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Regularity estimates for elliptic boundary value problems with smooth data on polygonal domains

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Abstract — We consider the model Dirichlet problem for Poisson's equation on a plane polygonal convex domain Ω with data f in a space smoother than L^2. The regularity and the critical case of the problem depend on the measure of the maximum angle of the domain. Interpolation theory and multilevel theory are used to obtain estimates for the critical case. As a consequence, sharp error estimates for the corresponding discrete problem are proved. Some classical shift estimates are also proved using the powerful tools of interpolation theory and mutilevel approximation theory. The results can be extended to a large class of elliptic boundary value problems.

Keywords: interpolation spaces, finite element method, multilevel decomposition, shift theorems, subspace interpolation

1. INTRODUCTION

Regularity estimates of the solutions of elliptic boundary value problems in terms of Sobolev norms of fractional order are known as shift theorems or shift estimates. The shift estimates for the Laplace operator with Dirichlet boundary conditions with non-smooth data on polygonal domains are well known (see, e.g, [2,21,23,27]). The classical regularity estimate for the case when Ω is a convex polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$, is as follows: If u is the variational solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

then, $u \in H^2(\Omega)$ and

$$\|u\|_{H^2} \leqslant C\|f\|_{H^0} \quad \forall f \in H^0(\Omega) = L^2(\Omega). \tag{1.2}$$

Let ω be the radian measure of the largest corner of Ω ($\omega < \pi$), and let $s_0 = \min\{1, \pi/\omega - 1\}$. If u is the variational solution of (1.1), then for $0 < s < s_0$, it

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is known (see e.g. [16]) that

$$\|u\|_{H^{2+s}} \leqslant C(s)\|f\|_{H^s} \quad \forall f \in H^s(\Omega). \tag{1.3}$$

Here, $H^s(\Omega)$ is the interpolation space between $H^1(\Omega)$ and $L^2(\Omega)$ $s-$distance from $L^2(\Omega)$. In this paper we will prove a stronger version of (1.3) and approach the problem in the critical value situation $s = s_0$. We prove that, for $s_0 < 1$, the following estimate holds.

$$\|u\|_{B_\infty^{2+s_0}(\Omega)} \leqslant c\|f\|_{B_1^{s_0}(\Omega)} \quad \forall f \in B_1^{s_0}(\Omega) \tag{1.4}$$

where $B_\infty^{2+s_0}(\Omega)$ and $B_1^{s_0}(\Omega)$ are standard interpolation spaces defined in Section 3. It is worth noting here that the space $B_1^{s_0}(\Omega)$ contains all the Hilbert spaces H^s with $s > s_0$. The estimate (1.4) leads to new finite element convergence estimates. The method presented in this paper can be extended to other boundary conditions such as Neumann or mixed Neumann–Dirichlet conditions.

The technique involved in proving shift results is the real method of interpolation of Lions and Peetre ([3,24,25]), combined with multilevel approximation theory. The cases $q = 2$, $q = 1$ and $q = \infty$, where q is the second index of interpolation, are of special importance for our problem. The following type of interpolation problem is essential for our approach. If X and Y are Sobolev spaces of integer order and X_K is a subspace of codimension one of X, then how can one characterize the interpolation spaces between X_K and Y for $q = 2$ and $q = \infty$? The problem was studied in [20,21] and [1] for $q = 2$ and particular spaces X and X_K. This paper gives an interpolation result of this type for the case $q = \infty$.

The remaining part of the paper is structured as follows. The interpolation results presented in Section 2 give a formulas for the norms on the intermediate subspaces $[X_K, Y]_{s,2}$ and $[X_K, Y]_{s,\infty}$ when X_K is of codimension one. In Section 3 the main result concerning the codimension-one subspace interpolation problem is presented. Shift theorems for the Poisson equation on polygonal domains are considered in Section 4. In the last section, a straightforward application of the interpolation results is shown to lead to some new estimates for finite element approximations.

2. INTERPOLATION RESULTS

In this section we give some definitions and results concerning interpolation spaces via real method of interpolation of Lions and Peetre (see [3,4,24]).

2.1. Interpolation between Banach spaces

Let X, Y be Banach spaces satisfying for some positive constant c,

$$\begin{cases} X \text{ is a dense subset of Y and} \\ \|u\|_Y \leqslant c\|u\|_X \quad \forall u \in X \end{cases} \tag{2.1}$$

where $\|u\|_X$ and $\|u\|_Y$ are the norms on X and Y, respectively.

The interpolation spaces $[X,Y]_{s,q}$, $1 \leqslant q \leqslant \infty$, and $s \in (0,1)$ are defined using the K function, where for $u \in Y$ and $t > 0$,

$$K(t,u) := \inf_{u_0 \in X} (\|u_0\|_X^2 + t^2 \|u - u_0\|_Y^2)^{1/2}.$$

Then, for $q < \infty$, the space $[X,Y]_{s,q}$ consists of all $u \in Y$ such that

$$\int_0^\infty (t^{-s} K(t,u))^q \, \frac{\mathrm{d}t}{t} < \infty$$

and $[X,Y]_{s,\infty}$ consists of all $u \in Y$ such that

$$\sup_{t>0} t^{-2s} K(t,u)^2 < \infty.$$

The norm on $[X,Y]_{s,q}$ is defined by

$$\|u\|_{[X,Y]_{s,q}}^2 := \int_0^\infty (t^{-s} K(t,u))^q \, \frac{\mathrm{d}t}{t}$$

and the norm on $[X,Y]_{s,\infty}$ is defined by

$$\|u\|_{[X,Y]_{s,\infty}} := \sup_{t>0} t^{-s} K(t,u).$$

Remark 2.1. Since $K(t,u) \leqslant t\|u\|_Y$ for all $u \in Y$, the interval $(0,\infty)$ in the above definitions can be replaced by any subinterval (A,∞). The new norms obtained are equivalent with the original norms.

2.2. Interpolation between Hilbert spaces

An extended Hilbert interpolation theory can be found in [24]. For completeness and consistence of notation we present in this section the interpolation results used in this paper.

Let X,Y be separable Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively, and satisfying (2.1). Let $D(S)$ denote the subset of X consisting of all elements u such that

$$v \to (u,v)_X, \quad v \in X \tag{2.2}$$

is continuous in the topology induced by Y.

