 .

Note that no tetrahedron in the initial decomposition is of type \mathbf{R}^4 . (However, the region Λ_0 is divided into tetrahedra of type \mathbf{R}^4 to yield \mathcal{T}'_0 , the initial tetrahedralization.)

5. INTERPOLATION AND APPROXIMATION ON STANDARD SIMPLICES

Let $T \subset \mathbb{R}^3$ be a tetrahedron with vertices A, B, C , and D . Let $L_m(T) \subset T$ be the set of nodes corresponding to the “linear m simplex” (in the terminology of [26]). In affine coordinates $[\lambda_0, \lambda_1, \lambda_2, \lambda_3] \in \mathbb{R}^3$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$, we have $L_m(T) = \{m^{-1}[k_0, \dots, k_3] \in T, k_j \in \mathbb{Z}_+\}$. Thus, for $m = 1$, we have $L_1(T) = \{A, B, C, D\}$, which corresponds to piecewise linear interpolation.

For $u \in \mathcal{C}(T)$, we shall denote by $I_{T,m}(u)$ the Lagrange interpolant associated to $L_m(T)$. It is the unique polynomial of order m such that $I_{T,m}(u)(x) = u(x)$ for all $x \in L_m(T)$ [26]. If T and m are clear from the context, we shall also write $u_I = I(u) = I_{T,m}(u)$.

Let us consider a prism $ABCA'B'C' \subset \mathbb{R}^3$. We assume that ABC and $A'B'C'$ are congruent triangles lying in parallel planes such that AA' , BB' , and CC' are perpendicular to the planes ABC and $A'B'C'$. In particular, AA' , BB' , and CC'

parallel and congruent (*i.e.*, of the same length). A triangular prism with these properties will be called a *straight triangular prism*.

We can choose our coordinate system such that A is the origin and ABC lies in the coordinate plane $0xy$. Then A' will be on the Oz axis. We shall choose our coordinate system so that the z component of A' is positive. We divide $ABCA'B'C'$ into the three tetrahedra $A'ABC'$, $ABCC'$, and $A'B'BC'$ and let $\hat{\sigma}$ denote any of these tetrahedra. This is the division that we obtain if we consider the prism $ABCA'B'C'$ as being marked by the choice of the diagonal BC' . This would be a good choice of mark if AA' was part of an edge of our polyhedral domain Ω . See Figure 5.1.

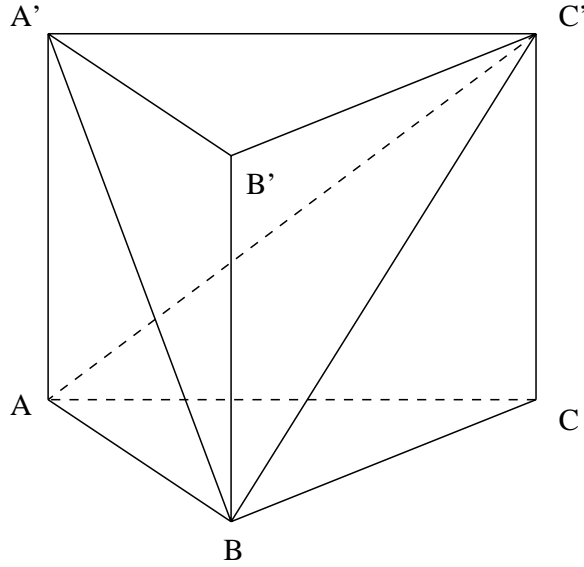


FIGURE 5.1. Marking a prism: $BC' = \text{mark}$, $AA' \parallel BB' \parallel CC' \perp ABC$ and $A'B'C'$

The following result should be compared to Lemmas 2.2 and 2.3 in Apel's book [5] (see especially Equations (2.19) and (2.23)).

