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Partition of unity method on nonmatching grids for the Stokes problem

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Abstract — We consider the Stokes problem on a plane polygonal domain $\Omega \subset \mathbb{R}^2$. We propose a finite element method for overlapping or nonmatching grids for the Stokes problem based on the partition of unity method. We prove that the discrete inf-sup condition holds with a constant independent of the overlapping size of the subdomains. The results are valid for multiple subdomains and any spatial dimension.

Keywords: non-matching grid, finite element method, partition of unity, Stokes problem

1. INTRODUCTION

In the present literature the study of finite element method applied to overlapping grids is done mainly in the framework of mortar method or Lagrange multiplier (see [2,4,10,15,17]). Using a partition of unity method which has roots in [3], a new finite element discretization for elliptic boundary value problems was introduced by Huang and Xu in [16]. A significant amount of literature was dedicated to numerical solutions of the Stokes problem (see e.g., [8,14] and the references of this two books). By our knowledge not to much was done for solving discretization of the Stokes problem when overlapping grids or nonmatching grids are involved. In this paper, following the ideas of Huang and Xu, we shall introduce a conforming finite element method, using a partition of unity type argument for the steady-state Stokes problem. The method is based also on the idea of building special Mini-type elements [1] for the overlapping region. In fact, for the first Mini-type stable pair,

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in the case of the matching grids, we recover the standard stable pair given by the Mini-element technique. The rest of the paper is organized as follows. In Section 2 we describe the problem and the discrete spaces for the simple case of two overlapping subdomains with non-matching grids. In Section 2 and Section 3 we present two ways of constructing stable pairs of discrete spaces for velocity and pressure in the case of non-matching grids on overlapping subdomains. In Section 5 we analyze a non-matching non-overlapping grid case and describe a way of construction of stable spaces with the help of the theory presented for the overlapping case. The last section presents a few remarks and conclusions.

2. THE CONTINUOUS STOKES PROBLEM AND OVERLAPPING SUBDOMAINS DISCRETIZATION

Even though the results hold in a more general context and for a general dimension, for clarity, we present the main ideas of the discretization method in case of two subdomains in $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega$ and let Γ be a closed subset of $\partial\Omega$. By $H_0^1(\Omega;\Gamma)$ we denote the closure in H^1-topology of $C^\infty(\overline{\Omega})$ functions that vanish in a neighborhood of Γ.

The steady-state Stokes problem in the velocity-pressure formulation is:

Find the vector-valued function $\mathbf{u}$ and the scalar-valued function p satisfying

$$\begin{cases} -\Delta\mathbf{u} - \nabla p = \mathbf{F} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \\ \int_\Omega p\,\mathrm{d}x = 0. & \end{cases} \tag{2.1}$$

Let $(\cdot,\cdot)_\Omega$, or simply $(\cdot,\cdot)$, denote the $L^2(\Omega)$-inner product applied to a pair of either scalar or vector functions. Similarly, let $\|\cdot\|_{0,\Omega}$, or simply $\|\cdot\|$ denote the $L^2(\Omega)$-norm. Define $\mathbf{V} = (H_0^1(\Omega))^2$ and $P = L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting of functions with zero mean value on Ω. The variational formulation of the problem (2.1) is:

Find $(\mathbf{u},p) \in (\mathbf{V},P)$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{F},\mathbf{v}) & \forall\mathbf{v}\in\mathbf{V} \\ b(\mathbf{u},q) = 0 & \forall q\in P \end{cases} \tag{2.2}$$

where a is the Dirichlet form on Ω defined by

$$a(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{2}\int_\Omega \nabla u_i\cdot\nabla v_i\,\mathrm{d}x$$

and

$$b(\mathbf{v},q) = (q,\nabla\cdot\mathbf{v}).$$

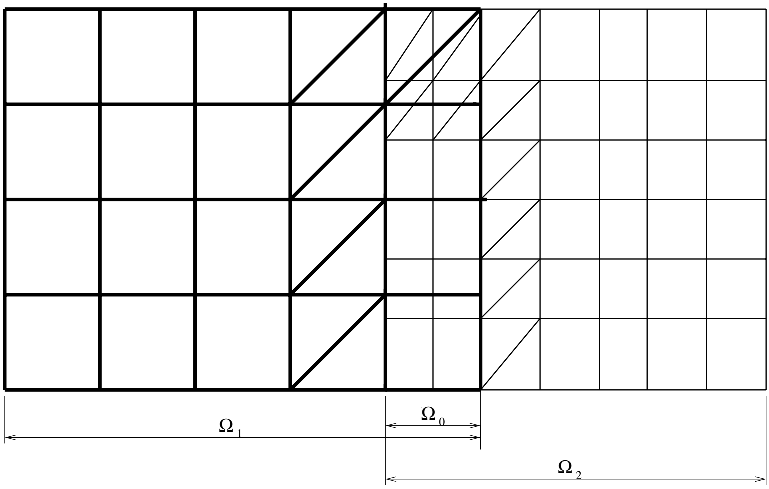

Figure 1. Overlapping grids.