For any u in $D(S)$ the anti-linear form (2.2) can be uniquely extended to a continuous anti-linear form on Y. Then by Riesz representation theorem, there exists an element Su in Y such that

$$(u,v)_X = (Su,v)_Y \quad \forall v \in X. \tag{2.3}$$

In this way S is a well defined operator with domain $D(S)$ in Y. The next result illustrates the properties of S (see [24]).

Proposition 2.1. *The domain $D(S)$ of the operator S is dense in X and consequently $D(S)$ is dense in Y. The operator $S : D(S) \subset Y \to Y$ is a bijective, self-adjoint and positive definite operator. The inverse operator $S^{-1} : Y \to D(S) \subset Y$ is a bounded symmetric positive definite operator and*

$$(S^{-1}z, u)_X = (z, u)_Y \quad \forall z \in y,\ u \in X. \tag{2.4}$$

The next lemma provides the relation between $K(t,u)$ and the connecting operator S. A proof of the lemma is given in [2].

Lemma 2.1. *For all $u \in Y$ and $t > 0$,*

$$K(t,u)^2 = t^2 \left((I + t^2 S^{-1})^{-1} u, u\right)_Y.$$

For the special case $q = 2$ (X, Y Hilbert spaces), due to the spectral or multilevel representation of the norm on $[X,Y]_{s,2}$, the definition of the norm is slightly changed as follows:

$$\|u\|^2_{[X,Y]_{s,2}} := \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} K(t,u)^2 \, \mathrm{d}t$$

where

$$\mathbf{c}_s := \left(\int_0^\infty \frac{t^{1-2s}}{t^2+1} \, \mathrm{d}t \right)^{-1/2} = \sqrt{\frac{2}{\pi} \sin(\pi s)}.$$

With the new definition for the norm of $[X,Y]_{s,2}$ it is natural (see the multilevel representation case in Section 3) to define

$$[X,Y]_{0,2} := X, \quad [X,Y]_{1,2} := Y.$$

Remark 2.2. Lemma 2.1 yields other expressions for the norms on $[X,Y]_{s,2}$ and $[X,Y]_{s,\infty}$. Namely,

$$\|u\|^2{}_{[X,Y]_{s,2}} = \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left((I + t^2 S^{-1})^{-1} u, u\right)_Y \, \mathrm{d}t \tag{2.5}$$

and

$$\|u\|^2{}_{[X,Y]_{s,\infty}} = \sup_{t>0} t^{2(1-s)} \left((I + t^2 S^{-1})^{-1} u, u\right)_Y. \tag{2.6}$$

2.3. Interpolation between subspaces of a Hilbert space

Let $\mathscr{K} = \mathrm{span}\{\varphi\}$ be a one-dimensional subspace of X and let $X_{\mathscr{K}}$ be the orthogonal complement of $\mathscr{K}$ in X in the $(\cdot,\cdot)_X$ inner product. We are interested in determining the interpolation spaces of $X_{\mathscr{K}}$ and Y, where on $X_{\mathscr{K}}$ we consider again the $(\cdot,\cdot)_X$ inner product. To apply the interpolation results from the previous section we need to check that the density part of the condition (2.1) is satisfied for the pair $(X_{\mathscr{K}}, Y)$.

For $\varphi \in \mathscr{K}$, define the linear functional $\Lambda_\varphi : X \to C$, by

$$\Lambda_\varphi u := (u, \varphi)_X, \quad u \in X.$$

The following result is an extension of Kellogg's lemma [21]. The proof can be found in [1].

Lemma 2.2. *Let $\mathscr{K}$ be a closed subspace of X and let $X_{\mathscr{K}}$ be the orthogonal complement of $\mathscr{K}$ in X in the $(\cdot,\cdot)_X$ inner product. The space $X_{\mathscr{K}}$ is dense in Y if and only if the following condition is satisfied:*

$$\left\{ \begin{array}{l} \Lambda_\varphi \text{ is not bounded in the topology of } Y \\ \quad \text{for all } \varphi \in \mathscr{K},\ \varphi \neq 0. \end{array} \right. \tag{2.7}$$

For the remaining part of this section we assume that Λ_φ is not bounded in the topology of Y, so the condition (2.1) is satisfied for the pair $(X_{\mathscr{K}}, Y)$. We denote $X_{\mathscr{K}}$ by X_φ. It follows from the previous section that the operator $S_\varphi : D(S_\varphi) \subset Y \to Y$ defined by

$$(u, v)_X = (S_\varphi u, v)_Y \quad \forall v \in X_\varphi \tag{2.8}$$

has the same properties as S. Consequently, the norms on the intermediate spaces $[X_\varphi, Y]_{s,2}$ and $[X_\varphi, Y]_{s,\infty}$ are given by:

$$\|u\|^2{}_{[X_\varphi,Y]_{s,2}} = \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left((I + t^2 S_\varphi^{-1})^{-1} u, u\right)_Y \, \mathrm{d}t \tag{2.9}$$

and

$$\|u\|^2{}_{[X_\varphi,Y]_{s,\infty}} := \sup_{t>0} t^{2(1-s)} \left((I + t^2 S_\varphi^{-1})^{-1} u, u\right)_Y. \tag{2.10}$$

Our aim in this section is to determine sufficient conditions for φ such that the norm $[X_\varphi, Y]_{s,q}$ (for $q = 2$ and $q = \infty$) can be compared with more familiar intermediate norms which are independent of φ. First, we note that the operators S_φ and S are related by the following identity:

$$S_\varphi^{-1} = (I - Q_\varphi) S^{-1} \tag{2.11}$$

where $Q_\varphi : X \to \mathscr{K}$ is the orthogonal projection onto $\mathscr{K} = \mathrm{span}\{\varphi\}$. The proof of (2.11) follows easily from the definitions of the operators involved.

Next, (2.11) leads to a new formula for the norms on $[X_\varphi, Y]_{s,\infty}$ and $[X, Y]_{s,2}$.