Theorem 5.1. *Let $\omega := ABCA'B'C'$ be a straight prism divided into three tetrahedra. Let $\hat{\sigma}$ be any of these tetrahedra and $m \geq 2$. Let $u \in C^1(\hat{\sigma})$ and $I(u) = u_I$ be interpolant associated to the linear m -simplex. Then there exists a constant $C > 0$ such that any $u \in H^{m+1}(\omega)$ satisfies*

- (1) $\partial_z u = 0$ implies $\partial_z I(u) = 0$;
- (2) $\|\partial_z(u - u_I)\|_{L^2(\hat{\sigma})} \leq \hat{C} |\partial_z u|_{H^m(\hat{\sigma})}$;
- (3) $\partial_x u = 0$ and $\partial_y u = 0$ imply $\partial_x I(u) = 0$ and $\partial_y I(u) = 0$;
- (4) $\|\partial_x(u - u_I)\|_{L^2(\hat{\sigma})} + \|\partial_y(u - u_I)\|_{L^2(\hat{\sigma})} \leq \hat{C} (|\partial_x u|_{H^m(\hat{\sigma})} + |\partial_y u|_{H^m(\hat{\sigma})})$.

Proof. Let I_2 be the two dimensional interpolant associated with triangle ABC (the basis of our prism). Thus $I_2(u)$ is a polynomial of degree m whose values coincides with those of u at $L_m(ABC)$. Let $u \in H^{m+1}(\hat{\sigma}) \subset C^1(\hat{\sigma})$, for one of the three reference tetrahedra $\hat{\sigma}$. Assume that $\partial_z u = 0$. Then u is independent of z . We then notice that the projection of $\hat{\sigma}$ onto the xy -coordinate plane is the triangle

ABC and that the nodes of $\hat{\sigma}$ project onto the nodes of ABC . It follows that the function $w(x, y, z) := I_2(u|_{ABC})(x, y)$ is such $w = u$ at the nodes of $\hat{\sigma}$. Therefore $I(u) = w$, which is independent of z as well. This proves Part (1).

Part (3) is proved similarly, but with the triangle ABC replaced with the segment AA' of the Oz axis.

Assume now (1) and let $v = \partial_z u$. Then $\partial_z(u - u_I)$ depends only on v in the sense that if $\partial_z u = \partial_z u_1$, then $\partial_z(u - u_I) - \partial_z(u_1 - u_{1I}) = \partial_z((u - u_1) - (u - u_1)_I) = 0$. Define then $F : H^m(\hat{\sigma}) \rightarrow L^2(\hat{\sigma})$ by $F(v) = \partial_z(u - u_I)$, where $u \in H^m(\hat{\sigma})$ is any function such that $v = \partial_z u$. If v is a polynomial of order $m - 1$, we let $u(x, y, z) = \int_0^z v(x, y, \zeta) d\zeta$, which is a polynomial of degree at most m satisfying $\partial_z u = v$. Then $u = u_I$, and hence we have $F(v) = 0$ for any polynomial v of order $m - 1$. Since F is continuous (for $m \geq 2$), we have by the Bramble-Hilbert lemma that

$$\|\partial_z(u - u_I)\|_{L^2(\hat{\sigma})} := \|F(v)\|_{L^2(\hat{\sigma})} \leq C|v|_{H^m(\hat{\sigma})}.$$

This proves Part (2).

The proof of Part (4) is similar. Let us consider the map $Du = (\partial_x u, \partial_y u)$, with $u \in H^{m+1}(\hat{\sigma})$. Let $V \subset H^m(\hat{\sigma}) \oplus H^m(\hat{\sigma})$ be the closed subspace of pairs (f, g) such that $\partial_y f = \partial_x g$. Then $Du \in V$. We define a map $F : V \rightarrow V$ by $F(f, g) = (\partial_x(u - u_I), \partial_y(u - u_I))$, where u is such that $\partial_x u = f$ and $\partial_y u = g$. By Part (3), this definition is independent of u . In fact, we can choose

$$u(x, y, z) = \int_0^1 (f(tx, ty, z)x + g(tx, ty, z)y) dt = \int_{(0,0,z)}^{(x,y,z)} (f dx + g dy),$$

