arXiv:1605.07907v1 [math.AP] 25 May 2016

DIFFERENTIAL OPERATORS ON DOMAINS WITH CONICAL POINTS: PRECISE UNIFORM REGULARITY ESTIMATES

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May 26, 2016

Abstract. We study families of strongly elliptic, second order differential operators with singular coefficients on domains with conical points. We obtain uniform estimates on their inverses and on the regularity of the solutions to the associated Poisson problem with mixed boundary conditions. The coefficients and the solutions belong to (suitable) weighted Sobolev spaces. The space of coefficients is a Banach space that contains, in particular, the space of smooth functions. Hence, our results extend classical well-posedness results for strongly elliptic equations in domains with conical points to problems with singular coefficients. We furthermore provide precise uniform estimates on the norms of the solution operators.

Contents

Date: May 26, 2016.
C. Bacuta has been partially supported by NSF Grant DMS-1522454. H. Li has been partially supported by the NSF Grant DMS-1418853 and by the Wayne State University Grants Plus Program. V. Nistor has been partially supported by ANR-14-CE25-0012-01 (SINGSTAR).
Manuscripts available from **http://iecl.univ-lorraine.fr/Victor.Nistor/**
AMS 2000 Subject classification: Primary 35J25; Secondary 35R05, 65N30, 58J32, 52B70.
Key words and phrases: Polyhedral domain, elliptic equations, mixed boundary conditions, weighted Sobolev spaces, well-posedness, parametric family, Legendre approximation.

1. Introduction

We consider mixed boundary value problems on a bounded, d-dimensional domain Ω with conical points, $d \geq 1$. The associated differential operators belong to suitable families of strongly elliptic, second order differential operators with singular coefficients. We show that considering suitable singular coefficients is natural even if one is interested only in the case of regular coefficients. Using appropiate weighted Sobolev spaces, we obtain uniform estimates on the norm and on the regularity of the solutions. In addition, we provide weighted Sobolev space conditions on the coefficients that ensure a regular dependence of the solution on the coefficients.

To better explain our results, it is useful to put them into perspective. A classical result in Partial Differential Equations states that a second order, strongly elliptic partial differential operator P induces an isomorphism

$$P \,:\, H^{m+1}(G) \cap \{u|_{\partial\Omega} = 0\} \xrightarrow{\sim} H^{m-1}(G)\,, \tag{1}$$

for all $m \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, provided that G is a smooth, bounded domain in some euclidean space. See, for example, [1, 32, 23, 25] and the references therein. This result has many applications and extensions. However, it does not extend directly to non-smooth domains. In fact, on non-smooth domains, the solution u of $Pu = F$ will have singularities, even if the right hand side F is smooth. See Kondratiev's fundamental 1967 paper [27] for the case of domain with conical points and Dauge's comprehensive Lecture Notes [20] for the case of polyhedral domains. See [4, 6, 7, 9, 17, 24, 28, 29, 30, 36, 39] for a sample of related results. These theoretical results have been a critical ingredient in developing effective numerical methods approximating singular solutions. See for example [5, 11]. In addition, we mention that estimates for equations on conical manifolds can also be obtained using the method of layer potentials (see, for example, [13, 22, 26, 34, 38] and references therein).

For polygonal domains (and, more generally, for domains with conical points), Kondratiev's results mentioned above extend the isomorphism in (1) to polygonal domains by replacing the usual Sobolev spaces $H^m(\Omega)$ with the Kondratiev type Sobolev spaces. Let Ω be then a curvilinear polygonal domain (see Definition 3.1, in particular, the sides are not required to be straight), and $r_\Omega > 0$ be a smooth function on Ω that coincides with the distance to its vertices when close to the vertices. We let

$$\mathcal{K}_a^m(\Omega) \;:=\; \{\, u : \Omega \to \mathbb{C} \mid r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega),\; |\alpha| \leq m \,\}\,, \tag{2}$$

where $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$, and $\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d}$. Kondratiev's results [27] (see also [17, 28]) give that the Laplacian $\Delta := \sum_{i \leq d} \partial_i^2$ induces an isomorphism $\Delta \,:\, \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to \mathcal{K}_{a-1}^{m-1}(\Omega)$ for all $m \in \mathbb{Z}_+ = \{0, 1, \ldots\}$ and all $|a| < \pi/\alpha_{MAX}$, where α_{MAX} is the maximum angle of Ω. That is,

$$\Delta : \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \xrightarrow{\sim} \mathcal{K}_{a-1}^{m-1}(\Omega) \tag{3}$$

is a continuous bijection with continuous inverse for the indicated values of a and all non-negative integers m. One can extend this result by interpolation to the usual range of values for m, [32]. A similar result holds also for more general strongly elliptic operators [28]. In [9], this result of Kondratiev was extended to three dimensional polyhedral domains and in [8] it was extended to general d-dimensional polyhedral domains. In three dimensions and higher, this type of results is not enough for numerical methods. Thus, in [10], an anisotropic regularity and well-posedness result was proved for three dimensional polyhedral domains. See also [17] for further references and for related results, including analytic regularity.

In this paper, we generalize Kondratiev's result by allowing low-regularity coefficients and by describing the dependence of the solution on these coefficients. To state our main result, let us fix some notation. Let $\beta := (a_{ij}, b_i, c)$ be the coefficients of

$$p_\beta u := -\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j u) + \sum_{i=1}^{d} b_i\partial_i u - \sum_{i=1}^{d} \partial_i(b_{d+i}u) + cu\,, \tag{4}$$

a second order differential operator in divergence form on some domain $\Omega \subset \mathbb{R}^d$. Many concepts discussed in the paper make sense for any dimension $d \geq 1$. Nevertheless, the main results we prove are for $d = 2$. Thus, we assume for the rest of this introduction that Ω is a two-dimensional curvilinear polygonal domain. The coefficients β of the operator p_β are obtained using weighted $\mathcal{W}^{m,\infty}$-type space defined by

$$\mathcal{W}^{m,\infty}(\Omega) := \{\, u : \Omega \to \mathbb{C} \mid r_\Omega^{|\alpha|}\partial^\alpha u \in L^\infty(\Omega),\ |\alpha| \leq m \,\}\,, \tag{5}$$

where r_Ω is as in Equation (2) (that is, it is equal to the distance function to the conical points when close to those points). We fix for the rest of the introduction $m \in \mathbb{Z}_+ := \{0, 1, \ldots\}$ and we assume that $a_{ij}, r_\Omega b_i, r_\Omega^2 c \in \mathcal{W}^{m,\infty}(\Omega)$. We let

$$\|\beta\|_{Z_m} := \max\{\|a_{ij}\|_{\mathcal{W}^{m,\infty}(\Omega)},\ \|r_\Omega b_i\|_{\mathcal{W}^{m,\infty}(\Omega)},\ \|r_\Omega^2 c\|_{\mathcal{W}^{m,\infty}(\Omega)}\,\}\,, \tag{6}$$

(notice the factors involving r_Ω!), and for $P = p_\beta$ and $V = H_0^1(\Omega)$, define

$$\rho(P) := \inf \frac{\Re(Pv, v)}{\|v\|^2_{H^1(\Omega)}}\,, \quad v \in V\,,\ v \neq 0\,, \tag{7}$$

where $\Re(z) = \Re z$ denotes the real part of z. Our main result *for Dirichlet boundary conditions in two dimensions* is as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygonal domain and $p_\beta : H_0^1(\Omega) \to H^{-1}(\Omega)$ be as in (4). If $\rho(\beta) := \rho(p_\beta) > 0$, then there exists $\eta > 0$ such that*

$$p_\beta : \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{\, u|_{\partial\Omega} = 0 \,\} \to \mathcal{K}^{m-1}_{a-1}(\Omega) \tag{8}$$

is an isomorphism for $|a| < \eta$ and $p_\beta^{-1} : \mathcal{K}^{m-1}_{a-1}(\Omega) \to \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\}$ depends analytically on the coefficients $\beta := (a_{ij}, b_i, c)$ and has norm

$$\|p_\beta^{-1}\| \leq C_m(\rho(\beta) - \gamma_1|a| - \gamma_2 a^2)^{-N_m-1}\|\beta\|_{Z_m}^{N_m}\,,$$

with C_m, γ_1, γ_2, and $N_m \geq 0$ independent of β.

Since the solution u of the equation $p_\beta u = F$, $u = 0$ on the boundary, is in $\mathcal{K}^{m+1}_{a+1}(\Omega)$ for $F \in \mathcal{K}^{m-1}_{a-1}(\Omega)$, $|a| < \eta$, we obtain the usual applications to the Finite Element Method on straight polygonal domains for $m \geq 1$ and $a > 0$.