Theorem 2.1. *For any $u \in Y$ we have,*

$$\|u\|^2{}_{[X_\varphi,Y]_{s,2}} = \|u\|^2{}_{[X,Y]_{s,2}} + \mathbf{c}_s^2 \int_0^\infty t^{-2s+3} \frac{|(u, \varphi)_{Y,t}|^2}{(\varphi, \varphi)_{X,t}} \, \mathrm{d}t \tag{2.12}$$

and

$$\|u\|^2_{[X_\varphi,Y]_{s,\infty}} = \sup_{t>0}\left(t^{2-2s}(u,u)_{Y,t} + t^{4-2s}\frac{|(u,\varphi)_{Y,t}|^2}{(\varphi,\varphi)_{X,t}}\right). \tag{2.13}$$

In particular for $u \in X_\varphi$ we have

$$\|u\|^2{}_{[X_\varphi,Y]_{s,2}} = \|u\|^2{}_{[X,Y]_{s,2}} + \mathbf{c}_s^2\int_0^\infty t^{-(2s+1)}\frac{|(u,\varphi)_{X,t}|^2}{(\varphi,\varphi)_{X,t}}\,\mathrm{d}t \tag{2.14}$$

and

$$\|u\|^2_{[X_\varphi,Y]_{s,\infty}} = \sup_{t>0}\left(t^{2-2s}(u,u)_{Y,t} + t^{-2s}\frac{|(u,\varphi)_{X,t}|^2}{(\varphi,\varphi)_{X,t}}\right) \tag{2.15}$$

where

$$(u,v)_{X,t} := \left((I+t^2S^{-1})^{-1}u,v\right)_X \quad \forall u,v \in X \tag{2.16}$$

and

$$(u,v)_{Y,t} := \left((I+t^2S^{-1})^{-1}u,v\right)_Y \quad \forall u,v \in Y. \tag{2.17}$$

Proof. The first two formulas follows immediately from Remark 2.2 and the following identity

$$\left((I+t^2S_\varphi^{-1})^{-1}u,u\right)_Y = (u,u)_{Y,t} + t^2\,\frac{|(u,\varphi)_{Y,t}|^2}{(\varphi,\varphi)_{X,t}}.$$

The proof of the identity is based on (2.11). A detailed proof of it can be found in [2]. Using (2.4) and a simple manipulation of the operator S we get

$$t^2(u,\varphi)_{Y,t} = (u,\varphi)_X - (u,\varphi)_{X,t}$$

which, for $u \in X_\varphi$ leads to (2.14) and (2.15). □

Theorem 2.1 can be easily extended to the case when K is of finite dimension(see [1]). Such an extension would be needed, for example, to treat the case of mixed Neumann–Dirichlet conditions or the case of the biharmonic problem.

3. MULTILEVEL REPRESENTATION OF INTERPOLATION SPACES

Let Ω be a domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Assume that

$$M_1 \subset M_2 \subset \cdots \subset M_k \subset \dots$$

is a sequence of finite dimensional subspaces of $H^1(\Omega)$ whose union is dense in $H^1(\Omega)$, and assume that an equivalent norm on $H^1(\Omega)$ is given by

$$\|u\|_1 := \left(\sum_{k=1}^{\infty}\lambda_k\|(Q_k - Q_{k-1})u\|^2\right)^{1/2} \tag{3.1}$$

where Q_k denotes the $L^2(\Omega)$ orthogonal projection onto M_k, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $Q_0 = 0$ and $\lambda_k = 4^{k-1}$.

The sequence $\{M_k\}$ can be taken for example the standard sequence of piecewise linear functions associated with a sequence of nested meshes. Proofs for the multilevel representation of the norm on H^1, for specific choices of the spaces M_k can be found in [1,10,28,30].

3.1. Scales of multilevel norms

On $H^1(\Omega)$ we consider the norm given by (3.1) and define $H^{-1}(\Omega)$ to be the dual of $H^1(\Omega)$. The elements of $L^2(\Omega)$ can be viewed as continuous linear functionals on $H^1(\Omega)$ and we have the natural continuous and dense embeddings

$$H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega).$$

The projection Q_k, $k = 1, 2, \ldots$, can be extended to be defined on $H^{-1}(\Omega)$ by

$$(Q_k u, v)_{L^2} = (u, v) \quad \forall u \in H^{-1}(\Omega),\ v \in M_k$$

where $(\cdot,\cdot)$ on the right hand side represents the duality between $H^{-1}(\Omega)$ and $H^1(\Omega)$.

One can easily check that the induced inner product on $H^\varepsilon(\Omega)$ is given by

$$(u, v)_\varepsilon := \sum_{k=1}^{\infty} \lambda_k^\varepsilon\ (q_k u, q_k v) \quad \forall u, v \in H^\varepsilon(\Omega), \quad \varepsilon = -1, 1$$

where $q_k = Q_k - Q_{k-1}$.

Then the pair $(H^1(\Omega), L^2(\Omega))$ satisfies the condition (2.1) and the operator S associated with the pair is given by

$$Su = \sum_{k=1}^{\infty} \lambda_k\ q_k u \quad \forall u \in D(S). \tag{3.2}$$

Thus, for $X = H^1(\Omega)$ and $Y = L^2(\Omega)$, we have

$$(u, v)_{Y,t} = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2} (q_k u, q_k v), \quad u, v \in Y$$

and

$$(u, v)_{X,t} = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\lambda_k + t^2} (q_k u, q_k v), \quad u, v \in X.$$

For any $s \in (0, 1)$, $q = 1$ or $q = \infty$, let

$$H^s(\Omega) := [H^1(\Omega), L^2(\Omega)]_{1-s,2}\ , \quad H^{2+s}(\Omega) := [H^3(\Omega), H^1(\Omega)]_{1-s,2}$$

and

$$B_q^s(\Omega) := [H^1(\Omega), L^2(\Omega)]_{1-s,q}, \quad B_q^{2+s}(\Omega) := [H^3(\Omega), H^1(\Omega)]_{1-s,q}.$$

By using (2.5) and (2.6), one can easily check that

$$\|u\|_{H^s(\Omega)}^2 = \sum_{k=1}^{\infty} \lambda_k^s \|q_k u\|^2 \tag{3.3}$$

and

$$\|u\|_{B_\infty^s(\Omega)}^2 = \sup_{t>0} \left(t^{2s} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2} \|q_k u\|^2 \right). \tag{3.4}$$

One can verify using the above formula that $B_\infty^0(\Omega) = L^2(\Omega)$ and $B_\infty^1(\Omega) = H^1(\Omega)$. In addition we have

$$\sup_{t>0} \frac{t^{2s}}{\lambda_k + t^2} = s^s (1-s)^{1-s} \lambda_k^{-1+s}$$

which leads to

$$\|u\|_{B_\infty^s(\Omega)}^2 \leqslant s^s (1-s)^{1-s} \|u\|_{H^s(\Omega)}^2 \quad \forall u \in H^s(\Omega) \tag{3.5}$$

a well known embedding property.