which shows that if f and g are polynomials of degree $\leq m - 1$, then we can choose u to be a polynomial of degree $\leq m$. Consequently, $F(f, g) = 0$ if f and g are polynomials of degree at most $m - 1$. This shows that

$$\begin{aligned} \|\partial_x(u - u_I)\|_{L^2(\hat{\sigma})}^2 + \|\partial_y(u - u_I)\|_{L^2(\hat{\sigma})}^2 &:= \|F(f, g)\|_{L^2(\hat{\sigma})}^2 \\ &\leq C|(f, g)|_{H^m(\hat{\sigma})}^2 := |f|_{H^m(\hat{\sigma})}^2 + |g|_{H^m(\hat{\sigma})}^2, \end{aligned}$$

again by the Bramble-Hilbert lemma. This proves Part (4). \square

6. INTERPOLATION ON THIN TETRAHEDRA

We now use the results of the previous section to establish that the sequence of tetrahedralizations defined in Section 4 satisfies also Condition (ix). (All the other conditions required in Subsection 3.2 were already verified in Section 4.)

The verification of Condition (ix) essentially reduces to the verification of the analogous statement for each straight triangular prism Λ appearing in the initial decomposition \mathcal{T}_0 of Ω (we shall give details below). Let then $\Lambda = ABCA'B'C'$ be a straight triangular prism as in Section 5. (Recall from Section 5 that a straight triangular prism is a triangular prism such that AA' , BB' , and CC' are perpendicular to the planes of the triangles ABC and $A'B'C'$.) We shall choose our notation so that AA' lies on an edge of Ω . Then, recall from Subsection 1.2, that the $\bar{\Lambda}$ intersects the edges of Ω in AA' , so no other points of $\bar{\Lambda}$, except those on AA' , lie on an edge of Ω . Moreover, no vertex of Ω belongs to $\bar{\Lambda}$.

We can also assume that A is the origin of the coordinate systems and that A' is on the positive Oz coordinate semi-axis. (Then ABC is in the xy coordinate plane.)

6.1. Prism divisions and tetrahedralizations. We shall denote by $\mathcal{T}_n(\Lambda)$ and $\mathcal{T}'_n(\Lambda)$ the restrictions to Λ of the decompositions \mathcal{T}_n and of the tetrahedralizations \mathcal{T}'_n of Ω . Recall then that $\mathcal{T}_n(\Lambda)$ consists of straight prisms that are obtained from a triangulation τ_n of ABC and the division of the edges AA' , BB' , and CC' into 2^n equal segments. The triangulation τ_n consists of 2^{2n} triangles obtained by induction as explained in Subsection 4.1.

Recall from Subsection 4.1 that each triangle in τ_n is obtained by dividing each triangle in τ_{n-1} into four other triangles. There are two types of triangles in τ_{n-1} : type **ERR** and **R³**. A triangle of type **R³** in the triangulation τ_{n-1} is divided uniformly, that is, in four equal triangles, by dividing each side in two. A triangle of type **ERR** in the triangulation τ_{n-1} is still divided into four triangles, but these triangles will not be equal, unless $\kappa = 1/2$. Therefore, a triangle of type **ERR** is divided non-uniformly. This division is obtained by deforming the uniform division so that the points on each side are closer to the singular point and divide that side into the ration $\kappa(1 - \kappa)^{-1}$. The resulting non-uniform division is the same as the one considered in [20].

In any case, the resulting triangles in τ_n can be divided into four groups, each similar to one of the triangles of τ_0 . (The small triangle of τ_0 at the vertex of ABC is similar to ABC .) More precisely, consider the family of affine maps

$$(31) \quad \Phi' := \{\phi(x, y) = (\lambda x + x_0, \lambda y + y_0)\}, \quad \text{with } \lambda = \pm \kappa^{j-1} 2^{j-n}, \quad 1 \leq j \leq n.$$

Then each of the triangles defining the triangulation τ_n is mapped by one of the affine maps ϕ^{-1} , $\phi \in \Phi'$, to one of the four triangles of the triangulation τ_1 .