Theorem 1.1 is a consequence of Theorem 4.4, which deals with the mixed boundary value problem

$$\begin{cases} p_\beta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial_D\Omega \\ \partial^\beta_\nu u = h & \text{on } \partial_N\Omega\,, \end{cases} \tag{9}$$

where $(\partial_\nu^\beta v) := \sum_{i=1}^d \nu_i\big(\sum_{j=1}^d a_{ij}\partial_j v + b_{d+i}v\big)$. An exotic example to which Theorem 4.4 applies is that of the Schroedinger operator $H := -\Delta + cr_\Omega^{-2}$ on Ω with *pure Neumann* boundary conditions. The main novelties of Theorem 4.4 (and of the paper in general) are the following:

- (i) The precise estimate on the norm of the inverse of p_β seems to be new even in the smooth case.
- (ii) We deal with singular coefficients of a type that has not been systematically considered in the literature on non-smooth domains. Thus our coefficients have both singular parts at the corners of the form r_Ω^{-j} $(j \leq 2)$ and have limited regularity away from the corners.
- (iii) We provide a new method to obtain higher regularity in weighted Sobolev spaces using divided differences; a method that is, in fact, closer to the one used in the classical case of smooth domains.

The paper is organized as follows. In Section 2, we introduce the notation and necessary preliminary results for our problem in the usual Sobolev spaces. In particular, an enhanced Lax-Milgram Lemma (Lemma 2.6) provides uniform estimates for the solution of our problem (9) and analytic dependence of this solution on the coefficients β. In Section 3, we first define curvilinear polygonal domains (Definition 3.1). We then provide several equivalent definitions of the weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ and the form of our differential operators. Then, in Section 4, using local coordinate transformations, we derive our main result, the analytic dependence of the solution on the coefficients in high-order weighted Sobolev spaces (Theorem 4.4). Finally, Section 5 contains some consequences of Theorem 4.4 and some extensions. In particular, we consider a framework for the pure Neumann problem with inverse square potentials at vertices.

We thank Thomas Apel, Martin Costabel, Monique Dauge, Markus Hansen, Serge Nicaise, and Christoph Schwab for useful comments.

2. Coercivity in classical Sobolev spaces

In this section, we recall some needed results on coercive operators.

2.1. **Function spaces and boundary conditions.** Throughout the paper, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denotes a connected, bounded domain. Further conditions on Ω will be imposed in the next section. As usual, $H^m(\Omega)$ denotes the space of (equivalence classes of) functions on Ω with m derivatives in $L^2(\Omega)$. When we write $A \subset B$, we allow also $A = B$. In what follows, $\partial_D\Omega$ is a suitable closed subset of the boundary $\partial\Omega$, where we impose *Dirichlet* boundary conditions.

To formulate our problem (9), it is necessary to introduce the right spaces. We shall rely heavily on the weak formulation of this problem. Thus, let us recall that $H^{-1}(\Omega)$ is defined as the dual space of

$$H_0^1(\Omega) := \{\, u \in H^1(\Omega) \mid u|_{\partial\Omega} = 0 \,\}\,, \tag{10}$$

with pivot $L^2(\Omega)$. We introduce homogeneous essential boundary conditions abstractly, by considering a subspace V,

$$H_0^1(\Omega) \subset V \subset H^1(\Omega)\,, \tag{11}$$

such that V is a Banach space in its own topology and $H_0^1(\Omega)$ is a closed subspace of V. In many applications, V is closed in $H^1(\Omega)$, but this is not the case in our application to the Neumann problem with inverse square potentials at vertices (see Theorem 5.4). Let V^* be *the dual of* V *with pivot space* $L^2(\Omega)$. Therefore, by $(\ ,\ )$ we shall denote *both* the inner product $(f,g) = \int_\Omega f(x)\overline{g(x)}\,dx$ on $L^2(\Omega)$, and by continuous extension, also the duality pairing between V^* and V. Thus, $V^* = H^{-1}(\Omega)$ if $V = H^1_0(\Omega)$; otherwise, V^* will incorporate also non-homogeneous natural boundary conditions.

For Problem (9), we choose

$$V \;=\; H^1_D(\Omega) \;:=\; \{\, u \in H^1(\Omega) \mid u = 0 \text{ on } \partial_D\Omega \,\}\,, \tag{12}$$

and assume that the Neumann part of the boundary contains no adjacent edges.

2.2. **The weak formulation.** Recall from Equation (4) the differential operator $p_\beta u := -\sum_{i,j=1}^{d} \partial_i\big(a_{ij}\partial_j u\big) + \sum_{i=1}^{d} b_i\partial_i u - \sum_{i=1}^{d}\partial_i(b_{d+i}u) + cu$, which is used in our problem (9), where $a_{ij}, b_i, c : \Omega \to \mathbb{C}$ denote measurable *complex valued* functions as in (4) and β denotes the coefficients (a_{ij}, b_i, c). We shall make suitable further assumptions on these coefficients below.

Equation (9), makes sense as formulated only if u is regular enough (at least in $H^{3/2+\epsilon}$, to validate the Neumann derivatives at the boundary). In order to use the Lax-Milgram Lemma for the problem (9), we formulate our problem in a more general way that allows $u \in V$. To this end, let us introduce the Dirichlet form B^β associated to (9), that is, the sesquilinear form

$$B^\beta(u,v) \;:=\; \sum_{i,j=1}^{d}\big(a_{ij}\partial_j u, \partial_i v\big) + \sum_{i=1}^{d}\big(b_i\partial_i u, v\big) + \sum_{i=1}^{d}\big(b_{d+i}u, \partial_i v\big) + \big(cu, v\big) \tag{13}$$

$$= \int_\Omega \Big[\sum_{i=1}^{d}\Big(\sum_{j=1}^{d} a_{ij}(x)\partial_j u(x) + b_{d+i}(x)u(x)\Big)\partial_i\overline{v(x)} + \Big(\sum_{i=1}^{d} b_i(x)\partial_i u(x) + c(x)u(x)\Big)\overline{v(x)}\Big]\,dx\,,$$

where dx denotes the volume element in the Lebesgue integral on $\Omega \subset \mathbb{R}^d$.

Remark 2.1. Let $F(v) = \int_\Omega f(x)v(x)dx + \int_{\partial_N\Omega} h(x)v(x)dS$. Then the weak variational formulation of Equation (9) is: Find $u \in V$, such that

$$B^\beta(u,v) \;=\; F(\overline{v})\,, \quad \text{for all } \; v \in V\,. \tag{14}$$

We then define $P^\beta : V \to V^*$ by

$$(P^\beta u, v) := B^\beta(u,v)\,, \quad \text{for all } \; u, v \in V\,. \tag{15}$$

Thus, the weak formulation of Equation (9) is equivalent to

$$P^\beta u \;=\; F \in V^*\,. \tag{16}$$

We are interested in the dependence of u on F and on the coefficients $\beta := (a_{ij}, b_i, c)$ of P^β. We notice that if the Neumann part of the boundary $\partial_N\Omega$ is empty, then p_β and P^β can be identified, but this is not possible in general. In fact, we are looking for an analytic dependence of the solutions on the coefficients. For this reason, it is useful to consider complex Banach spaces and complex valued coefficients.

2.3. **Bounded forms and operators.** For two Banach spaces X and Y, let $\mathcal{L}(X;Y)$ denote the Banach space of continuous, linear maps $T : X \to Y$ endowed with the operator norm

$$\|T\|_{\mathcal{L}(X;Y)} := \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}. \tag{17}$$

We write $\mathcal{L}(X) := \mathcal{L}(X;X)$.

Let us define Z to be the set of coefficients $\beta = (a_{ij}, b_i, c)$ such that the form B^β is defined (and continuous) on $V \times V$, and we give Z the induced norm. Thus Z is given the induced topology from $\mathcal{L}(V;V^*)$.

Corollary 2.2. *The map $Z \ni \beta \to P^\beta \in \mathcal{L}(V;V^*)$ is well defined and continuous. For each $0 < r \leq \infty$, the set*

$$\{\, \beta \in Z,\ P^\beta \text{ is invertible and } \|(P^\beta)^{-1}\|_{\mathcal{L}(V^*;V)} < r \,\} \tag{18}$$

is open in Z.

Proof. By the definition of Z, $P^\beta : V \to V^*$ is a bounded operator and that the map $Z \ni \beta \to P^\beta \in \mathcal{L}(V;V^*)$ is continuous. Next, we know that the set $\mathcal{L}_{inv}(V;V^*)$ of invertible operators in $\mathcal{L}(V;V^*)$ is open and that the map $P \to P^{-1}$ is continuous on $\mathcal{L}_{inv}(V,V^*)$. Therefore the set $\{P \in \mathcal{L}(V;V^*) \mid \|P^{-1}\| < r\}$ is open in $\mathcal{L}(V;V^*)$. Our desired set is the inverse image of this set via the continuous map $\beta \to P^\beta$. Since the inverse image of an open set via a continuous map is open, the result follows. □

It will be convenient to use a slightly enhanced version of the well-known Lax-Milgram Lemma stressing the analytic dependence on the operator and on the data. We thus first review a few basic definitions and results on analytic functions [21].