Remark 3.1. If we assume that the sequence of subspaces $\{M_k\}$ is chosen so that the following approximation and inverse inequalities hold with a constant c independent of j:

(Ap) $$\|u - Q_j u\| \leqslant c\, 2^{-j} \|u\|_{H^1} \quad \forall u \in H^1(\Omega)$$

(Inv) $$\|u\| \leqslant c\, 2^j \|u\|_{H^1} \quad \forall u \in M_j$$

then, by a well known result of the approximation theory, an equivalent norm on $[H^1, L^2]_{1-s,q}$ is given by

$$\|\{2^{sj} \|q_j u\|\}_{j \geqslant 1}\|_{l_q}, \quad 0 < s < 1, \quad 1 \leqslant q \leqslant \infty.$$

In particular, an equivalent norm on $B_1^s := [H^1, L^2]_{1-s,1}$ is given by

$$\sum_{j=1}^{\infty} 2^{sj} \|q_j u\|, \quad 0 < s < 1.$$

One can verify now that we have the following norm-inequality:

$$\|u\|_{B_1^s} \leqslant c \varepsilon^{-1/2} \|u\|_{H^{s+\varepsilon}} \quad \forall u \in H^{s+\varepsilon} \tag{3.6}$$

with c independent of ε.

3.2. Subspace interpolation results

Next, we focus our attention on a specific case of subspace interpolation associated with the pair (X,Y), where $X = H^1(\Omega)$ and $Y = L^2(\Omega)$. For a fixed s_0 in the interval $(0,1)$, let $\vartheta_0 = 1 - s_0$ and $\varphi \in H^{-1}(\Omega)$. By the Riesz representation theorem there exists a function $\varphi \in H^1(\Omega)$ such that

$$(v,\varphi) = (v,\varphi)_1 = \sum_{k=1}^{\infty} \lambda_k (q_k\varphi, v) \quad \forall\, v \in H^1(\Omega).$$

Since $q_i q_j = 0$ for $i \neq j$, we deduce that in fact

$$q_k\varphi = \lambda_k^{-1} q_k\varphi. \tag{3.7}$$

Next, we assume that the function φ satisfies the following condition:

(C) There exist two positive constants c_1, c_2 such that

$$c_1\lambda_k^{s_0} \leqslant \|q_k\varphi\|^2 \leqslant c_2\lambda_k^{s_0}, \quad k = 1,2,\dots .$$

Note that the above condition is equivalent to

$$c_1\lambda_k^{-\vartheta_0} \leqslant \|q_k\varphi\|_1^2 \leqslant c_2\lambda_k^{-\vartheta_0}, \quad k = 1,2,\dots .$$

Lemma 3.1. *Let $\varphi \in H^{-1}(\Omega)$ satisfy* (C) *and let φ be the corresponding H^1-representation. Then the following conditions are also satisfied.*

(C.0) *H^1_φ is dense in $[H^1, L^2]_{1-s}$ for $s < s_0$. Here, H^1_φ is the kernel of φ.*

(C.1) *There exist two positive constants c_1, c_2 such that*

$$c_1 t^{-2\vartheta_0} \leqslant (\varphi,\varphi)_{X,t} = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\lambda_k + t^2} \| q_k\varphi \|^2 \leqslant c_2 t^{-2\vartheta_0}, \quad t > 1.$$

Proof. The constants involved in this proof might change at different occurrences. To prove (C.0), by Lemma 2.2, it is enough to show that the functional

$$u \to (u,\varphi)_1 = (u,\varphi), \quad u \in H^1(\Omega) \tag{3.8}$$

is not continuous in the topology induced by $H^s(\Omega)$, $(s < s_0)$. Let $\{u_n\}$ be the sequence in $H^1(\Omega)$ defined by

$$u_n := \sum_{k=1}^{n} \lambda_k^{-s_0}\, q_k\varphi.$$

Then,

$$(u_n,\varphi)_1 = (u_n,\varphi) = \sum_{k=1}^{n} \lambda_k^{-s_0} \|q_k\varphi\|^2 \to \infty \quad \text{as } n \to \infty.$$

On the other hand

$$\|u_n\|_{H^s}^2 = \sum_{k=1}^{n} \lambda_k^{s-2s_0} \|q_k\varphi\|^2$$

is uniformly bounded. Therefore, the functional defined in (3.8) is not continuous and (C.0) is proved.

To estimate $(\varphi,\varphi)_{X,t}$ we observe that $\|q_k\varphi\|^2 = \lambda_k^{-1}\|q_k\varphi\|_1^2$ and that

$$\sum_{k=1}^{\infty} \frac{\lambda_k^{1-\vartheta_0}}{\lambda_k + t^2} = t^{-2\vartheta_0} \sum_{k=1}^{\infty} \frac{(4^k/t^2)^{1-\vartheta_0}}{(4^k/t^2+1)}.$$

Using the standard convergence criteria for sums via integrals, the last sum can be estimated below and above by constants which are independent of t. □

Next theorem is the main subspace interpolation result of the paper.