The division $\mathcal{T}_n(\Lambda)$ of the prism $\Lambda = ABCA'B'C'$ is then obtained by first dividing $\mathcal{T}_n(\Lambda)$ into 2^{2n} straight triangular prisms with bases the triangles in τ_n and their counterparts in $A'B'C'$. Then each of these 2^{2n} thin straight triangular prisms is further divided into 2^n equal straight triangular prisms using equally spaced planes parallel to the bases.

Let us consider the family of affine maps

$$(32) \quad \Phi := \{\phi(x, y, z) = (\lambda x + x_0, \lambda y + y_0, 2^{1-n} z + z_0)\},$$

with $\lambda = \pm \kappa^{j-1} 2^{j-n}$ and $1 \leq j \leq n$ as in the definition of the family Φ' above. Then each of the prisms in $\mathcal{T}_n(\Lambda)$ is mapped by some affine map ϕ^{-1} , $\phi \in \Phi$, to a prism in $\mathcal{T}_1(\Lambda)$, with (x_0, y_0, z_0) a uniquely determined nodal point of $\mathcal{T}_n(\Lambda)$. The 12 tetrahedra of $\mathcal{T}'_1(\Lambda)$ will be called *standard simplices* and will be denoted by $\hat{\sigma}$. Our affine maps preserve the marks and hence can be used to map each simplex in $\mathcal{T}'_n(\Lambda)$ to one of the 12 standard simplices $\hat{\sigma}$.

The following proposition is the crucial step in checking condition (ix). The following lemma is the reason why we need to impose the condition $m \geq 2$ for our main results.

Lemma 6.1. *Let T be a tetrahedron in the level n decomposition $\mathcal{T}'_n(\Lambda)$. Also, let u_I be the interpolant of u associated to the Lagrange m -simplex, $m \geq 2$ and $0 < \kappa \leq 2^{-m/a}$. Then*

$$|u - u_I|_{H^1(T)} \leq C_0 2^{-nm} \|u\|_{\mathcal{D}_{a+1}^{m+1}(T)},$$

for a constant $C_0 > 0$ that depends only on the initial decomposition \mathcal{T}_0 of Ω , on m , and on κ , but not on T or n .

Proof. Let $\phi(x, y, z) = (\lambda x + x_0, \lambda y + y_0, 2^{1-n}z + z_0)$ be the affine map that sends one of the standard simplexes $\hat{\sigma}$ (i.e., one of the tetrahedra of $\mathcal{T}'_1(\Lambda)$) to T bijectively. Recall that $\lambda = \pm\kappa^{j-1}2^{j-n}$ and $1 \leq j \leq n$.

We shall write

$$\hat{u} := u \circ \phi \in \mathcal{D}_{a+1}^{m+1}(\hat{\sigma}),$$

for any $u \in \mathcal{D}_{a+1}^{m+1}(T)$. It is a standard fact that $\widehat{v}_I = (\widehat{v})_I$. We shall also write $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, and $\partial_3 = \partial_z$, and $\partial^{\alpha_\perp} = \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ if $\alpha_\perp = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. Let δ^{-1} be the Jacobian of ϕ .

Let us fix $u \in \mathcal{D}_{a+1}^{m+1}(T)$. Then Theorem 5.1(4) gives, for $|\alpha_\perp| = \alpha_1 + \alpha_2 = m - \gamma$,

$$\begin{aligned} (33) \quad & |\lambda|^2 \delta \sum_{i=1}^2 \|\partial_i u - \partial_i u_I\|_{L^2(T)}^2 = |\lambda|^2 \sum_{i=1}^2 \|\widehat{\partial_i u} - \widehat{\partial_i u}_I\|_{L^2(\hat{\sigma})}^2 \\ &= \sum_{i=1}^2 \|\partial_i \hat{u} - \partial_i \widehat{u}_I\|_{L^2(\hat{\sigma})}^2 = \sum_{i=1}^2 \|\partial_i \hat{u} - \partial_i (\hat{u})_I\|_{L^2(\hat{\sigma})}^2 \leq C \sum_{i=1}^2 |\partial_i \hat{u}|_{H^m(\hat{\sigma})}^2 \quad (\text{Thm. 5.1}) \\ &= C \sum_{i=1}^2 \sum_{\alpha_\perp} \|\partial^{\alpha_\perp} \partial_z^\gamma \partial_i \hat{u}\|_{L^2(\hat{\sigma})}^2 = C_1 \delta \sum_{i=1}^2 \sum_{\alpha_\perp} |\lambda|^{2|\alpha_\perp|+2} 2^{-2(n-1)\gamma} \|\partial^{\alpha_\perp} \partial_z^\gamma \partial_i u\|_{L^2(T)}^2, \end{aligned}$$