Let X and Y be Banach spaces. In what follows, $\mathcal{L}_i(Y;X)$ will denote the space of continuous, multi-linear functions $L : Y \times Y \times \ldots \times Y \to X$, where i denotes the number of copies of Y. The norm on the space $\mathcal{L}_i(Y;X)$ is

$$\|L\|_{\mathcal{L}_i(Y;X)} := \sup_{\|y_j\|\leq 1} \|L(y_1, y_2, \ldots, y_i)\|_X .$$

Of course, $\mathcal{L}_1(Y;X) = \mathcal{L}(Y;X)$, isometrically. We shall need analytic functions defined on open subsets of a Banach space. Let $U \subset Y$ be an open subset, then $\mathcal{C}^k(U;X)$, $k \in \mathbb{Z}_+ \cup \{\infty, \omega\}$, denotes the space of functions $v : U \to X$ with k continuous (Fréchet) derivatives $D^i v : U \to \mathcal{L}_i(Y;X)$, $i \leq k$, with $D^i_a v \in \mathcal{L}_i(Y;X)$ denoting the value of $D^i v$ at a. Similarly, $\mathcal{C}^k_b(U;X) \subset \mathcal{C}^k(U;X)$, $k \in \mathbb{Z}_+ \cup \{\infty, \omega\}$, denotes the subspace of those functions $v \in \mathcal{C}^k(U;X)$ for which the derivatives $D^i v$, $i \leq k$, are bounded on U. For each finite j, we let

$$\|v\|_{\mathcal{C}^j_b(U;X)} := \sup_{i\leq j,\, y\in U} \|D^i_y v\|_{\mathcal{L}_i(Y;X)} \tag{19}$$

denotes the natural Banach space norm on $\mathcal{C}^j_b(U;X)$.

The case $k = \omega$ refers to analytic functions, that is, $\mathcal{C}^\omega(U;X)$ denotes the space of functions $f : U \to X$ that have, for any $a \in U$, a uniformly convergent power series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k_a f(x-a, x-a, \ldots, x-a),$$

for x in a small, non-empty open ball centered at a. If k is not finite, that is, if $k = \infty$ or $k = \omega$, we endow $\mathcal{C}_b^k(U;X)$ with the Fréchet topology defined by the family of seminorms $\|\cdot\|_{\mathcal{C}_b^j(U;X)}$, $j \geq 1$. We shall need the following standard result.

Lemma 2.3. *Let Y_1, Y_2 be Banach spaces.*

(i) The map $\mathcal{L}(Y_1;Y_2) \times Y_1 \ni (T,y) \to Ty \in Y_2$ is analytic.

(ii) The map $T \to T^{-1} \in \mathcal{L}(Y_1)$ is analytic on the open set $\mathcal{L}_{\mathrm{inv}}(Y_1)$ of invertible operators in $\mathcal{L}(Y_1) := \mathcal{L}(Y_1;Y_1)$.

Proof. In (i), the desired map is bilinear, and hence analytic. To prove (ii), we simply write the Neumann series formula $(T-R)^{-1} = \sum_{n=0}^{\infty} T^{-1}(RT^{-1})^n$, which is uniformly and absolutely convergent for $\|R\|\|T^{-1}\| \leq 1-\epsilon$, $\epsilon > 0$. □

2.4. **An enhanced Lax-Milgram Lemma.** We now recall the classical Lax-Milgram Lemma, in the form that we will need.

Definition 2.4. Let $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. A continuous operator $P : V \to V^*$ is called *coercive on V* (or simply *coercive* when there is no danger of confusion) if

$$0 < \rho(P) := \inf_{v \in V \smallsetminus \{0\}} \frac{\Re(Pv,v)}{\|v\|_V^2} .$$

We shall usually write $\rho(\beta) = \rho(P^\beta)$, where $\rho(P^\beta)$ is as defined in Equation (7). For $P = P^\beta$, we thus have $\rho(\beta)\|v\|^2_{H^1(\Omega)} = \rho(P^\beta)\|v\|^2_{H^1(\Omega)} \leq \Re B^\beta(v,v)$, for all $v \in V$. We shall need the following simple observation:

Remark 2.5. If $P : V \to V^*$ is coercive on V and $P_1 : V \to V^*$ satisfies $\|P_1\| < \rho(P)$, then $P + P_1$ is also coercive on V and $\rho(P+P_1) \geq \rho(P) - \|P_1\|$. Indeed,

$$\Re\big((P+P_1)u,u\big) \geq \Re(Pu,u) - \|P_1\|\|u\|_V^2 \geq (\rho(P) - \|P_1\|)\|u\|_V^2 , \tag{20}$$

and hence the set $\mathcal{L}(V;V^*)_c$ of coercive operators is open in $\mathcal{L}(V;V^*)$.

Recall now the standard way of solving Equation (14) using the Lax-Milgram Lemma for coercive operators.

Lemma 2.6 (Analytic Lax-Milgram Lemma)**.** *Assume that $P : V \to V^*$ is coercive. Then P is invertible and $\|P^{-1}\| \leq \rho(P)^{-1}$. Moreover, the map $\mathcal{L}(V;V^*)_c \times V^* \ni (P,F) \to P^{-1}F \in V$ is analytic. Consequently,*

$$Z \cap \mathcal{L}(V;V^*) \times V^* \ni (\beta, F) \to (P^\beta)^{-1}F \in V$$

is analytic as well.

Proof. The first part is just the classical Lax-Milgram Lemma [12, 14, 35], which states that "coercivity implies invertibility" and gives the norm estimate. The second part follows from Lemma 2.3. Indeed, the map $\Phi : \mathcal{L}(V;V^*)_c \times V^* \to V$, $\Phi(\beta,F) := (P^\beta)^{-1}F$ is the composition of the maps

$$\mathcal{L}(V;V^*)_c \times V^* \times V^* \ni (\beta, F) \to (P^\beta, F) \in \mathcal{L}_{inv}(V,V^*) \times V^* ,$$

$$\mathcal{L}_{inv}(V;V^*) \times V^* \ni (P,F) \to (P^{-1},F) \in \mathcal{L}(V^*;V) \times V^* , \quad \text{and}$$

$$\mathcal{L}(V^*;V) \times V^* \ni (P^{-1},F) \to P^{-1}F \in V .$$

The first of these three maps is well defined and linear by the classical Lax-Milgram Lemma. The other two maps are analytic by Lemma 2.3. Since the composition of analytic functions is analytic, the result follows. □

Examples of coercive operators are obtained using "uniformly strongly elliptic" operators, whose definition we recall next.

Definition 2.7. Let $\beta \in Z$. The operator P^β is called *uniformly strongly elliptic* if there exists $C > 0$ such that

$$\sum_{ij=1}^{d} \Re\big(a_{ij}(x)\xi_i\xi_j\big) \, \geq \, C\|\xi\|^2 \,, \tag{21}$$

for all $\xi = (\xi_i) \in \mathbb{R}^d$ and all $x \in \overline{\Omega}$. Here $\|\cdot\|$ denote the standard euclidean norm on $\mathbb{R}^d$. The largest C with the property in (21) will be denoted $C_{use}(\beta)$.

Then, we have the following standard example.

Example 2.8. Let $\beta \in Z$, as in Definition 2.7. We shall regard a matrix $X := [x_{ij}]$, $(X)_{ij} = x_{ij}$, as a linear operator acting on $\mathbb{C}^d$ by the formula $X\zeta = \xi$, where $\xi_i = \sum_j x_{ij}\zeta_j$. We consider the adjoint and positivity with respect to the usual inner product on $\mathbb{C}^d$. We thus have $X \geq 0$ if, and only if $(X\xi, \xi) = \sum_{ij} x_{ij}\xi_j\overline{\xi_i} \geq 0$ for all $\xi \in \mathbb{C}^d$. Also, recall that X^*, the adjoint of the matrix X, has entries $(X^*)_{ij} = \overline{x_{ji}}$. Then P^β is uniformly strongly elliptic if, and only if, there exists $\gamma > 0$ such that the matrix $a(x) := [a_{ij}(x)]$ of highest order coefficients of P^β satisfies

$$a(x) + a(x)^* \, \geq \, \gamma I_d \,, \quad \text{for all } x \in \Omega \,, \tag{22}$$

where I_d denotes the unit matrix on $\mathbb{C}^d$. Assume also that $b_i = c = 0$. Then,

$$\begin{aligned} 2\Re(P^\beta u, u) \, &:= \, 2\Re\Big(\int_\Omega \sum_{i,j=1}^{d} a_{ij}(x)\partial_j u(x)\partial_i\overline{u(x)}\, dx\Big) \, = \, 2\Re(a\nabla u, \nabla u) \\ &= \, (a\nabla u, \nabla u) + (\nabla u, a\nabla u) \, = \, \big((a + a^*)\nabla u, \nabla u\big) \, \geq \, \gamma\|\nabla u\|^2_{L^2(\Omega)} \,. \end{aligned}$$

If, moreover, $\partial_D\Omega$ has positive measure, then there exists $c = c_{\Omega,\partial_D\Omega} > 0$ such that $\int_\Omega |\nabla v|^2 dx \, \geq \, c\|v\|^2_{H^1(\Omega)}$ for all $v \in H^1_D(\Omega)$, and hence P^β is coercive on $V = H^1_D(\Omega)$. (Recall that $H^1_D(\Omega)$ was defined in Equation (12). In particular, $v = 0$ on $\partial_D\Omega$ if $v \in H^1_D(\Omega)$.)