We assume that the inf-sup condition

$$c_0\|p\| \leqslant \sup_{\mathbf{v}\in\mathbf{V}} \frac{(p, \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}} \qquad \forall p \in P \tag{2.3}$$

holds for a positive constant c_0. Consequently, there is a unique solution $(\mathbf{u}, p) \in (\mathbf{V}, P)$ of (2.2). Let Ω be covered by a family of overlapping subdomains. For a better presentation of the main idea, we consider the case of two overlapping subdomains with polygonal shapes.

Let Ω_1 and Ω_2 be overlapping subdomains of Ω satisfying $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_0 = \Omega_1 \cap \Omega_2$. We further assume that Ω_1 and Ω_2 are partitioned by quasi-uniform finite element triangulations $\mathscr{T}_1$ and $\mathscr{T}_2$ of maximal mesh sizes h_1 and h_2 (which might not match on Ω_0). Again, just for the sake of simplicity, we assume that Ω_0 is a strip-type domain of width $d = O(h_1)$ and that $\partial\Omega_0$ is aligned with $\mathscr{T}_1$ and $\mathscr{T}_2$ (see Fig. 1).

Next, we let $\{\varphi_1, \varphi_2\}$ be a partition of unity subordinate to the covering partition $\{\Omega_1, \Omega_2\}$ of Ω, i.e., φ_1, φ_2 are continuous functions defined on Ω such that $\varphi_1 + \varphi_2 = 1$, $0 \leqslant \varphi_i \leqslant 1$, and $\|\nabla\varphi_i\|_{\infty,\Omega} \leqslant 1/d$. We further assume that $\varphi_1 \equiv 1$ on $\Omega_1\backslash\Omega_0$ and $\varphi_1 \equiv 0$ on $\Omega_2\backslash\Omega_0$, and $\varphi_2 \equiv 1$ on $\Omega_2\backslash\Omega_0$ and $\varphi_2 \equiv 0$ on $\Omega_1\backslash\Omega_0$.

To obtain a conforming discretization of the variational problem (2.2) we define first the following spaces

$$\begin{aligned}
P_{h_i}(\Omega_i) &:= \left\{p \in C^0(\Omega_i) :\ p|_T \in \mathbb{P}_1,\ T \in \mathscr{T}_i\right\}\\
\hat{P}_{h_i}(\Omega_i) &:= \left\{p \in P_{h_i}(\Omega_i) :\ p = 0 \text{ on } \partial\Omega_i\backslash\partial\Omega\right\}\\
\mathbf{V}_{h_i}(\Omega_i) &:= \left\{\mathbf{v} \in (H_0^1(\Omega_i; \partial\Omega\cap\partial\Omega_i))^2 :\ \mathbf{v}|_T \in (\mathbb{P}_1)^2,\ T \in \mathscr{T}_i\right\}
\end{aligned}$$

where $\mathbb{P}_1$ denotes the set of polynomials in two variables of degree at most one. Using the above spaces, we are interested in building stable pairs $(\mathbf{V}_h, P_h)$, where $\mathbf{V}_h \subset \mathbf{V}$ and $P_h \subset P$, i.e., pairs $(\mathbf{V}_h, P_h)$ which satisfy the discrete inf-sup condition

$$c_0\|p\| \leqslant \sup_{\mathbf{v}\in\mathbf{V}_h} \frac{(p, \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}} \qquad \forall p \in P_h. \tag{2.4}$$

If the above condition is satisfied then the discrete variational problem:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{F}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_h \\ \qquad\qquad b(\mathbf{u}_h, q) = 0 & \forall q \in P_h \end{cases} \tag{2.5}$$

has unique solution and the error satisfies,

$$|\mathbf{u}-\mathbf{u}_h|_{1,\Omega} + \|p-p_h\|_{0,\Omega} \leqslant C\left(\inf_{\mathbf{v}_h\in\mathbf{V}_h} |\mathbf{u}-\mathbf{v}_h|_{1,\Omega} + \inf_{q_h\in P_h} \|p-q_h\|_{0,\Omega}\right)$$

with C depending on c_0, but independent of h (or the spaces $\mathbf{V}_h$ and P_h). In the next two sections we build stable pairs $(\mathbf{V}_h, P_h)$ which have also good approximation properties.