where the second sum is for all $\alpha_\perp = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ such that $|\alpha_\perp| + \gamma = m$, as it will be the case below.

Let r be the distance to the line AA' (the line supporting an edge of our polyhedral domain Ω). Assume first that our tetrahedron T is not adjacent to the edge AA' . If $\lambda = \pm\kappa^{j-1}2^{j-n}$, this means that $C_r r \geq \kappa^j$ on T , for a constant that depends only on the initial decomposition \mathcal{T}_0 . Next we use, for $k \geq 1$ and $a \in (0, 1/2]$, the inequality $\|w\|_{L^2(T)} \leq C_r^{k-a} \kappa^{j(a-k)} \|r^{k-a} w\|_{L^2(T)}$ to obtain, with C_1 and δ as in Equation (33) (and C a generic constant that depends only on the initial decomposition \mathcal{T}_0),

$$\begin{aligned} (34) \quad & C_1 \delta \sum_{i=1}^2 \sum_{\alpha_\perp} |\lambda|^{2|\alpha_\perp|+2} 2^{-2(n-1)\gamma} \|\partial^{\alpha_\perp} \partial_z^\gamma \partial_i u\|_{L^2(T)}^2 \\ & \leq C \delta \sum_{i=1}^2 \sum_{|\alpha_\perp| > 0} |\lambda|^{2|\alpha_\perp|+2} 2^{-2(n-1)\gamma} \kappa^{2j(a-|\alpha_\perp|)} \|r^{|\alpha_\perp|-a} \partial^{\alpha_\perp} \partial_z^\gamma \partial_i u\|_{L^2(T)}^2 \\ & \quad + C \delta |\lambda|^2 2^{-2(n-1)m} \sum_{i=1}^2 \|\partial_z^m \partial_i u\|_{L^2(T)}^2 \\ & \leq C \delta |\lambda|^2 \kappa^{-2m} 2^{-2nm} \left(\sum_{|\alpha_\perp| \geq 2} \|r^{|\alpha_\perp|-a-1} \partial^{\alpha_\perp} \partial_z^\gamma u\|_{L^2(T)}^2 + \sum_{i=1}^2 \|\partial_z^m \partial_i u\|_{L^2(T)}^2 \right). \end{aligned}$$

(To estimate the complicated constants, we just compute, using $|\alpha_\perp| + \gamma = m$ and $\kappa \leq 2^{-m/a}$.)

Combining the Equations (33) and (34), we obtain

$$(35) \quad \sum_{i=1}^2 \|\partial_i u - \partial_i u_I\|_{L^2(T)}^2 \leq C_0 2^{-2nm} \left(\sum_{|\alpha_\perp| \geq 2} \|r^{|\alpha_\perp| - a - 1} \partial^{\alpha_\perp} \partial_z^\gamma u\|_{L^2(T)}^2 + \sum_{i=1}^2 \|\partial_z^m \partial_i u\|_{L^2(T)}^2 \right),$$

with C_0 depending on κ , on m , and on the initial decomposition \mathcal{T}_0 , but not on n or T (as long as $\bar{T} \cap AA' = \emptyset$).