We then have the following standard result, whose proof we recall for the benefit of the reader. (See also [15, 37].)

Proposition 2.9. *If $\beta = (a_{ij}, b_i, c) \in Z$ is such that P^β is coercive on $H^1_0(\Omega) \subset V \subset H^1(\Omega)$, then P^β is strongly elliptic, more precisely, the estimate* (21) *is satisfied for any $C \leq \rho(\beta) := \rho(P^\beta)$. Moreover, $P^\beta : V \to V^*$ is a continuous bijection and $(P^\beta)^{-1}F$ depends analytically on the coefficients β and on $F \in V^*$.*

Proof. The second part is an immediate consequence of the analytic Lax-Milgram Lemma. Let us concentrate then on the first part. Let us assume that P is coercive and let $\xi = (\xi_i) \in \mathbb{R}^d$. Also, let us choose an arbitrary smooth function ϕ with compact support D in Ω. We then define the function $\psi \in \mathcal{C}^\infty_c(\Omega) \subset V$ by the formula $\psi(x) := e^{\imath t\xi\cdot x}\phi(x) \in \mathbb{C}$, where $\imath := \sqrt{-1}$ and $\xi \cdot x = \sum_{k=1}^{d} \xi_k x_k$. Then $\partial_j\psi(x) = \imath t\xi_j e^{\imath t\xi\cdot x}\phi(x) + e^{\imath t\xi\cdot x}\partial_j\phi(x)$, and hence $\imath t\xi_j e^{\imath t\xi\cdot x}\phi(x)$ is the dominant term in $\partial_j\psi(x)$ as $t \to \infty$. Taking into account all the indices j and computing the squares of the L^2-norms, we obtain

$$\lim_{t\to\infty} t^{-2}\|\psi\|^2_{H^1(D)} \;=\; \sum_{j=1}^{d} \xi_j^2 \int_D |\phi(x)|^2\,dx \;=\; \|\xi\|^2 \int_D |\phi(x)|^2\,dx\,. \tag{23}$$

Similarly, the coefficients a_{ij} of P^β, are estimated using "oscillatory testing"

$$\lim_{t\to\infty} t^{-2}(P^\beta\psi,\psi) \;=\; \int_D \sum_{i,j=1}^{d} a_{ij}(x,y)\xi_i\xi_j|\phi(x)|^2\,dx\,. \tag{24}$$

We then use Definition 2.4 for $v=\psi$ and we pass to the limit as $t\to\infty$. By coercivity and the definition of $\rho(\beta) := \rho(P^\beta)$, we have that $\rho(\beta)\|\psi\|^2_{H^1(D)} \le \Re(P^\beta\psi,\psi)$. Dividing this inequality by t^{-2} and taking the limit as $t\to\infty$, we obtain from Equations (23) and (24) that

$$\rho(\beta)\|\xi\|^2 \int_D |\phi(x)|^2\,dx \;\le\; \Re \int_D \sum_{ij} a_{ij}(x,y)\xi_i\overline{\xi}_j|\phi(x)|^2\,dx\,.$$

Since ϕ is an arbitrary compactly supported smooth function on D, it follows that, for all $x\in D$,

$$\rho(\beta)\|\xi\|^2 \;\le\; \Re \sum_{ij} a_{ij}(x)\xi_i\xi_j\,.$$

Since ξ is arbitrary, we obtain Equation (21) with $C=\rho(P)$. □

An immediate corollary of Proposition 2.9 is

Corollary 2.10. *We have* $\rho(P) \le C_{use}$.

This inequality will be used in the form $C_{use}^{-1} \le \rho(P)^{-1}$ in the following sections.

3. Polygonal domains, operators, and weighted Sobolev spaces

In this section, we introduce the domains, the weighted Sobolev spaces, and the differential operators that we shall use. We also provide several equivalent definitions of the weighted Sobolev spaces and prove some intermediate results.

3.1. **Polygonal domains and defining local coordinates.**

In this section, we let Ω be a *curvilinear polygonal domain,* although our method works without significant change for domains with conical points.

Let us describe in detail our domain Ω as a Dauge-type corner domain, with the purpose of fixing the notation and of introducing some useful local coordinate systems – called "defining coordinates" – that will be used in the proofs below. Let B_j denote the open unit ball in $\mathbb{R}^j$. Thus B_0 is reduced to one point, $B_1=(-1,1)$, and $B_2=\{(x,y)\subset\mathbb{R}^2, x^2+y^2<1\}$.

Definition 3.1. A *curvilinear polygonal domain* $\Omega\subset\mathbb{R}^2$ is an open, bounded subset of $\mathbb{R}^2$ with the property that for every point $p\in\overline{\Omega}$ there exists $j\in\{0,1,2\}$, a neighborhood U_p of p in $\mathbb{R}^2$, and a smooth map $\phi_p:\mathbb{R}^2\to\mathbb{R}^2$ that defines a diffeomorphism $\phi_p:U_p\to B_j\times B_{2-j}\subset\mathbb{R}^2$, $\phi_p(p)=0$, satisfying the following conditions:

(i) If $j=2$, then $U_p\subset\Omega$;
(ii) If $j=1$, then $\phi_p(U_p\cap\Omega)=B_1\times(0,1)$ or $\phi_p(U_p\cap\Omega)=B_1\times(B_1\smallsetminus\{0\})$;

(iii) If $j = 0$, then $\phi_p(U_p \cap \Omega) = \{(r\cos\theta, r\sin\theta) | \text{ with } r \in (0,1), \theta \in I_p\}$, for some finite union I_p of open intervals in S^1.

For $p \in \overline{\Omega}$, we let j_p the *largest* j for which p satisfies one of the above properties.

These are essentially the *corner domains* in [20]. The definition above was generalized to arbitrary dimensions in [8]. See also [28, 29, 33, 36]. The second case in (ii) corresponds to cracks in the domain. We continue with some remarks.

Remark 3.2. We notice that in the two cases (i) and (iii) of Definition 3.1 ($j = 2$ and $j = 0$), the spaces $\phi_p(U_p) = B_j \times B_{2-j}$ will be the same (up to a canonical diffeomorphism), but the spaces $\phi_p(U_p \cap \Omega)$ will not be diffeomorphic.

Remark 3.3. Let Ω be a curvilinear polygonal domain and $p \in \overline{\Omega}$. Then p satisfies the conditions of the definition for *exactly* one value of j, *except* the case when p is on a smooth part of the boundary, when a choice of $j = 1$ or $j = 0$ is possible. This is the case exactly when $j_p = 1$. If $j = 0$ is chosen, then I_p is half a circle.

Remark 3.4. The set $\mathcal{V}_g := \{p \in \overline{\Omega} | \, j_p = 0\}$ is finite and is contained in the boundary of Ω. It is the set of *geometric vertices*.

Let us choose for each point $p \in \overline{\Omega}$ a value $j = i_p$ that satisfies the conditions of the definition. If $j_p = 1$, we choose $i_p = j_p = 1$, except possibly for finitely many points $p \in \overline{\Omega}$. These points will be called *artificial vertices*. The set of all vertices (geometric and artificial) is finite, which will be denoted by $\mathcal{V}$, and *will be fixed in what follows*. We assume that all points where the boundary conditions change are in $\mathcal{V}$. We also fix the resulting polar coordinates $r \circ \phi_p$ and $\theta \circ \phi_p$ on U_p, for all $p \in \mathcal{V}$.

Definition 3.5. The coordinate charts $\phi_p : U_p \to B_j \times B_{2-j}$ of Definition 3.1 that were chosen such that $j = i_p$ are called the *defining* coordinate charts of the curvilinear polygonal domain Ω. (Recall that $j = i_p = 0$ if, and only if $p \in \mathcal{V}$.)