Theorem 3.1. *Let* $\varphi \in H^{-1}(\Omega)$ *satisfy* (C). *Then*

$$[H_\varphi^1, L^2]_{1-s,2} = [H^1, L^2]_{1-s,2} := H^s(\Omega), \quad s < s_0 \tag{3.9}$$

and

$$B_1^{s_0} := [H^1, L^2]_{1-s_0,1} \subset [H_\varphi^1, L^2]_{1-s_0,\infty}. \tag{3.10}$$

Proof. Recall that $L^2(\Omega) = Y$ and $H^1(\Omega) = X$. In order to prove (3.9) it is enough (by the density property (C.0), (2.14) and Remark 2.1) to prove that

$$I := \int_1^\infty t^{-(2(1-s)+1))} \frac{|(u,\varphi)_{X,t}|^2}{(\varphi,\varphi)_{X,t}}\, dt \leqslant c\|u\|_{H^s}^2 \quad \forall u \in X_\varphi \tag{3.11}$$

for $u \in H^s(\Omega)$ denote $\tilde{u}_k := \lambda_k^{s/2}\|q_k u\|$ and $\tilde{u} := \{\tilde{u}_k\}$. Then we have

$$\|u\|_{H^s} = \|\tilde{u}\|_{l_2}.$$

Here $(\cdot,\cdot)$ is simply the $L^2(\Omega)$ inner product. Then, we have

$$(u,\varphi)_{X,t} = \left((I + t^2 S^{-1})^{-1} u, \varphi\right)_X = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2} (q_k u, q_k \varphi)_X.$$

For $u \in X_\varphi$ we have $(u,\varphi)_X = 0$. Then

$$\sum_{k=1}^{\infty} (q_k u, \varphi)_X = 0.$$

Consequently,

$$(u,\varphi)_{X,t} = -t^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k + t^2}(q_k u, q_k \varphi).$$

Thus, using the Cauchy–Schwarz inequality and (C) we obtain the estimate

$$|(u,\varphi)_{X,t}| \leqslant c_2 t^2 \sum_{k=1}^{\infty} \frac{\lambda_k^{s_0/2}}{\lambda_k + t^2} \|q_k u\|. \tag{3.12}$$

Now we are prepared to estimate the integral I. The constant c, to be used next, may have different values at different places in which it appears. Let $s_1 = s_0 - s$. Then, by (C.1) and the estimate (3.12), we have

$$\begin{aligned} I &\leqslant c \int_1^{\infty} t^{-3+2-2s_0+4} \left(\sum_{k=1}^{\infty} \frac{\lambda_k^{1-s_0/2}}{\lambda_k + t^2} \|q_k u\| \right)^2 \mathrm{d}t \\ &\leqslant c \int_1^{\infty} t^{3-2s_1} \left(\sum_{m,n=1}^{\infty} \frac{(\lambda_m \lambda_n)^{s_0/2}}{(\lambda_m + t^2)(\lambda_n + t^2)} \|q_m u\| \|q_n u\| \right) \mathrm{d}t \\ &= c \sum_{m,n=1}^{\infty} (\lambda_m \lambda_n)^{s_0/2} \|q_m u\| \; \|q_n u\| \int_1^{\infty} \frac{t^{3-2s_1}}{(\lambda_m + t^2)(\lambda_n + t^2)} \mathrm{d}t. \end{aligned}$$

Next, we use the formula

$$\int_0^{\infty} \frac{t^{3-2\vartheta}}{(a+t^2)(b+t^2)} \mathrm{d}t = \frac{1}{\mathbf{c}_{\vartheta}^2} \frac{a^{1-\vartheta} - b^{1-\vartheta}}{a-b}, \quad 0 < \vartheta < 2, \;\; \vartheta \neq 1, \;\; a,b > 0. \tag{3.13}$$

The integral can be calculated by elementary calculus methods. If $a = b$, then the right side of the above identity is replaced by $\frac{1-\vartheta}{\mathbf{c}_{\vartheta}^2} a^{-\vartheta}$. Thus,

$$\int_1^{\infty} \frac{t^{3-2s_1}}{(\lambda_m + t^2)(\lambda_n + t^2)} \mathrm{d}t \leqslant \int_0^{\infty} \frac{t^{3-2s_1}}{(\lambda_m + t^2)(\lambda_n + t^2)} \mathrm{d}t = \mathbf{c}_{s_1}^{-2} \frac{\lambda_m^{1-s_1} - \lambda_n^{1-s_1}}{\lambda_m - \lambda_n}.$$

Combining the above inequalities, we get

$$I \leqslant c \sum_{m,n=1}^{\infty} (\lambda_m \lambda_n)^{s_1/2} \, \frac{\lambda_m^{1-s_1} - \lambda_n^{1-s_1}}{\lambda_m - \lambda_n} \lambda_m^{s/2} \|q_m u\| \lambda_n^{s/2} \|q_n u\|.$$

Let

$$l_{mn} = (\lambda_m \lambda_n)^{s_1/2} \, \frac{\lambda_m^{1-s_1} - \lambda_n^{1-s_1}}{\lambda_m - \lambda_n}.$$

Then, the above estimate becomes

$$I \leqslant c \sum_{m,n=1}^{\infty} l_{mn} \tilde{u}_m \tilde{u}_n.$$

An elementary calculation gives

$$l_{mn} = \frac{2^{(m-n)(1-s_1)} - 2^{-(m-n)(1-s_1)}}{2^{(m-n)} - 2^{-(m-n)}} \leqslant 2^{-|m-n|s_1}, \quad m,n = 1,2,\ldots$$

and by elementary estimates we obtain

$$I \leqslant c\|\tilde{u}\|_{l_2}^2 = c\| u\|_{H^s}^2$$

which proves (3.9).

Next, we prove (3.10). Let $u \in B_1^{s_0}$. Then by (2.13) and Remark 2.1 we have that

$$\|u\|^2_{[X_\varphi,Y]_{1-s_0,\infty}} \leqslant c\left(\sup_{t>1} t^{2s_0}(u,u)_{Y,t} + \sup_{t>1} t^{4-2\vartheta_0}\frac{|(u,\varphi)_{Y,t}|^2}{(\varphi,\varphi)_{X,t}}\right).$$

Note that

$$\sup_{t>1} t^{2s_0}(u,u)_{Y,t} \leqslant \sup_{t>0} t^{2s_0}(u,u)_{Y,t} = \|u\|^2_{B^{s_0}_\infty} \leqslant c\|u\|^2_{B^{s_0}_1}$$

and, with the help of (C.1) we have

$$\sup_{t>1} t^{4-2\vartheta_0}\frac{|(u,\varphi)_{Y,t}|^2}{(\varphi,\varphi)_{X,t}} \leqslant c \sup_{t>1} t^4|(u,\varphi)_{Y,t}|^2. \tag{3.14}$$