3. FIRST MINI-TYPE STABLE PAIR

We introduce a space B of bubble functions associated with the 'union' partition $\mathscr{T} := \mathscr{T}_1 \cup \mathscr{T}_2$ as follows. For a triangle T we define the bubble function B_T supported on T as the product of the nodal functions associated with the vertices of T.

If $K = T_1 \cap T_2 \in \mathscr{T}_1 \cup \mathscr{T}_2$ we define

$$B_K := B_{T_1} \cdot B_{T_2}.$$

If $K = T_i$ for some $T_i \in \mathscr{T}_i$, $i = 1, 2$, then we just take $B_K := B_{T_i}$ (see Fig. 2). A composite, conforming finite element space for velocity can be defined by

$$\mathbf{V}_h \equiv \mathbf{V}_h(\Omega) := \varphi_1 \mathbf{V}_{h_1} + \varphi_2 \mathbf{V}_{h_2} + \mathbf{B}$$

where

$$\mathbf{B} := \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : \; b_1, b_2 \in B \right\}. \tag{3.1}$$

The discrete pressure space we associate with $\mathbf{V}_h$ is

$$P_h := (\hat{P}_{h_1}(\Omega_1) + \hat{P}_{h_2}(\Omega_2)) \cap P.$$

Let $h := h_1 \geqslant h_2 = rh_1$, for some positive constant r. Before we state the main result of this section we introduce the following assumption:

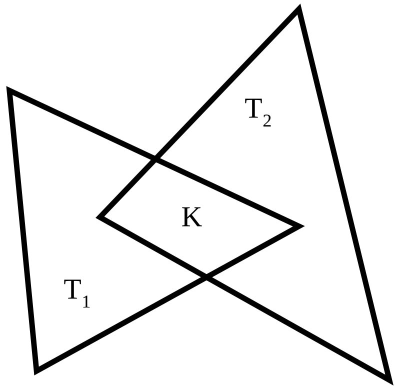

Figure 2. Overlapping triangles.

(A1) *There exists a positive constant c such that*
$|K| \cong ch^2$ *for any* $K \in \mathscr{T}$,

where $|K|$ denotes the Lebesgue measure of $K \in \mathscr{T}$.

Theorem 3.1. *If* (A1) *is satisfied, then the pair* $(\mathbf{V}_h, P_h)$ *defined above is a stable pair.*

Proof. We will construct two operators $\Pi_1 : \mathbf{V} \to \mathbf{V}_h$, $\Pi_2 : \mathbf{V} \to \mathbf{V}_h$ with the following properties:

$$\left|\mathbf{v} - \Pi_1\mathbf{v}\right|_{1,\Omega} \lesssim |\mathbf{v}|_{1,\Omega} \qquad \forall \mathbf{v} \in \mathbf{V} \tag{3.2}$$

$$\left|\Pi_2(I - \Pi_1)\mathbf{v}\right|_{1,\Omega} \lesssim |\mathbf{v}|_{1,\Omega} \qquad \forall \mathbf{v} \in \mathbf{V} \tag{3.3}$$

$$b(\mathbf{v} - \Pi_2\mathbf{v}, q) = 0 \qquad \forall q \in P_h, \ \ \mathbf{v} \in \mathbf{V}. \tag{3.4}$$

Having constructed Π_1 and Π_2, the operator $\Pi_h = \Pi_1 + \Pi_2(I - \Pi_1)$ satisfies the the hypothesis of Proposition II 2.8 in [8], for example, and the inf-sup condition follows according to this proposition.

For $i = 1,2$, let $\mathbf{V}_i := (H_0^1(\Omega_i; \partial\Omega \cap \partial\Omega_i))^2$ and define $\Pi_1^i : \mathbf{V}_i \to \mathbf{V}_{h_i}$ to be good regularization operators. For example, we can take Π_1^i to be Clement operators. Thus,

$$\|\mathbf{v} - \Pi_1^i\mathbf{v}\|_{0,\Omega_i} \lesssim h_i|\mathbf{v}|_{1,\Omega_i} \qquad \forall \mathbf{v} \in \mathbf{V}_i \tag{3.5}$$

and

$$|\mathbf{v} - \Pi_1^i\mathbf{v}|_{1,\Omega_i} \lesssim |\mathbf{v}|_{1,\Omega_i} \qquad \forall \mathbf{v} \in \mathbf{V}_i. \tag{3.6}$$

We define Π_1 as follows:

$$\Pi_1\mathbf{v} := \varphi_1\Pi_1^1(\mathbf{v}|_{\Omega_1}) + \varphi_2\Pi_1^2(\mathbf{v}|_{\Omega_2}).$$