Remark 3.6. Artificial vertices are useful, for instance, in the case when we have a change in boundary conditions or if there are point singularities in the coefficients, see [29, 30] and the references therein. The right framework is, of course, that of a stratified space [8], with j_p denoting the dimension of the stratum to which p belongs, but we do not need this in the simple case at hand.

Remark 3.7. It follows from Definition 3.1 that if Ω is a curvilinear polygonal domain, then the set $\partial\Omega \smallsetminus \mathcal{V}$ is the union of finitely many smooth, open curves $e_j : (-1, 1) \to \partial\Omega$. The curves e_j have as image the *open edges* of $\partial\Omega$ and we shall sometimes identify e_j with its image. The curves e_j are disjoint and have no self-intersections. The closure of (the image of e_j) is called a *closed edge*. Thus, the vertices are not contained in the open edges (but they are, of course, contained in the closed edges). Our assumption that all points where the boundary conditions change are in $\mathcal{V}$ implies that $\partial_D\Omega$ consists of a union of closed edges of Ω.

3.2. **Equivalent definitions of weighted spaces.** In this section, we discuss some equivalent definitions of weighted Sobolev spaces. We adapt to our setting the results in [2], to which we refer for more details.

We shall fix, from now on, a *finite* set of defining coordinate charts $\phi_k = \phi_{p_k}$, for some $p_k \in \Omega$, $1 \leq k \leq N$, so that $U_k := U_{p_k}$, $1 \leq k \leq N$, defines a finite covering of $\overline{\Omega}$. Thus, for $p = p_k$ such that $j_p \neq 0$, the coordinates are $(x, y) \in \mathbb{R}^2$.

Otherwise, these coordinates will be denoted by $(r, \theta) \in (0, 1) \times S^1$. We may relabel these points such that p_k is a vertex if, and only if, $1 \leq k \leq N_0$. We then have the following alternative definition of the weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$. We denote

$$X_k u := \partial_x u \quad \text{and} \quad Y_k u := \partial_y u\,, \quad \text{for } N_0 < k \leq N\,, \tag{25}$$

in the coordinate system defined by $\phi_k = \phi_{p_k} = (x, y) \in \mathbb{R}^2$ that corresponds to one of the chosen points p_k, provided that is *not* a vertex. If, however, p_k is a vertex, then we let

$$X_k u := r\partial_r u \quad \text{and} \quad Y_k u := \partial_\theta u\,, \quad \text{for } 1 \leq k \leq N_0\,, \tag{26}$$

in the coordinate system defined by $\phi_k = (r, \theta) \in (0, 1) \times S^1$. Note the appearance of r in front of ∂_r!

Remark 3.8. Assuming that the coefficients are locally Lipschitz, we can express the differential operator $r_\Omega^2 p_\beta$ in any of the coordinate systems $\phi_k : U_k \to \mathbb{R}^2$. That means that, for each $1 \leq k \leq N$, we can find coefficients $c, c_1, c_2, c_{11}, c_{12}, c_{22}$ such that

$$p_\beta u = (c_{11} X_k^2 + c_{12} X_k Y_k + c_{22} Y_k^2 + c_1 X_k + c_2 Y_k + c) u \quad \text{on } U_k\,, \tag{27}$$

with the vector fields X_k and Y_k introduced in Equations (25) and (26).

For each open subset $U \subset \Omega$, let us denote

$$\|u\|^2_{\mathcal{K}_a^m(U)} := \sum_{|\alpha| \leq m} \|r_\Omega^{|\alpha| - a} \partial^\alpha u\|^2_{L^2(U)}\,. \tag{28}$$

Thus, if $U = \Omega$, $\|u\|_{\mathcal{K}_a^m(U)} = \|u\|_{\mathcal{K}_a^m(\Omega)}$ is simply the norm on $\mathcal{K}_a^m(\Omega)$. Note that the weight r_Ω is not intrinsic to the set U, but depends on Ω, which is nevertheless not indicated in the notation, in order not to overburden it. We define the spaces $\mathcal{W}^{m,\infty}(U)$ similarly as in (5) with the same weight r_Ω. We then have the following result that, in particular, provides an alternative definition of the weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ introduced in Equation (2).

Proposition 3.9. *Let $u : \Omega \to \mathbb{C}$ be a measurable function and $U \subset \Omega$ and open subset. We have that $u \in \mathcal{K}_a^m(U)$ if, and only if, $r_\Omega^{-a} X_k^i Y_k^j u \in L^2(U \cap U_k)$, where $U_k = U_{p_k}$, for all k and all $i + j \leq m$. Moreover, the $\mathcal{K}_a^m(U)$–norm is equivalent to the norm*

$$|||u|||'_U := \sum_{k=1}^N \sum_{i+j \leq m} \|r_\Omega^{-a} X_k^i Y_k^j u\|_{L^2(U \cap U_k)}\,.$$

Proof. This follows right away from the definition of the $\mathcal{K}_a^m(U)$-norm. Indeed, away from the vertices, both the $|||\cdot|||'_U$-norm and the $\mathcal{K}_a^m$-norm coincide with the usual H^m-norm. On the other hand, near a vertex, or more generally on an angle $\Xi := \{(r, \theta) |\, \alpha < \theta < \beta\}$, both norms are given by $\{u |\, r^{-a}(r\partial_r)^i \partial_\theta^j u \in L^2(\Xi)\}$. For the $\mathcal{K}_a^m(U)$-norm this is seen by writing ∂_x and ∂_y in polar coordinates, more precisely, from

$$r\partial_x = (\cos\theta) r\partial_r - (\sin\theta)\partial_\theta \quad \text{and} \quad r\partial_y = (\sin\theta) r\partial_r + (\cos\theta)\partial_\theta\,. \tag{29}$$

See [3, 30] for more details. □

We finally have the following corollary.

Corollary 3.10. *The norm $\|u\|_{\mathcal{K}_a^{m+1}(\Omega)}$ is equivalent to the norm*

$$|||u||| := \|u\|_{\mathcal{K}_a^m(\Omega)} + \sum_{k=1}^{N} \big(\|X_k u\|_{\mathcal{K}_a^m(U_k)} + \|Y_k u\|_{\mathcal{K}_a^m(U_k)}\big)\,.$$

Proof. In the definition of $|||u|||'$, Proposition 3.9, with m replaced by $m+1$, we collect all the terms with $i+j \le m$ and notice that they give the norm for $\mathcal{K}_a^m$. The rest of the terms will contain at least one differential X_k or one differential Y_k and thus are of the form $\|r_\Omega^{-a} X_k^i Y_k^j Y_k u\|_{L^2(\Omega)}$ or $\|r_\Omega^{-a} X_k^i Y_k^j X_k u\|_{L^2(\Omega)}$, $i+j \le m$, since the differential operators X_k and Y_k commute on U_k. □

3.3. **The differential operators.** We include in this subsection the definition of our differential operators and three needed intermediate results (lemmas).

We introduce now our set of coefficients. Recall the norm $\|\beta\|_{Z_m}$ introduced in Equation (6) and let

$$Z_m := \{\, \beta = (a_{ij}, b_i, c) \mid \text{such that } \|\beta\|_{Z_m} < \infty \,\}\,. \tag{30}$$

Note that for example, the Schroedinger operator $-\Delta + r^{-2}$ is an operator of the form P^β for suitable $\beta \in Z_m$.

Below, we shall often use inequalities of the form $A \le CB$, where A and B are expressions involving u and β and $C \in \mathbb{R}$. We shall say that C is an *admissible bound* if it does not depend on u and β, and then we shall write $A \le_c B$.

Lemma 3.11. *Let $\beta = (a_{ij}, b_i, c) \in Z_m$, $m \ge 1$, and let us express p_β as in Remark 3.8. Then $c, c_1, c_2, c_{11}, c_{12}, c_{22} \in \mathcal{W}^{m-1,\infty}(U_k)$. Moreover,*

$$\|c\|_{\mathcal{W}^{m-1,\infty}(U_k)} + \|c_1\|_{\mathcal{W}^{m-1,\infty}(U_k)} + \ldots + \|c_{22}\|_{\mathcal{W}^{m-1,\infty}(U_k)} \le_c \|\beta\|_{Z_m}\,.$$

If p_β is moreover uniformly strongly elliptic, then $|c_{22}^{-1}| \le_c C_{use}^{-1}$ on U_k.

Proof. We first notice that since $m \ge 1$, we can convert our operator to a non-divergence form operator. Indeed, one can simply replace a term of the form $\partial_i a \partial_j u$ with $a\partial_i\partial_j u + (\partial_i a)\partial_j u$, where $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ and $r_\Omega \partial_i a \in \mathcal{W}^{m-1,\infty}(\Omega)$. We deal similarly with the terms of the form $\partial_i(b_i u)$. This accounts for the loss of one derivative in the regularity of the coefficients of $c, \ldots, c_{22}$.