For $u \in Y$, we have

$$(u,\varphi)_{Y,t} = \sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k+t^2}(q_k u, q_k\varphi) = \sum_{k=1}^{\infty}\frac{1}{\lambda_k+t^2}(q_k u, q_k\varphi)$$

and by using condition (C), we obtain

$$|(u,\varphi)_{Y,t}| \leqslant c_2\sum_{k=1}^{\infty}\frac{\lambda_k^{s_0/2}}{\lambda_k+t^2}\|q_k u\|. \tag{3.15}$$

The function

$$t \to t^2|(u,\varphi)_{Y,t}| = t^2\sum_{k=1}^{\infty}\frac{\lambda_k^{s_0/2}}{\lambda_k+t^2}\|q_k u\|$$

is an increasing function of t. As $t \to \infty$, the limit of the above function is exactly $\|u\|_{B_1^{s_0}}$. Therefore,

$$\|u\|^2_{[X_\varphi,Y]_{1-s_0,\infty}} \leqslant c\|u\|^2_{[X,Y]_{1-s_0,1}}$$

and the proof is complete. □

4. A SHIFT THEOREM FOR THE LAPLACE OPERATOR ON CONVEX POLYGONAL DOMAINS

In this section we will prove a stronger version of the estimate (1.3) and (1.4).

4.1. Shift estimates

Let Ω be a convex polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$ and no right angles. Let $V^k(\Omega) := H^k(\Omega) \cap H_0^1(\Omega)$, $k = 2$ and $k = 3$. It is well known that for $f \in L^2(\Omega)$ the variational problem has a unique solution $u \in V^2(\Omega)$ and (1.2) holds. If Ω is an acute triangular domain then one can prove that for $f \in H^1(\Omega)$ the solution of (1.1) belongs to $V^3(\Omega)$ and (1.3) holds for $s = 1$. Using (1.2) and interpolation we obtain

$$\|u\|_{H^{2+s}} \leqslant c(s)\|u\|_{[V^3,V^2]_{1-s,2}} \leqslant C(s)\|f\|_{H^s} \quad \forall f \in H^s(\Omega),\ 0 \leqslant s \leqslant 1.$$

Thus, without restricting the generality of the problem, we can assume that there exist one corner of measure ω with $\pi/2 < \omega < \pi$. In fact, by a partition of unity type argument and using the regularity results for domains with smooth boundary, we can reduce the problem to the case when Ω is a domain with only one corner of measure ω with $\pi/2 < \omega < \pi$. We will call this the 'ω-corner' and we will assume that the vertex of the ω-corner coincides with the origin of polar system of coordinates. Let $\alpha = \pi/\omega$ and $s_0 = \alpha - 1$. Given $f \in H^1(\Omega)$, we consider the Dirichlet problem (1.1). The variational formulation of (1.1) is :

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in H_0^1(\Omega). \tag{4.1}$$

Let $\zeta \in \mathscr{D}(\Omega)$ be a cut-off function, which depends only on the distance r to the ω-corner, is identically equal to one in a neighborhood of the ω-corner and is identically equal to zero close to the part of $\partial\Omega$ which does not contain the sides of the ω-corner. Let $\psi = \varphi + u^R$, $\varphi(r,\vartheta) = \zeta\ r^{-\alpha}\sin(\alpha\vartheta)$ and $u^R \in H_0^1(\Omega)$ be the variatonal solution of (1.1) with $f = \Delta\varphi$. One can check without difficulty that $\psi \in H^{-1}(\Omega)$. Let H_ψ^1 be defined as the kernel of ψ as linear functional on H^1. Then, (see e.g., Theorem 9.8 in [16]) for $f \in H_\psi^1$ the (variational) solution of (1.1) Belongs to $V^3(\Omega)$ and

$$\|u\|_{H^3(\Omega)} \leqslant c\|f\|_{H^1(\Omega)} \quad \forall u \in H_\psi^1(\Omega). \tag{4.2}$$

Thus, by interpolation, we have

$$\|u\|_{[V^3(\Omega),V^2(\Omega)]_{1-s,q}} \leqslant c\|f\|_{[H^1(\Omega)_\psi,L^2(\Omega)]_{1-s,q}} \tag{4.3}$$

for all $f \in [H^1(\Omega)_\psi, L^2(\Omega)]_{1-s,q}$. Then, we have the following theorem.

Theorem 4.1. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with no right angles and with ω the measure of the largest angle and let $s_0 = \min\{1, \pi/\omega - 1\}$. If u is the variational solution of* (1.1) *then, for $0 < s < s_0$ there exist positive constant $c(s)$ and $C(s)$ such that*

$$\|u\|_{H^{2+s}} \leqslant c(s)\|u\|_{[V^3,V^2]_{1-s,2}} \leqslant C(s)\|f\|_{H^s} \quad \forall f \in H^s(\Omega) \tag{4.4}$$

and for $0 < s_0 < 1$,

$$\|u\|_{B_\infty^{2+s_0}(\Omega)} \leqslant c(s_0)\|u\|_{[V^3,V^2]_{1-s_0,\infty}} \leqslant C(s_0)\|f\|_{B_1^{s_0}(\Omega)} \quad \forall f \in B_1^{s_0}(\Omega). \tag{4.5}$$

Proof. Use (4.3) and apply Theorem 3.1 with $\varphi = \psi$. The proof that (C) is satisfied is given later. The lower part of (4.4) or (4.5) follows by comparing the K functions associated with the two intermediate spaces. □

4.2. Proving condition (C)

Let Ω be a polygonal convex domain with the only one vertex O of measure ω with $\pi/2 < \omega < \pi$ and the remaining vertices denoted by $S_1, S_2, \dots S_n$. Let

$$\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\tau_i}$$

where, for $i = 1, \dots, n$, τ_i is a triangular domain with vertices S_i, O, S_{i+1} and O is taken to be the origin of a Cartesian system of coordinates in the plane. For $i = 1, \dots, n+1$, let Γ_i denote the segment $[O, S_i]$. We assume, without loss of generality, that S_1 lies on the positive semi-axis (see Fig. 1, the case $n = 2$).

Let $\mathscr{T}_1 = \{\tau_1, \dots, \tau_n\}$ be the initial triangulation of Ω. We define multilevel triangulations recursively. For $k > 1$, the triangulation $\mathscr{T}_k$ is obtained from $\mathscr{T}_{k-1}$ by splitting each triangle in $\mathscr{T}_{k-1}$ into four triangles by connecting the midpoints of the edges. The space M_k is defined to be the space of all functions which are piecewise linear with respect to $\mathscr{T}_k$, vanish on $\partial\Omega$ and are continuous on Ω. Let Q_k denote the $L^2(\Omega)$ orthogonal projection onto M_k.