Note that $\mathbf{v}|_{\Omega_i} \in \mathbf{V}_i$ and $\Pi_1\mathbf{v} \in \mathbf{V}_h$. Thus Π_1 is well defined. In order to simplify the notation we denote $\Pi_i^1(\mathbf{v}|_{\Omega_i})$ simply by $\Pi_i^1\mathbf{v}$. Next, we verify that the operator Π_1 satisfies (3.2). We will prove first that,

$$\|\mathbf{v} - \Pi_1\mathbf{v}\|_{0,\Omega} \lesssim h|\mathbf{v}|_{1,\Omega} \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{3.7}$$

Indeed,

$$\begin{aligned}\|\mathbf{v} - \Pi_1\mathbf{v}\|_{0,\Omega} &= \left\|\sum_{i=1}^{2}\varphi_i(\mathbf{v} - \Pi_1^i\mathbf{v})\right\|_{0,\Omega} \leqslant \sum_{i=1}^{2}\left\|\varphi_i(\mathbf{v} - \Pi_1^i\mathbf{v})\right\|_{0,\Omega_i} \\ &\leqslant \sum_{i=1}^{2}\left\|\mathbf{v} - \Pi_1^i\mathbf{v}\right\|_{0,\Omega_i} \lesssim \sum_{i=1}^{2} h_i|\mathbf{v}|_{1,\Omega_i} \lesssim h|\mathbf{v}|_{1,\Omega}.\end{aligned}$$

The justification of (3.2) is also straightforward.

$$\begin{aligned}|\mathbf{v} - \Pi_1\mathbf{v}|_{1,\Omega} &\leqslant \sum_{i=1}^{2}|\varphi_i(\mathbf{v} - \Pi_1^i\mathbf{v})|_{1,\Omega_i} \\ &\leqslant \sum_{i=1}^{2}|\nabla\varphi_i(\mathbf{v} - \Pi_1^i\mathbf{v})|_{0,\Omega_i} + \sum_{i=1}^{2}|\mathbf{v} - \Pi_1^i\mathbf{v}|_{1,\Omega_i} \\ &\lesssim d^{-1}\sum_{i=1}^{2}|\mathbf{v} - \Pi_1^i\mathbf{v}|_{0,\Omega_i} + \sum_{i=1}^{2}|\mathbf{v}|_{1,\Omega_i} \lesssim |\mathbf{v}|_{1,\Omega}.\end{aligned}$$

Next, we define Π_2. For $\mathbf{v} \in \mathbf{V}$ and $K \in \mathscr{T}$ define

$$\Pi_2\mathbf{v}|_K := \alpha B_K$$

where $\alpha = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}$ is determined such that $\int_K(\mathbf{v} - \Pi_2\mathbf{v})\,\mathrm{d}x = 0$, i.e.,

$$\alpha = \frac{\int_K \mathbf{v}\,\mathrm{d}x}{\int_K B_K\,\mathrm{d}x}.$$

For $\mathbf{v} \in \mathbf{V}$ and $q \in P_h$ we have

$$b(\mathbf{v} - \Pi_2\mathbf{v}, q) = -(\mathbf{v} - \Pi_2\mathbf{v}, \nabla q) = -\sum_{K\in\mathscr{T}}\nabla q\int_K(\mathbf{v} - \Pi_2\mathbf{v})\,\mathrm{d}x = 0.$$

Thus (3.4) holds and to end the proof we have to verify that (3.3) holds. Let us note that

$$|\Pi_2\mathbf{v}|_{1,K} \lesssim h^{-1}|\mathbf{v}|_{0,K} \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{3.8}$$

The proof of (3.8) is a consequence of the following two estimates.

$$|\Pi_2\mathbf{v}|_{1,K}^2 = (\alpha_1^2 + \alpha_2^2)\int_K|\nabla B_K|^2\,\mathrm{d}x \lesssim |\alpha|^2|K|h^{-2}$$

and

$$|\alpha|^2 = \frac{|\int_K \mathbf{v}\,dx|^2}{|\int_K B_K\,dx|^2} \lesssim \frac{|K||\mathbf{v}|_{0,K}^2}{h^4}.$$

Thus, from (3.8) and (3.7) we obtain

$$\begin{aligned}|\Pi_2(I-\Pi_1)\mathbf{v}|_{1,\Omega}^2 &= \sum_{K\in\mathscr{T}} |\Pi_2(I-\Pi_1)\mathbf{v}|_{1,K}^2 \\ &\lesssim \sum_{K\in\mathscr{T}} h^{-2}|(I-\Pi_1)\mathbf{v}|_{0,K}^2 \lesssim |\mathbf{v}|_{1,\Omega}^2\end{aligned}$$

which proves that (3.3) holds and concludes the proof of the theorem. □

Remark 3.1. In the special case when any $K \in \mathscr{T}$ is a triangle in either T_1 or T_2 (any $T_1 \in \mathscr{T}_1$, $T_1 \subset \Omega_0$ is an union of triangles of $\mathscr{T}_2$ and any $K \in \mathscr{T}$ which is not subset of Ω_0 belongs to either T_1 or T_2), we have that (A1) is satisfied. Moreover we have:

(A1′) *There exists a positive constant c such that* $|T_i| \cong ch_i^2$ *for any* $K = T_i \in \mathscr{T}$.