We now treat the $\partial_z u$ term. We continue to assume that T is not adjacent to the edge AA' . We proceed similarly, but using Theorem 5.1(2) instead of Theorem 5.1(4). We continue to denote $|\alpha_\perp| = \alpha_1 + \alpha_2 = m - \gamma$. Then

$$(36) \quad \begin{aligned} 2^{-2(n-1)} \delta \|\partial_z u - \partial_z u_I\|_{L^2(T)}^2 &= 2^{-2(n-1)} \|\widehat{\partial_z u} - \widehat{\partial_z u_I}\|_{L^2(\hat{\sigma})}^2 \\ &= \|\partial_z \hat{u} - \partial_z \widehat{u_I}\|_{L^2(\hat{\sigma})}^2 = \|\partial_z \hat{u} - \partial_z(\hat{u})_I\|_{L^2(\hat{\sigma})}^2 \leq C |\partial_z \hat{u}|_{H^m(\hat{\sigma})}^2 \quad (\text{Thm. 5.1}) \\ &= C \sum_{\alpha_\perp} \|\partial^{\alpha_\perp} \partial_z^{\gamma+1} \hat{u}\|_{L^2(\hat{\sigma})}^2 = C_1 \delta \sum_{\alpha_\perp} |\lambda|^{2|\alpha_\perp|} 2^{-2(n-1)(\gamma+1)} \|\partial^{\alpha_\perp} \partial_z^{\gamma+1} u\|_{L^2(T)}^2 \\ &\leq C \delta \left(\sum_{|\alpha_\perp| \geq 2} |\lambda|^{2|\alpha_\perp|} 2^{-2(n-1)(\gamma+1)} \kappa^{2j(a+1-|\alpha_\perp|)} \|r^{|\alpha_\perp| - a - 1} \partial^{\alpha_\perp} \partial_z^{\gamma+1} u\|_{L^2(T)}^2 \right. \\ &\quad \left. + |\lambda|^2 2^{-2(n-1)m} \sum_{i=1}^2 \|\partial_i \partial_z^m u\|_{L^2(T)}^2 + 2^{-2(n-1)(m+1)} \|\partial_z^{m+1} u\|_{L^2(T)}^2 \right) \\ &\leq C \delta \kappa^{-2m} 2^{-2(n-1)} 2^{-2nm} \left(\sum_{|\alpha_\perp| \geq 2} \|r^{|\alpha_\perp| - a - 1} \partial^{\alpha_\perp} \partial_z^{\gamma+1} u\|_{L^2(T)}^2 \right. \\ &\quad \left. + \sum_{i=1}^2 \|\partial_i \partial_z^m u\|_{L^2(T)}^2 + \|\partial_z^{m+1} u\|_{L^2(T)}^2 \right). \end{aligned}$$

Next, we include the factor κ^{-m} into the constant C , which will be then denoted C_0 and will hence depend on κ . Then we multiply the inequality (36) with $2^{2(n-1)} \delta^{-1}$ and add with the inequality of Equation (35), to obtain (for the last inequality we also use Lemma 1.7)

$$(37) \quad |u - u_I|_{H^1(T)}^2 := \sum_{i=1}^3 \|\partial_i u - \partial_i u_I\|_{L^2(T)}^2 \leq C_0 2^{-2nm} \sum_{i=1}^2 \left(\sum_{|\alpha_\perp| \geq 2} \|r^{|\alpha_\perp| - a - 1} \partial^{\alpha_\perp} \partial_z^{\gamma+1} u\|_{L^2(T)}^2 + \sum_{i=1}^2 \|\partial_i \partial_z^m u\|_{L^2(T)}^2 + \|\partial_z^{m+1} u\|_{L^2(T)}^2 \right) \leq C_0 2^{-2nm} \|u\|_{D_{a+1}^{m+1}(T)}^2,$$

for a possibly larger constant C_0 (still independent of n and T). This proves the desired result as long as T is not adjacent to AA' .

Assume now that T is adjacent to AA' . Then $\lambda = \kappa^{n-1}$, so $j = n$. We shall proceed as in the proof of Theorem 3.2, more precisely, we shall use the idea used in the proof of Equation (28).