We need to show that the coefficients $c, \ldots, c_{22}$ are in $\mathcal{W}^{m-1,\infty}(\Omega)(U_k)$ and that they have the indicated bounds. To this end, we consider the two possible cases: when U_k contains no vertices of Ω (equivalently, if $k > N_0$) and the case when U_k is centered at a vertex.

If $k > N_0$, then the coefficients $c, \ldots, c_{22}$ can be expressed using ϕ_k and its derivatives linearly in terms of the coefficients β on the closure of U_k. Since there is a finite number of such neighborhoods and ϕ_k and its derivatives are bounded on the closure of U_k, the bound for the coefficients $c, \ldots, c_{22}$ in terms of $\|\beta\|_{Z_m}$ on U_k follows using a compactness argument. In particular, the bound $|c_{22}^{-1}| \le_c C_{use}^{-1}$ follows from the uniform ellipticity of p_β on $\overline{U_k}$.

If, on the other hand, $k \le N_0$ (that is, U_k is centered at a vertex). Let us concentrate on the highest order terms, for simplicity. We then have, up to lower order terms (denoted $l.o.t$)

$$\begin{aligned} r^2\partial_x^2 &= (\cos\theta)^2(r\partial_r)^2 - 2(\sin\theta\cos\theta)r\partial_r\partial_\theta + (\sin\theta)^2\partial_\theta^2 + l.o.t. \\ r^2\partial_x\partial_y &= (\sin\theta\cos\theta)(r\partial_r)^2 + (\cos^2\theta - \sin^2\theta)r\partial_r\partial_\theta + (\sin\theta\cos\theta)\partial_\theta^2 + l.o.t. \\ r^2\partial_y^2 &= (\sin\theta)^2(r\partial_r)^2 + 2(\sin\theta\cos\theta)r\partial_r\partial_\theta + (\cos\theta)^2\partial_\theta^2 + l.o.t. \end{aligned}$$

The bound on the coefficients $c, \ldots, c_{22}$ follows since $\sin\theta$ and $\cos\theta$ are in $\mathcal{W}^{m,\infty}(U_k)$ for all m. This gives also that $c_{22} = a_{11}\cos^2\theta + 2a_{12}\cos\theta\sin\theta + a_{22}\sin^2\theta \geq C_{use}$ for the coefficient c_{22} of $Y_k^2 = \partial_\theta^2$. (Thus $|c_{22}^{-1}| \leq C_{use}^{-1}$ on U_k, for $k \leq N_0$.) □

For instance, for the Laplacian in polar coordinates, we have

$$r_\Omega^2\Delta \;=\; (r\partial_r)^2 + \partial_\theta^2 \;=\; X_k^2 + Y_k^2$$

in the neighborhood U_k of the vertex p_k.

The following lemma will be used in the proof of Theorem 4.4 and explains some of the calculations there.

Lemma 3.12. *For two functions b and c, we have*

(i) $\|bc\|_{\mathcal{K}_a^m(\Omega)} \leq C\|b\|_{\mathcal{W}^{m,\infty}(\Omega)}\|c\|_{\mathcal{K}_a^m(\Omega)}$.
(ii) $\|bc\|_{\mathcal{W}^{m,\infty}(\Omega)} \leq C\|b\|_{\mathcal{W}^{m,\infty}(\Omega)}\|c\|_{\mathcal{W}^{m,\infty}(\Omega)}$, *therefore $\mathcal{W}^{m,\infty}(\Omega)$ is an algebra.*
(iii) If $b \in \mathcal{W}^{m,\infty}(\Omega)$ is such that $b^{-1} \in L^\infty(\Omega) = \mathcal{W}^{0,\infty}(\Omega)$, then b is invertible in $\mathcal{W}^{m,\infty}(\Omega)$ and

$$\|b^{-1}\|_{\mathcal{W}^{m,\infty}(\Omega)} \;\leq\; C\,\|b^{-1}\|_{L^\infty(\Omega)}^{m+1}\,\|b\|_{\mathcal{W}^{m,\infty}(\Omega)}^m\,.$$

The parameter C depends only on m and Ω.

Proof. This is a direct calculation. Indeed, the first two relations are based on the rule $\partial^\alpha(bc) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\,\partial^\beta b\,\partial^{\alpha-\beta}c$. The last one is obtained from the relation $\partial^\alpha(b^{-1}) = b^{-1-|\alpha|}Q$, where $Q = Q(b, \partial_1 b, \partial_2 b, \ldots, \partial^\alpha b)$ is a polynomial of degree $|\alpha|$ in all derivatives $\partial^\beta b$, with $0 \leq \beta \leq \alpha$. This relation is proved by induction on $|\alpha|$. □

For further reference, we shall need the following version of "Nirenberg's trick," (see, for instance, [1, 23]).

Lemma 3.13. *Let $T : X \to Y$ be a continuous, bijective operator between two Banach spaces X and Y. Let $S_X(t)$ and $S_Y(t)$ be two c_0 semi-groups of operators on X, respectively Y, with generators denoted by A_X and, respectively, A_Y. We assume that for any $t > 0$, there exists $T_t \in \mathcal{L}(X;Y)$ such that $S_Y(t)T = T_tS_X(t)$. Assume that $t^{-1}(T_t - T)$ converges strongly as $t \to 0$ to a bounded operator B. Then T maps bijectively the domain of A_X to the domain of A_Y and we have that $A_XT^{-1}\xi = T^{-1}\big(A_Y\xi - BT^{-1}\xi\big)$, for all ξ in the domain of A_Y. Consequently,*

$$\|A_XT^{-1}\xi\|_X \;\leq\; \|T^{-1}\|\,\big(\,\|A_Y\xi\|_Y + \|B\|\|T^{-1}\xi\|_X\,\big)\,.$$

Proof. We have that $\xi \in X$ is in the domain of A_X, the generator of S_X if, and only if, the limit $A_X\xi := \lim_{t\to 0} t^{-1}\big(S_X(t) - 1\big)\xi$ exists. The definition of T_t gives

$$t^{-1}\big(S_Y(t) - 1\big)T\xi \;=\; t^{-1}\big(T_t - T\big)S_X(t)\xi \,+\, t^{-1}T\big(S_X(t) - 1\big)\xi\,.$$

Since $t^{-1}(T_t - T)\zeta \to B\zeta$ for all vectors $\zeta \in X$ and $B : X \to Y$ is bounded, we obtain that the limit $\lim_{t\to 0} t^{-1}\big(S_Y(t) - 1\big)T\xi$ exists if, and only if, the limit $\lim_{t\to 0} t^{-1}\big(S_X(t) - 1\big)\xi$ exists. This shows that T maps bijectively the domain of A_X to the domain of A_Y and that $A_YT = B + TA_X$. Multiplying by T^{-1} to the left and to the right gives the desired result. □

One can use Lemma 3.13 as a regularity estimate.

4. Higher regularity in weighted Sobolev spaces

In this section, we prove our main result, Theorem 4.4. Theorem 1.1 is an immediate consequence of this theorem and of Remark 4.3.

4.1. The higher regularity problem.

We now come back to the study of our mixed problem, as formulated in Equation (9). We are interested in solutions with more regularity than the ones provided by the space V appearing in its weak formulation, Equation (14) or Equation (16). While for the weak formulation the classical Sobolev spaces suffice, the higher regularity is formulated in the framework of the weighted Sobolev spaces considered by Kondratiev [27] and others, see also [18, 19].

We thus introduce

$$\begin{aligned} V_m(a) &:= \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \quad \text{for} \quad m \in \mathbb{Z}_+ = \{0,1,2,\ldots\} \quad \text{and} \\ V^-_m(a) &:= \mathcal{K}^{m-1}_{a-1}(\Omega) \oplus \mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega) \quad \text{for} \quad m \in \mathbb{N} = \{1,2,\ldots\}. \end{aligned} \tag{31}$$

The spaces $\mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)$, $m \geq 1$, are the spaces of traces of functions in $\mathcal{K}^m_a(\Omega)$, in the sense that the restriction at the boundary defines a continuous, surjective map $\mathcal{K}^m_a(\Omega) \to \mathcal{K}^{m-1/2}_{a-1/2}(\partial_N\Omega)$ [3]. The space $\mathcal{K}^m_a(\partial_N\Omega)$ can be defined directly for $m \in \mathbb{Z}_+$ in a manner completely analogous to the usual Kondratiev spaces. For non-integer regularity, they can be obtained by interpolation, [2, 3].

We recall that differentiation defines continuous maps $\partial_j : \mathcal{K}^m_a(\Omega) \to \mathcal{K}^{m-1}_{a-1}(\Omega)$. In the same way, the combination of the normal derivative at the boundary $(\partial^\beta_\nu v) := \sum_{i=1}^d \nu_i\big(\sum_{j=1}^d a_{ij}\partial_j v + b_{d+i}v\big)$ and restriction at the boundary define a continuous, surjective map $\partial^\beta_\nu : \mathcal{K}^m_a(\Omega) \to \mathcal{K}^{m-3/2}_{a-3/2}(\partial_N\Omega)$, $m \geq 2$.