Following the proof of the above theorem in this particular case, we deduce that the constants which are involved in (3.2) and (3.3) are also independent of the ratio $r = h_2/h_1$. Consequently, the inf-sup condition holds with a constant independent of h_2, h_1, and r.

Remark 3.2. According to [16] the space $\mathbf{V}_h$ has the following approximation property:

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{v}-\mathbf{v}_h\|_{1,\Omega} \lesssim h_1\|v\|_{2,\Omega_1} + h_2\|v\|_{2,\Omega_2}, \quad \mathbf{v}\in (H^2(\Omega)\cap H_0^1(\Omega))^2.$$

If $\hat{P}_{h_1}(\Omega_1)+\hat{P}_{h_2}(\Omega_2)$ is a linear space which contains the constant function then, by using Bramble–Hilbert lemma [5], we have that the space P_h has the following approximation property

$$\inf_{p_h\in P_h} \|p-p_h\|_{0,\Omega} \lesssim h_1\|p\|_{1,\Omega} + h_2\|p\|_{1,\Omega} \forall\ p \in H^1(\Omega)\cap L_0^2(\Omega).$$

Therefore the pair $(\mathbf{V}_h, P_h)$ has good approximation properties and is a stable pair.

On the other hand, if $\hat{P}_{h_1}(\Omega_1)+\hat{P}_{h_2}(\Omega_2)$ is a linear space which does not have good approximation properties we can consider for the discrete pressure space P_h a partition of unity type space and modify accordingly the velocity space. This is the subject of the next section.

4. SECOND MINI-TYPE STABLE PAIR

A discrete pressure space P_h with good approximation properties (see [16]) is the space

$$P_h := (\varphi_1 P_{h_1}(\Omega_1) + \varphi_2 P_{h_2}(\Omega_2)) \cap P.$$

Since the pressure space is enriched (on the overlapping region), in order to have satisfied the inf-sup condition, we have to enrich the velocity space also. As in the previous section we define a bubble space B. For each $K \in \mathscr{T}$, $K \subset \Omega_0$ we construct two bubble functions $B_{j,K}$, $j = 1,2$, supported on K. These functions are used to define a Fortin-type operator [13], and prove the well-posedness of the discrete problem. For each one of the remaining regions $K \in \mathscr{T}$ we consider only one bubble function defined as in the first case. The rest of the construction of the second pair of mini element is standard. We let B to be the span of all these bubble functions and define the discrete space $\mathbf{V}_h$ as before:

$$\mathbf{V}_h := \varphi_1 \mathbf{V}_{h_1} + \varphi_2 \mathbf{V}_{h_2} + \mathbf{B}$$

where $\mathbf{B}$ is defined by (3.1) with the new space B yet to be specified.

4.1. A preliminary construction

Before we turn to the formulation and the proof of a stability theorem, let us introduce a simple construction, which will be needed in the proof of the Theorem 4.1. In this paragraph we shall restrict ourselves to the case of square K, define appropriate bubble functions, and prove two inequalities.

We consider two square domains with side 2ℓ: a square R, centered at a fixed point in the plane (x_0, y_0) and $R_0 = [-\ell, \ell] \times [-\ell, \ell]$. Clearly there is an affine mapping $F : R_0 \mapsto R$, such that

$$F = Q\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad Q = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

for some $\vartheta \in [-\pi, \pi]$. We aim to construct two functions $B_{1,R}$ and $B_{2,R}$, supported in R, such that, the following relations are satisfied:

$$|\nabla B_{j,R}| \leqslant \max\{2, \ell^{-1}\}, \quad j = 1,2 \tag{4.1}$$

and

$$|(B_{1,R}, 1)(B_{2,R}, (x - x_0)) - (B_{2,R}, 1)(B_{1,R}, (x - x_0))| \geqslant \tfrac{16}{81} \cos\vartheta\, \ell^5. \tag{4.2}$$