We verify that the function ψ satisfies the condition (C). To begin with, we will prove that the function φ satisfies (C). First we will prove that there exist a function $w_h \in M_k$ which is supported in a ball of radius $H = 1/2^{k-1} = 2h$ centered at origin and w_h is orthogonal on the space M_{k-1}. Let $\varphi_1, \dots, \varphi_7$ be the nodal functions in M_{k-1} corresponding to the nodal points in $\mathscr{T}_{k-1}$ marked by '$\circ$' in the figure. Next, we consider the eight nodal functions $\varphi_1, \dots \varphi_8$ corresponding to the nodes marked by '$\star$'). We define w_h to be a linear combination w_h of $\varphi_1, \dots, \varphi_8$, with coefficients independent of h such that

$$(w_h, \varphi_j) = 0, \quad j = 1, \dots, 7, \quad (w_h, \varphi) \neq 0.$$

Hence, w_h is orthogonal on the space M_{k-1} and consequently $q_k w_h = w_h$.

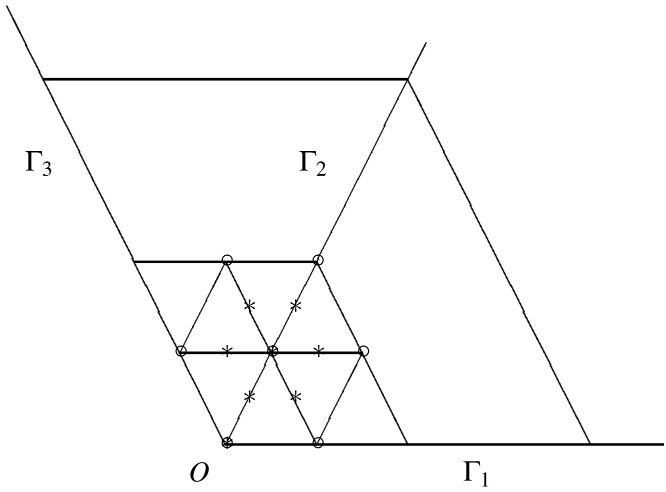

Figure 1. The ω-corner of Ω.

Then,

$$\|q_k\varphi\| = \sup_{v\in L^2(\Omega)} \frac{(q_k\varphi, v)}{\|v\|} = \sup_{v\in L^2(\Omega)} \frac{(\varphi, q_k v)}{\|v\|} \geqslant \frac{|(\varphi, q_k w_h)|}{\|w_h\|} = \frac{|(\varphi, w_h)|}{\|w_h\|}.$$

Note further that

$$\|w_h\| \leqslant ch$$

with c independent of h and by making the change of variable $x = h\hat{x}$ in the integral which defines the inner product (φ, w_h) we get

$$|(\varphi, w_h)| \geqslant ch^{2-\alpha}$$

for another constant c. Combining the above estimates we conclude that the lower part of the condition (C) holds.

For the upper part of (C) we first note that $\|q_k\varphi\|_0 = \lambda_k^{1/2}\|q_k\varphi\|_{-1}$. To estimate $\|q_k\varphi\|_{-1}$ we let $\eta = \eta_h$ to be a cut off function which depends only on r and satisfies

$$\eta_h(r) = 1 \;\text{ for } r \leqslant h/2, \quad \eta_h(r) = 0 \;\text{ for } r \geqslant h$$

$$|\eta_h'(r)| \leqslant c/h, \;\; |\eta_h''(r)| \leqslant c/h^2 \quad \forall\, h/2 \leqslant r \leqslant h$$

for some positive constant c. Then,

$$\begin{aligned}\|q_k\varphi\|_{-1} = \sup_{v\in H^1(\Omega)} \frac{(q_k\varphi, v)}{\|v\|_1} \;&\leqslant\; \sup_{v\in H^1(\Omega)} \frac{(q_k(\eta\varphi), v)}{\|v\|_1} + \sup_{v\in H^1(\Omega)} \frac{(q_k((1-\eta)\varphi), v)}{\|v\|_1} \\ &=\; M_1 + M_2.\end{aligned}$$

Using polar coordinates we have

$$(\eta\varphi, q_k v) = \int_0^{\omega} \left(\int_0^h r^{-\alpha+1} \eta \; q_k v \right) \mathrm{d}r \sin(\alpha\vartheta)\, \mathrm{d}\vartheta.$$

Next, we integrate by parts with respect to the r variable (write $r^{1-\alpha}$ as the derivative of $r^{2-\alpha}/(2-\alpha)$). Using the Cauchy–Schwarz inequality and the estimate for η' we get

$$|(\eta\varphi, q_k v)| \leqslant c \left(h^{-\alpha+1} \|q_k v\| + h^{-\alpha+2} \|(q_k v)_r\| \right).$$

The L^2 and H^1-stability of the L^2 projection give

$$\|q_k v\| \leqslant ch\|v\|_1, \quad \|q_k v\|_1 \leqslant c\|v\|_1$$

with c independent of h (or k). Thus,

$$|(\eta\varphi, q_k v)| \leqslant ch^{2-\alpha}\|v\|_1, \quad M_1 \leqslant ch^{2-\alpha}.$$

To estimate M_2, we observe first that $(1-\eta_h)\varphi \in H^2(\Omega)$. Let $\Pi_h : H^2(\Omega) \to M_k$ be the interpolant associated with $\mathscr{T}_h = T_k$. By applying standard approximation properties and (1.2) we obtain

$$\begin{aligned} M_2 = \|q_k(1-\eta_h)\varphi\|_{-1} &\leqslant h\|q_k(1-\eta_h)\varphi\| \leqslant h\|(I-Q_{k-1})(1-\eta)\varphi\| \\ &\leqslant ch\|(I-\Pi_h)(1-\eta_h)\varphi\| \leqslant ch^3\|(1-\eta_h)\varphi\|_{H^2(\Omega)} \leqslant ch^3\|\Delta(1-\eta_h)\varphi)\|_{L^2(\Omega)}. \end{aligned}$$

Using a simple computation in polar coordinates, and the estimates for the derivative of η_h, we get

$$\|\Delta(\eta_h\varphi)\|_{L^2(\Omega)} \leqslant ch^{-1-\alpha}.$$

Combining the above inequalities, we have that

$$M_2 \leqslant ch^{2-\alpha}.$$

Hence, the upper part of the condition (C) holds and consequently, (C) holds for the function φ. Since the function u_R belongs to $H^1(\Omega)$, we have

$$\|q_k u_R\|^2 \leqslant c\lambda_k^{-1}, \quad k = 1, 2, \dots.$$

Therefore, the function ψ satisfies condition (C).