Let χ be a smooth function on Ω such that $\chi = 0$ in a neighborhood of the edges and such that $\chi = 1$ at all nodal points of \mathcal{T}'_0 not on the edges. Let $u \in \mathcal{D}_{a+1}^{m+1}(T)$. Define $v \in \mathcal{D}_{a+1}^{m+1}(T)$ by $\hat{v} = \chi \hat{u}$. Then we still have $C_r r \geq \kappa^j = \kappa^n$, with C_r depending only on the initial decomposition \mathcal{T}_0 . Exactly the same proof as above then extends to v in place of u to show that

$$(38) \quad |v - v_I|_{H^1(T)} \leq C_0 2^{-nm} \|v\|_{\mathcal{D}_{a+1}^{m+1}(T)}.$$

We again notice that $v_I = u_I$ (since u and v coincide at the node points). Let χ_0 be such that $\hat{\chi}_0 = \chi$. So $v = \chi_0 u$. We can assume that the restriction of χ_0 to our prism Λ is independent of z . Then we obtain $\|\chi_0\|_{m+1,\infty} = \|\chi\|_{m+1,\infty}$, and hence Lemma 1.6 gives

$$\|v\|_{\mathcal{D}_{a+1}^{k+1}(T)} = \|\chi_0 u\|_{\mathcal{D}_{a+1}^{k+1}(T)} \leq C_0 \|u\|_{\mathcal{D}_{a+1}^{k+1}(T)}, \quad 0 \leq k \leq m,$$

with C_0 depending only on the initial decomposition \mathcal{T}_0 . Taking also into account that $\kappa^{na} \leq 2^{-nm}$, we obtain, using also Equation (20),

$$(39) \quad \begin{aligned} |u - u_I|_{H^1(T)} &\leq |u - v|_{H^1(T)} + |v - v_I|_{H^1(T)} \leq C_0 (|u|_{H^1(T)} + 2^{-nm} \|v\|_{\mathcal{D}_{a+1}^{m+1}(T)}) \\ &\leq C_0 \left(\kappa^{na} \sum_{i=1}^3 \|r^{-a} \partial_i u\|_{L^2(T)} + 2^{-nm} \|v\|_{\mathcal{D}_{a+1}^{m+1}(T)} \right) \\ &\leq C_0 2^{-nm} (\|u\|_{\mathcal{D}_{a+1}^1(T)} + \|u\|_{\mathcal{D}_{a+1}^{m+1}(T)}) \leq C_0 2^{-nm} \|u\|_{\mathcal{D}_{a+1}^{m+1}(T)}. \end{aligned}$$

□

We are ready now to prove one of our main results, stating that Condition (ix) is satisfied for $m \geq 2$ for our sequence of tetrahedralizations.

Theorem 6.2. *Let \mathcal{T}_n be the sequence of decompositions obtained by applying the procedure outlined in Subsection 4.2 using uniform, semi-uniform, and non-uniform refinements. Let $X := \Omega \setminus \cup_P \mathcal{V}_{P,1}$, $m \geq 2$, and $\kappa \leq 2^{-m/a}$. Denote by $u_{I,n}$ the Lagrange interpolant associated to \mathcal{T}'_n and the “ m -simplex.” Then*

$$|u - u_{I,n}|_{H^1(X)} \leq C_0 2^{-nm} \|u\|_{\mathcal{D}_{a+1}^{m+1}(X)},$$

for a constant C_0 that depends only on the initial tetrahedralization \mathcal{T}'_0 , on m , and on κ , but not on n or u .

Proof. We now turn to the proof of Condition (ix). Let $X := \Omega \setminus \cup_P \mathcal{V}_{P,1}$, (as in Condition (ix)). Recall from Corollary 4.2, that X consists of a union of regions Λ that are either marked prisms and or tetrahedra of type \mathbf{R}^4 . To these region we apply either a level n or $n - 1$ of uniform or semi-uniform refinement. Let $u_{I,n}$ be the interpolant associated to \mathcal{T}'_n (which is a mesh on Λ) on each of these regions Λ . If $\Lambda \subset X := \Omega \setminus \cup_P \mathcal{V}_{P,1}$ is a prism, then adding the inequalities of Lemma 6.1 for all tetrahedra T , we obtain

$$(40) \quad |u - u_I|_{H^1(\Lambda)} \leq C_0 2^{-nm} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\Lambda)},$$

with C_0 depending only on \mathcal{T}_0 , m , and κ (but not on n).