Since some of the considerations, which we make below, depend on the orientation of R, we shall consider in detail the case $|\vartheta| \notin [\pi/4, 3\pi/4]$. The proof of the inequalities (4.1), (4.2) for the other values of ϑ is merely an interchange of $\xi \leftrightarrow \eta$

everywhere below and cos to sin in (4.2). The functions $B_{j,R}$ are obtained by first defining their counterparts on R_0, as follows

$$B_{1,0}(\xi,\eta) = \begin{cases} 4\ell^{-4}\xi(\ell-\xi)(\ell^2-\eta^2), & (\xi,\eta)\in R_0, \quad \xi>0 \\ 0, & (\xi,\eta)\in R_0, \quad \xi\leqslant 0 \\ 0, & (\xi,\eta)\notin R_0 \end{cases} \tag{4.3}$$

and setting $B_{2,0} = B_{1,0}(-\xi,\eta)$. Then $B_{j,R}$ are defined in standard way as

$$B_{j,R}(x,y) = B_{j,0}(F^{-1}(\xi,\eta)), \quad j=1,2. \tag{4.4}$$

To prove (4.1), (4.2) is rather straightforward computation, using the change of variable (4.4).

4.2. Stability theorem

The next theorem gives the stability of the constructed composite spaces and the well-posedness of the discrete problem. The assumption needs to be made about the shape of $K\in\mathscr{T}$, $K\subset\Omega_0$ as stated in the theorem below.

Theorem 4.1. *Assume that for each such K there exists a square with side $2\ell\approx h$, which is completely contained in K. Then the new pair $(\mathbf{V}_h, P_h)$ defined above is a stable pair.*

Proof. We follow the a construction procedure similar to the one used in the proof of Theorem 3.1. The operator Π_1 is the same as defined in the proof of Theorem 3.1. Next, we define Π_2 such that (3.3) and (3.4) are satisfied. Let $\varphi_1 := \varphi$ and $\varphi_2 := 1-\varphi$. To simplify the computation we will assume that φ is a linear function in only one variable, say x. Thus, for any $q\in P_h$ and any $K\in\mathscr{T}$, $K\subset\Omega_0$ we have that

$$\begin{aligned}\nabla q|_K &\in \operatorname{span}\left\{\begin{pmatrix}x\\0\end{pmatrix},\begin{pmatrix}y\\x\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\\ &= \operatorname{span}\left\{\begin{pmatrix}x-x_0\\0\end{pmatrix},\begin{pmatrix}y-y_0\\x-x_0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\end{aligned}$$

where (x_0,y_0) are the coordinates of the center of the square $R\subset K$. We then define Π_2 as follows

$$\Pi_2\mathbf{v}|_K := \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} B_{1,K} + \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} B_{2,K}$$

if $K\in\mathscr{T}$, $K\subset\Omega_0$, where $B_{j,K} = B_{j,R}$ as defined in Subsection 4.1. In the case when $K\in\mathscr{T}$ and $K\subset\Omega_i\backslash\Omega_0$, $i=1,2$, we define $\Pi_2\mathbf{v}|_K := \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} B_K$. The constants $\alpha = \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$ and $\beta = \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}$ are determined such that

$$\int_K (\mathbf{v}-\Pi_2\mathbf{v})\cdot\nabla q\,\mathrm{d}x = 0, \quad q\in P_h.$$

Here, in the second case, B_K is the bubble function defined in the previous section. Obviously, (3.4) holds. The justification of (3.3) is similar and we only need to prove that (3.8) holds. For $K \in \mathscr{T}$ and $K \subset \Omega_i \backslash \Omega_0$, $i = 1,2$, the proof was done in the previous section. We will focus now on the case $K \in \mathscr{T}$, $K \subset \Omega_0$. From the definition of Π_2 and the condition $\int_K (\mathbf{v} - \Pi_2 \mathbf{v}) \cdot \nabla q \, \mathrm{d}x = 0$ we obtain the following system of linear equations for the coefficients α_1, α_2, β_1, and β_2:

$$
\left\{
\begin{array}{rcl}
\alpha_1 (B_{1,K}, x - x_0) + \beta_1 (B_{2,K}, x - x_0) & = & (v_1, x - x_0) \\
\alpha_1 (B_{1,K}, y - y_0) + \beta_1 (B_{2,K}, y - y_0) & & \\
+ \alpha_2 (B_{1,K}, x - x_0) + \beta_2 (B_{2,K}, x - x_0) & = & (v_1, y - y_0) + (v_2, x - x_0) \\
\alpha_1 (B_{1,K}, 1) + \beta_1 (B_{2,K}, 1) & = & (v_1, 1) \\
\alpha_2 (B_{1,K}, 1) + \beta_2 (B_{2,K}, 1) & = & (v_2, 1)
\end{array}
\right.
$$

where v_1, v_2 are the components of the velocity vector $\mathbf{v}$ restricted to K. The above linear system will have a unique solution, if and only if

$$
\mathrm{det}_K := \det \begin{pmatrix} (B_{1,K}, x - x_0) & (B_{2,K}, x - x_0) \\ (B_{1,K}, 1) & (B_{2,K}, 1) \end{pmatrix} \neq 0. \tag{4.5}
$$