Lemma 4.1. *We have continuous maps*

$$\begin{aligned} P^\beta(m,a) &:= (p_\beta, \partial^\beta_\nu) : V_m(a) \to V^-_m(a), \quad m \geq 1, \\ P^\beta(m,a)(u) &= \Big(\sum_{ij} \partial_i(a_{ij}\partial_j u) + \sum_i b_i\partial_i u + cu, \sum_{ij} \nu_i a_{ij}\partial_j u|_{\partial_N\Omega} \Big). \end{aligned} \tag{32}$$

Therefore the operators $P^\beta(m,a)$, $m \in \mathbb{N}$, $a \in \mathbb{R}$, are given by the same formula (but have different domains and ranges).

Remark 4.2. Let us assume for this remark that $a = 0$ and discuss this case in more detail. If $\partial_N\Omega$ contains no adjacent edges, the the Hardy inequality [9, 28] shows that the natural inclusion

$$\mathcal{K}^1_1(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \to H^1_D(\Omega) := H^1(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \tag{33}$$

is an isomorphism (that is, it is continuous with continuous inverse). We thus consider $V := V_0(0)$ in general (for all $\partial_N\Omega$). For symmetry, we also let $V^-_0(0) := V^*$ and

$$P^\beta(0,0) := P^\beta : V_0(0) = V \to V^-_0(0) := V^*, \tag{34}$$

which is, of course, nothing but the operator studied before.

We then have

$$V_{m+1}(0) \subset V_m(0) \quad \text{and} \quad V^-_{m+1}(0) \subset V^-_m(0) \quad \text{for all } m \geq 0.$$

This is trivially true for $m > 0$. For $m = 0$, in which case we need to construct the natural inclusion $\Phi : V^-_m(0) \to V^-_0(0)$, $m \geq 1$. The map Φ associates to

$(f,h) \in V_m^-(0) := \mathcal{K}_{-1}^{m-1}(\Omega) \oplus \mathcal{K}_{-1/2}^{m-1/2}(\partial_N \Omega)$ the linear functional $F := \Phi(f,h)$ on V, $F \in V^*$ defined by the formula

$$F(v) \,=\, \Phi(f,h)(v) \,:=\, \int_\Omega f\, v\, dx + \int_{\partial_N \Omega} h\, v\, dS\,, \tag{35}$$

where dx is the volume element on Ω and dS is the surface element on $\partial\Omega$. With this definition of the inclusion $\Phi : V_m^-(0) \to V_0^-(0) := V^*$, we obtain that $P^\beta(m,0)$ is the restriction of $P^\beta(0,0)$ to $V_m(0)$. In other words, we have the commutative diagram

$$\begin{array}{ccc} V_m(0) & \xrightarrow{P^\beta(m,0)} & V_m^-(0) \\ \big\downarrow & & \big\downarrow \\ V_0(0) := V & \xrightarrow{P^\beta(0,0):=P^\beta} & V_0^-(0) \end{array} \tag{36}$$

with the operators P^β introduced in Equations (32) and (34).

See also Remark 2.1. We now return to the general case $a \in \mathbb{R}$.

Remark 4.3. We then notice that we have

$$V_m(a) \,=\, r_\Omega^a V_m(0) \quad \text{for} \quad m \geq 0 \quad \text{and} \quad V_m^-(a) \,=\, r_\Omega^a V_m^-(0) \quad \text{for } m > 0\,.$$

We then let

$$V_0^-(a) \,:=\, r_\Omega^a V_0^-(0) \,=\, r_\Omega^a V^*\,.$$

By symmetry, we obtain

$$V_{m+1}(a) \subset V_m(a) \quad \text{and} \quad V_{m+1}^-(a) \subset V_m^-(a) \quad \text{for all } m \geq 0\,, \tag{37}$$

in general (for all a). In fact, the relation between the spaces above for different values of a allows us to reduce to the case $a = 0$ since, if $\beta \in Z_m$, then there exists $\beta(a) \in Z_m$ such that

$$P^\beta(m,a) \,=\, r_\Omega^a P^{\beta(a)}(m,0) r_\Omega^{-a}\,, \quad m \geq 1\,. \tag{38}$$

This can be seen from $r^a(\partial_j)r^{-a}u = \partial_j u - a x_j r^{-1} u$ and $x_j/r \in \mathcal{W}^{m,\infty}(\Omega)$ for all m. In particular, $\beta(a) = \beta + a\gamma_1 + a^2\gamma_2$, with $\gamma_1, \gamma_2 \in Z_m$, whenever $\beta \in Z_m$. (This explains why it is crucial to consider coefficients in weighted spaces of the form $\mathcal{W}^{m,\infty}(\Omega)$ as well as in terms of the form $\partial_i(b_i u)$ in the definition of p_β.) We use Equation (38) to define $P^\beta(0,a)$ for all a. Of course, $P^\beta(0,0) = P^\beta : V \to V^*$.

Our *higher regularity problem* is then to establish conditions for $P^\beta(m,a)$ to be an isomorphism, which is achieved in Theorem 4.4.

4.2. **Extension of Theorem 1.1 and its proof.** For its proof, it will be convenient to extend the differential operators X_k, Y_k from U_k to the whole domain Ω. We choose these extensions so that

(i) If p_k is a vertex, then all X_j, Y_j, $j \neq k$, vanish in a neighborhood of p_k.
(ii) For all k, X_k (regarded as a vector field) is tangent to all edges (if X_k vanishes at a point on an edge, it is considered to be tangent to the edge at that point).

Recall that $\rho(P) := \inf_{v\neq 0} \Re(Pv,v)/\|v\|_V$, for any linear map $P : V \to V^*$, that $\rho(\beta) := \rho(P^\beta)$, and that $C_{use}^{-1} \leq \rho(\beta)^{-1}$.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, curvilinear polygonal domain and $\beta = (a_{ij}, b_i, c) \in Z_m$. If $P^{\beta(a)} : V \to V^*$ is coercive, then $P^\beta(m, a) : V_m(a) \to V_m^-(a)$ is invertible for all $m \geq 0$. Moreover, $P^\beta(m, a)^{-1}$ depends analytically on the coefficients β and there exists $C_m = C_m(\Omega, \partial_D \Omega)$ and $N_m \geq 0$ such that*

$$\|P^\beta(m, a)^{-1}\|_{\mathcal{L}(V_m^-; V_m)} \ \leq \ C_m \left(\rho(\beta(a))\right)^{-N_m - 1} \|\beta(a)\|_{Z_m}^{N_m} .$$

Proof. In view of Remark 4.3 and of the relation in Equation (38), we can reduce the proof of this theorem to the case $a = 0$. Because of this, *we shall assume for the rest of this section that $a = 0$ and we shall write $V_m(0) = V_m$ and $V_m^-(0) = V_m^-$.* We also denote $\|(P^\beta)^{-1}\|_m := \|(P^\beta)^{-1}\|_{\mathcal{L}(V_m^-; V_m)}$.

For $m = 0$, we can just take $C_0 = 1$ and $N_0 = 0$ and then the result reduces to the Lax-Milgram Lemma 2.6. In general, we adapt to our setting the classical method based on finite differences (see for example [23, 32, 16]), which was used in similar settings in [9, 10, 31, 37]. We thus give a summary of the argument. For simplicity, we drop Ω from the notations of the norms. In this proof, as throughout the paper, C is a parameter that is independent of β or F, and hence it depends only on Ω, $\partial_N \Omega$, m, and the choice of the vector fields X_k and Y_k (and of their initial domains U_k). However, we write $A \leq_c B$ instead of $A \leq CB$, if C is such a bound.