5. APPLICATIONS TO FINITE ELEMENT CONVERGENCE ESTIMATES

Let Ω be a convex polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$ and no right angles. Let ω be the measure of the largest corner and let $s_0 = \min\{1, \pi/\omega - 1\}$ and let $u \in H_0^1(\Omega)$ be the variational solution of (4.1) with $f \in L^2(\Omega)$. We let V_h to be a finite dimensional approximation subspace of $H_0^1(\Omega)$ and consider the discrete problem:

Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in V_h. \tag{5.1}$$

Further, let us assume that

$$\|u - u_h\|_{H^1(\Omega)} \leqslant ch\|u\|_{H^2(\Omega)} \quad \forall u \in V^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \tag{5.2}$$

and

$$\|u - u_h\|_{H^1(\Omega)} \leqslant ch^2\|u\|_{H^3(\Omega)} \quad \forall u \in V^3(\Omega) = H^3(\Omega) \cap H_0^1(\Omega). \tag{5.3}$$

By interpolation with $p = 2$ and $0 < s < 1$, from (5.3) and (5.2), we obtain that

$$\|u - u_h\|_{H^1(\Omega)} \leqslant ch^{1+s}\|u\|_{[V^3(\Omega),V^2(\Omega)]_{1-s,2}} \tag{5.4}$$

for all $u \in [V^3(\Omega), V^2(\Omega)]_{1-s,2}$, where c is a constant independent of h. Interpolating with $p = \infty$ and $s = 1 - s_0$ we have

$$\|u - u_h\|_{H^1(\Omega)} \leqslant ch^{1+s_0}\|u\|_{[V^3(\Omega),V^2(\Omega)]_{1-s_0,\infty}} \tag{5.5}$$

for all $u \in [V^3(\Omega), V^2(\Omega)]_{1-s_0,\infty}$ where again c is a constant independent of h.

We thus have the following result.

Theorem 5.1. *Let u, u_h be the variational solutions of problem* (4.1) *and* (5.1)*, respectively. Then, there exists a constant c independent of h such that*

$$\|u - u_h\|_{H^1} \leqslant ch^{1+s}\|f\|_{H^s} \quad \forall f \in H^s,\ 0 < s < s_0 \tag{5.6}$$

$$\|u - u_h\|_{H^1} \leqslant ch^{1+s_0}\|f\|_{B_1^{s_0}} \quad \forall f \in B_1^{s_0}. \tag{5.7}$$

Furthermore, for $s_0 < s \leqslant 1$ there exists a constant c independent of h and s such that

$$\|u - u_h\|_{H^1(\Omega)} \leqslant c(s - s_0)^{-1/2}h^{1+s_0}\|f\|_{H^s(\Omega)} \quad \forall f \in H^s(\Omega) \tag{5.8}$$

and for $h \leqslant e^{-1/(2(1-s_0))}$,

$$\|u - u_h\|_{H^1(\Omega)} \leqslant ch^{1+s_0}(\log 1/h)^{1/2}\|f\|_{H^{s_0}(\Omega)} \quad \forall f \in H^{s_0}(\Omega). \tag{5.9}$$

Proof. The inequalities (5.6) and (5.7) follows from (5.4) and (5.5), respectively, as a direct consequence of Theorem 4.1. The estimate (5.8) follows from

(5.7) and (3.6) with $\varepsilon = s - s_0$. The inequality (5.9) is obtained from (5.8) as follows. Let $f \in H^{s_0}(\Omega)$ and write $f = f - Q_h f + Q_h f$. Next, we denote by $u^h \in H_0^1(\Omega)$ the solution of

$$\int_\Omega \nabla u^h \cdot \nabla v \, \mathrm{d}x = \int_\Omega Q_h f v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega).$$

Since $u - u^h$ is the solution of (4.1) with $f - Q_h f$ instead of f we have

$$\|u - u^h\|_{H^1(\Omega)} \leqslant c\|f - Q_h f\|_{H^{-1}(\Omega)}$$

and from standard approximation properties we get

$$\|f - Q_h f\|_{H^{-1}(\Omega)} \leqslant ch^{1+s_0}\|f\|_{H^{s_0}(\Omega)}.$$

Combining the two inequalities we have

$$\|u - u^h\|_{H^1(\Omega)} \leqslant ch^{1+s_0}\|f\|_{H^{s_0}(\Omega)}. \tag{5.10}$$

Next, using the estimate (5.8), a standard inverse inequality and the stability of the L^2 projection in $H^{s_0}(\Omega)$, we get

$$\begin{aligned} \|u^h - u_h\|_{H^1(\Omega)} &\leqslant c(s - s_0)^{-1/2} h^{1+s_0} \|Q_h f\|_{H^{-1+s}(\Omega)} \\ &\leqslant ch^{1+s_0}(s - s_0)^{-1/2} h^{s_0 - s} \|Q_h f\|_{H^{s_0}(\Omega)} \\ &\leqslant ch^{1+s_0}(s - s_0)^{-1/2} h^{s_0 - s} \|f\|_{H^{s_0}(\Omega)} \end{aligned}$$

where c is a constant independent of of h and s. The minimum of the function $s \to (s - s_0)^{-1/2} h^{s_0 - s}$ on the interval $(s_0, 1]$ is $(2e \log 1/h)^{1/2}$ and is attained for $s = s_0 + (2\log 1/h)^{-1}$, with $h \leqslant e^{-1/(2(1-s_0))}$. Thus,

$$\|u^h - u_h\|_{H^1(\Omega)} \leqslant ch^{1+s_0}(\log 1/h)^{1/2}\|f\|_{H^{s_0}(\Omega)}. \tag{5.11}$$

Finally, (5.9) follows from (5.10) and (5.11). □

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