From (4.2) we conclude that our choice of $B_{1,K}$ and $B_{2,K}$ guarantees that

$$
|det_K| \gtrsim h^5 \qquad \forall K \in \mathscr{T}, \;\; K \subset \Omega_0. \tag{4.6}
$$

Then one can solve for the coefficients α and β and taking for example α_1 we have

$$
\alpha_1 = \frac{1}{\mathrm{det}_K} \left((B_{2,K}, 1)(v_1, x - x_0) - (B_{2,K}, x - x_0)(v_1, 1) \right). \tag{4.7}
$$

Using that $|K| \lesssim h^2$ and (4.1), we get

$$
|\Pi_2 \mathbf{v}|^2_{1,K} \leqslant (\alpha_1^2 + \alpha_2^2) \int_K |\nabla B_{1,K}|^2 \, \mathrm{d}x + (\beta_1^2 + \beta_2^2) \int_K |\nabla B_{2,K}|^2 \, \mathrm{d}x \lesssim (|\alpha|^2 + |\beta|^2). \tag{4.8}
$$

On the other hand from $|x - x_0| \leqslant h$ and (4.7) we may conclude that

$$
|\alpha_1| \leqslant \frac{1}{|\mathrm{det}_K|} h^2 |v_1|_{0,K} h^2
$$

and hence

$$
|\alpha_1| \lesssim h^{-1} |\mathbf{v}|_{0,K}.
$$

Similar estimates hold for α_2, β_1, and β_2. Finally, via (4.8), we have that (3.8) holds and consequently (3.3) is satisfied. □

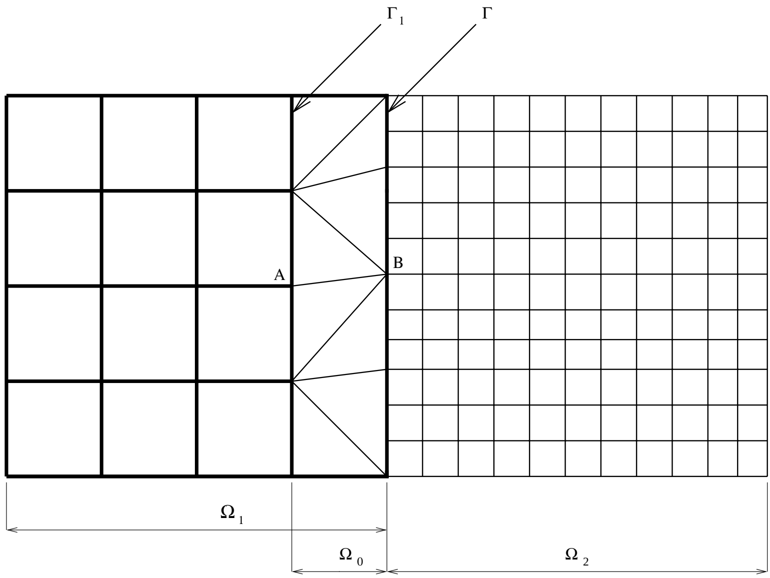

Figure 3. Modifying the the grid on Ω_1.

5. NONOVERLAPPING NONMATCHING GRIDS

Let Ω be split into two nonoverlapping subdomains Ω_1 and Ω_2 and let Γ be the interface between Ω_1 and Ω_2. As in the overlapping case we assume that Ω_1 and Ω_2 are partitioned by quasi-uniform finite element triangulations $\mathscr{T}_1$ and $\mathscr{T}_2$ of maximal mesh sizes h_1 and h_2 ($h_1 \geqslant h_2$). The grid on the interface Γ might not match the two partitions. We can extend the mesh of Ω_2 inside Ω_1 so that the overlapping meshed region is a strip Ω_0 of size $d = O(h_1)$ which matches the mesh of Ω_1. In this way we have reduced the setting to the case of overlapping subdomains. Nevertheless, if the extension is not careful done, the condition (A1) might not be satisfied (or poorly satisfied). One way to construct a good extension is by slightly modify the mesh on one subdomain. Next, we describe a way of constructing such a grid extension. Let us assume that the mesh of Ω_2 is extended inside Ω_1 for a 'strip' with $d = O(h_1)$ (see Fig. 3). Let Γ_1 be the border of our extension which lies inside Ω_1. We assume that Γ_1 matches the mesh on Ω_1 and ignore the mesh $\mathscr{T}_1$ between Γ_1 and Γ. We present in Fig. 3 a case with $h_1/h_2 = r = 11/4$.