Equation (40) holds also for $\Lambda \subset X$ a tetrahedron of type \mathbf{R}^4 , since all tetrahedra obtained by uniformly dividing Λ are similar to a fixed number of tetrahedra and all edges are of the order 2^{-n} .

Adding all the equations (40) for all the regions $\Lambda \subset X$, we obtain the desired inequality. The proof is now complete. \square

An averaged interpolant like in [5, 27, 56] will probably allow us to remove the condition $m \geq 2$ in the above theorem.

7. CONCLUSION

We summarize here our main results. Let \mathcal{T}_0 be the initial decomposition of our polyhedral domains in straight triangular prisms, tetrahedra of types **VERR** and **VR**³, and an interior region Λ_0 , as in Subsection 1.2. (The tetrahedra of type **VERR** and **VR**³ are the ones having a vertex in common with the vertices of Ω .) We mark the prisms of \mathcal{T}_0 following a given set of rules that will insure the conformity of the resulting meshes. The marks will allow us to tetrahedralize Ω by dividing each prisms in three tetrahedra as determined by the mark and then by tetrahedralizing Λ_0 without introducing additional edges on the boundary of Λ_0 (but allowing additional internal edges and vertices). We then apply uniform, semi-uniform, and non-uniform refinements to obtain the decompositions \mathcal{T}_n of Ω into marked prisms and tetrahedra, as summarized in Subsection 4.2 and then explained in detail in the later subsections. The meshes \mathcal{T}_n are obtained by dividing each prism into three tetrahedra as determined by the mark (see Figure 5.1). Then the sequence of decomposition \mathcal{T}_n and the sequence of meshes \mathcal{T}'_n satisfy the conditions (i–viii) of Subsection 3.2, Theorem 4.1. For $m \geq 2$, the sequences \mathcal{T}_n and \mathcal{T}'_n satisfy also Condition (ix) of Subsection 3.2, by Theorem 6.2.

Let S_n be the Finite Element spaces of continuous, piecewise polynomials of degree at most m on our sequence of meshes \mathcal{T}'_n and $u_n \in S_n$ the discrete (Finite Element) solution of the Poisson problem (1) (recall that this main boundary value problem is $-\Delta u = f$, $u = 0$ at the boundary). Theorem 2.2 then guarantees that there exists $a > 0$, depending only on Ω , and a constant $C > 0$, depending on a and Ω , such that $\|u\|_{\mathcal{D}_{\frac{a+1}{a}}^{m+1}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)}$. Combining this result with Theorem 3.3, we obtain our main result:

Theorem 7.1. *Let S_n be the Finite Element spaces of continuous, piecewise polynomials of degree $m \geq 2$ associated to the tetrahedralization \mathcal{T}'_n of Ω . Let $a > 0$ be as in Theorem 2.2 and $\kappa \leq 2^{-m/a}$. Then there exists $C_\kappa > 0$ such that*

$$\|u - u_n\|_{H^1(\Omega)} \leq C_\kappa \dim(S_n)^{-m/3} \|f\|_{H^{m-1}(\Omega)},$$

for all $f \in H^{m-1}(\Omega)$ and all $n \in \mathbb{N}$. Moreover, $\dim(S_n) \sim 2^{3n}$.

By $\dim(S_n) \sim 2^{3n}$, we mean that $\dim(S_n)2^{-3n} \in [c^{-1}, c]$ for some $c > 0$ independent of n .

The interpolation results of this paper hold only for $m \geq 2$, because of this, the final results, including Theorem 7.1, are proved only for $m \geq 2$ (it is possible that Theorem 7.1 remains true for $m = 1$, but the proof would require a different type of interpolant).

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