If $A \in \Gamma_1$ is an arbitrary nodal point in $\mathscr{T}_1$, we first connect A with the closest nodal point in $\mathscr{T}_2$ situated on Γ (point B in our picture). After all 'horizontal connections' are made we subdivide all the quadrilaterals new formed by connecting two diagonal points. This produces a new mesh on Ω_0 which together with the unchanged mesh on Ω_1 creates a new mesh on Ω_1 denoted $\tilde{\mathscr{T}}_1$. Next we refine the triangulation on Ω_0 to a quasi-uniform triangulation of size h_2 which matches the mesh on Ω_2 (not showed in our picture).

Let $\tilde{\Omega}_2$ be the domain made up by Ω_2 and Ω_0 and let $\tilde{\mathscr{T}}_2$ be the mesh which extends $\mathscr{T}_2$ with the fine mesh on Ω_0 of size h_2. We note that the meshes $\tilde{\mathscr{T}}_1$ and

$\tilde{\mathscr{T}}_2$ corresponding to the subdomain Ω_1 and $\tilde{\Omega}_2$, respectively, satisfy the special case presented in Remark 3.1. Therefore, the mini-type stable pair presented in Section 3 can be involved for solving the discrete Stokes problem if nonmatching grids are provided.

6. CONCLUSIONS

- The method can be extended with no difficulties to the more subdomains case or the multidimensional case.
- If the discrete approximation spaces are spaces of continuous piecewise linear functions then the partition of unity functions can be chosen to be piecewise linear functions also.
- The condition (A1) is too restrictive. In practice, we can slightly change the mesh by moving points of the mesh towards other close points or edges.
- We conjecture that other classical stable pairs for subdomains, for example $\mathbb{P}_2 - \mathbb{P}_1$ 'subspaces', could be glued by partition of unity method in order to construct stable pairs with good approximation properties.

REFERENCES

1. D. Arnould, F. Brezzi, and M. Fortin, A stable finite element for Stokes equations. *Calcolo* (1984) **21**, 337 – 344.
2. Y. Achdou and Y. Maday, The mortar element method with overlapping subdomains. *SIAM J. Numer. Anal.* (2002) **40**, No. 2, 601 – 628.
3. I. Babuška and J. M. Melenk, The partition of unity finite element method: basic theory and applications. *Comp. Meth. Appl. Mech. Engrg.* (1996) **139**, No. 1 – 4, 289 – 314.
4. F. B. Belgacem, A stabilized domain decomposition method with nonmatching grids for the Stokes problem in three dimensions. *SIAM J. Numer. Anal.* (2004) **42**, 667 – 685.
5. J. H. Bramble and S. R. Hilbert, Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. *SIAM J. Numer. Anal.* (1970) **7**, 113 – 124.
6. C. Bacuta, J. H. Bramble, and J. Pasciak, New interpolation results and applications to finite element methods for elliptic boundary value problems. *Numer. Lin. Algebra Appl.* (2003) **10**, No. 1 – 2, 33 – 64.
7. J. H. Bramble, J. Pasciak, and J. Xu, Parallel multilevel preconditioners. *Math. Comp.* (1990) **55**, 1 – 22.
8. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.
9. S. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.
10. X. Cai, M. Dryja, and M. Sarkis, Overlapping nonmatching grid mortar element methods for elliptic problems. *SIAM J. Numer. Anal.* (1999) **36**, No. 2, 581 – 606.
11. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*. North Holland, Amsterdam, 1978.

12. Ph. Clement, Approximation by finite element functions using local regularization. *R.A.I.R.O Anal. Numer.* (1975) **9**, 77 – 84.

13. M. Fortin, An analysis of the convergence of mixed finite element methods. *R.A.I.R.O Anal. Numer.* (1977) **11**, R3, 341 – 354.

14. V. Girault and P. A. Raviart, *Finite Element Methods for Navier–Stokes Equations*. Springer-Verlag, Berlin, 1986.

15. V. Girault, B. Riviére, and M. F. Wheeler, A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier–Stokes problems. *Math. Comp.* (2005) **74**, 53 – 84.

16. Y. Huang and J. Xu, A new finite element method for non-matching grids based on partition of unity. *Math. Comp.* (to appear).

17. Yu. Kuznetsov, Overlapping domain decomposition with non-matching grids. In: *Proc. of the 9th Inter. Conference on Domain Decomposition Methods*. (Eds. P. Bjostad, M. Espedal, and D. Keyes) Domain Decomposition press, 1998, pp. 64 – 76.

18. J. Xu, Iterative methods by space decomposition and subspace correction. *SIAM Review* (1992) **34**, 581 – 613.