Let us notice that since $\|P^\beta(m, 0)\| \|P^\beta(m, 0)^{-1}\| \geq \|P^\beta(m, 0) P^\beta(m, 0)^{-1}\| \geq 1$ and since $\|P^\beta(m, 0)\|_m \leq_c \|\beta\|_{Z_m}$ we have that

$$\|\beta\|_{Z_m} \|(P^\beta)^{-1}\|_m \geq 1/C > 0 .$$

When $m = 0$, we also have $\rho(\beta)^{-1} \geq \|(P^\beta)^{-1}\| =: \|(P^\beta)^{-1}\|_0$, and hence

$$R(\beta) \ := \ \|\beta\|_{Z_m} \rho(\beta)^{-1} \ \geq \ \|\beta\|_{W^{0,\infty}} \|(P^\beta)^{-1}\|_0 \geq 1/C > 0 . \tag{39}$$

To show that the operator $P^\beta(m, 0) : V_m(0) \to V_m^-(0)$ is invertible and to obtain estimates on $\|(P^\beta)^{-1}\|_m := \|P^\beta(m, 0)\|$, we proceed by induction on m. As we have explained above, for $m = 0$, this has already been proved. We thus assume that $P^\beta(m-1, 0)$ is invertible and that it satisfies the required estimate, which we write as

$$\|(P^\beta)^{-1}\|_{m-1} \ := \ \|P^\beta(m-1, 0)^{-1}\|_{\mathcal{L}(V_m^-; V_m)} \ \leq \ C_{m-1} \frac{R(\beta)^{N_{m-1}}}{\rho(\beta)} .$$

Let $F \in V_m^-$ be arbitrary but fixed. We know by the induction hypothesis that $u := (P^\beta)^{-1} F = P^\beta(m-1, 0)^{-1} F \in V_{m-1}$, but we need to show that it is in fact in V_m and to estimate its norm in terms of $\|F\|_{V_m^-}$. Since $V_m := \mathcal{K}_1^{m+1} \cap V$, it is enough to show that $u \in \mathcal{K}_1^{m+1}$ and to estimate $\|u\|_{\mathcal{K}_1^{m+1}} = \|(P^\beta)^{-1} F\|_{\mathcal{K}_1^{m+1}}$.

First of all, by Corollary 3.10, it is enough to estimate $\|X_k u\|_{\mathcal{K}_1^m}$ and $\|Y_k u\|_{\mathcal{K}_1^m}$. Indeed,

$$\|u\|_{\mathcal{K}_1^{m+1}} \ \leq_c \ \|u\|_{\mathcal{K}_1^m} + \sum_{k=1}^N \|X_k u\|_{\mathcal{K}_1^m(U_k)} + \sum_{k=1}^N \|Y_k u\|_{\mathcal{K}_1^m(U_k)} , \tag{40}$$

and the first term on the right hand side is estimated by induction on m by

$$\|u\|_{\mathcal{K}_1^m} \ \leq \ \frac{C_{m-1} R(\beta)^{N_{m-1}}}{\rho(\beta)} \|F\|_{V_{m-1}^-} \ \leq \ \frac{C_{m-1} R(\beta)^{N_{m-1}}}{\rho(\beta)} \|F\|_{V_m^-} . \tag{41}$$

(Note that the other terms in Equation (40) are computed on smaller subsets U_k.)

Let us estimate now the other terms in the sum appearing on the right hand side of the inequation (40). First, since X_k is tangent to all edges of Ω, it integrates to a one parameter family of diffeomorphisms of Ω, and hence to strongly continuous one-parameter groups of continuous operators on $X := V_{m-1}$ and $Y := V_{m-1}^-$, due to the particular form of boundary conditions used to define these spaces. Let us denote by $S_X(t) : X \to X$ and $S_Y(t) : Y \to Y$, $t \in \mathbb{R}$, the operators defining these one-parameter groups of operators. We have that

$$B := X_k P^\beta - P^\beta X_k = \lim_{t\to 0} t^{-1}(S_X(t)P^\beta S_Y(-t) - P^\beta) = P^{\beta'},$$

and hence $\beta' \in Z_{m-1}$ is obtained by taking derivatives of β. Therefore $B : X \to Y$ is bounded by Lemma 4.1. The assumptions of Lemma 3.13 are therefore satisfied. Moreover, $\|B\| \le_c \|\beta'\|_{Z_{m-1}} \le_c \|\beta\|_{Z_m}$, which allows us to conclude that

$$\|X_k u\|_{\mathcal{K}_1^m} \le_c \|(P^\beta)^{-1}\|_{m-1}\big(\|X_k F\|_{V_{m-1}^-} + \|\beta\|_{Z_m}\|(P^\beta)^{-1}\|_{m-1}\|F\|_{V_{m-1}^-}\big). \tag{42}$$

Using also the relation $\|\beta\|_{Z_m}\|P^\beta(m-1,0)^{-1}\| \ge 1/C$ of Equation (39), we obtain

$$\begin{aligned} \|X_k u\|_{\mathcal{K}_1^m} &\le_c \|(P^\beta)^{-1}\|_{m-1}\big(1 + \|(P^\beta)^{-1}\|_{m-1}\|\beta\|_{Z_m}\big)\|F\|_{V_m^-} \\ &\le_c \frac{R(\beta)^{2N_{m-1}+1}}{\rho(\beta)}\|F\|_{V_m^-}. \end{aligned} \tag{43}$$

We now turn to the study of the terms $\|Y_k u\|_{\mathcal{K}_1^m}$, for which we need to use the strong ellipticity of P^β (as in the classical methods [32, 23]) together with Lemmas 3.11 and 3.12. First of all, Lemma 3.11 provides us with the decomposition $c_k Y_k^2 u = r_\Omega^2 P^\beta u - Q_k u$, where $c_k \in \mathcal{W}^{m,\infty}(U_k)$ and Q_k is a sum of differential operators of the form $Y_k X_k$ and X_k^2 and lower order differential operators generated by X_k and Y_k with coefficients in $\mathcal{W}^{m,\infty}(U_k)$. This gives using first the general form of the $\|\cdot\|_{\mathcal{K}_1^m(U_k)}$-norm

$$\begin{aligned} \|Y_k u\|_{\mathcal{K}_1^m(U_k)} &\le_c \|Y_k u\|_{\mathcal{K}_1^{m-1}} + \|X_k Y_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k^2 u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\le_c \|u\|_{\mathcal{K}_1^m} + \|Y_k X_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k^2 u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\le_c \|u\|_{\mathcal{K}_1^m} + \|X_k u\|_{\mathcal{K}_1^m} + \|c_k^{-1}(r_\Omega^2 p_\beta - Q_k)u\|_{\mathcal{K}_1^{m-1}(U_k)}. \end{aligned} \tag{44}$$

The first term in the last line of Equation (44) is estimated by the induction hypothesis in Equation (41). The second one is estimated in Equation (43). To estimate the third term, we obtain directly from Lemma 3.11 the following

1. each $c_k \in \mathcal{W}^{m,\infty}(U_k)$ is bounded in terms of $\|\beta\|_{Z_m}$,
2. the coefficients of X_k^2, $X_k Y_k$, X_k, and Y_k and the free term of Q_k (which is no longer in divergence form) are in $\mathcal{W}^{m-1,\infty}(U_k)$ and are also bounded in terms of $\|\beta\|_{Z_m}$,
3. $\|c_k^{-1}\|_{L^\infty} \le_c C_{use}^{-1} \le_c \rho(\beta)^{-1}$.

Hence

$$\begin{aligned} \|c_k^{-1}\|_{\mathcal{W}^{m-1,\infty}(U_k)} &\le_c \|c_k^{-1}\|^m_{L^\infty(U_k)}\|c_k\|^{m-1}_{\mathcal{W}^{m-1,\infty}(U_k)} \\ &\le_c \rho(\beta)^{-m}\|\beta\|^{m-1}_{\mathcal{W}^{m,\infty}} = \rho(\beta)^{-1}R(\beta)^{m-1}, \end{aligned} \tag{45}$$

where the first inequality is by Lemma 3.12(iii).

We have, successively

$$\|r_\Omega^2 p_\beta u\|_{\mathcal{K}_1^{m-1}(U_k)} \le_c \|p_\beta u\|_{\mathcal{K}_{-1}^{m-1}(U_k)} \le_c \|p_\beta u\|_{\mathcal{K}_{-1}^{m-1}} \le_c \|F\|_{V_{m-1}^-}. \tag{46}$$

Similarly, let ν be the $\mathcal{W}^{m-1,\infty}(U_k)$ norm of the coefficients of Q_k, then $\nu \leq_c \|\beta\|_{\mathcal{W}^{m,\infty}}$ and hence

$$\begin{aligned}(47)\quad \|Q_k u\|_{\mathcal{K}_1^{m-1}(U_k)} &\leq_c \nu\Big(\|X_k^2 u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k X_k u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\quad + \|X_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|Y_k u\|_{\mathcal{K}_1^{m-1}(U_k)} + \|u\|_{\mathcal{K}_1^{m-1}(U_k)}\Big) \\ &\leq_c \|\beta\|_{Z_m}\Big(\|X_k u\|_{\mathcal{K}_1^m} + \|u\|_{\mathcal{K}_1^m}\Big) \\ &\leq_c (R(\beta)^{N_{m-1}+1} + R(\beta)^{2N_{m-1}+2})\|F\|_{V_{m-1}^-} \leq_c R(\beta)^{2N_{m-1}+2}\|F\|_{V_{m-1}^-},\end{aligned}$$

where we have used also Equations (41) and (43). Consequently,

$$\begin{aligned}(48)\quad \|c_k^{-1}(r_\Omega^2 p_\beta - Q_k)u\|_{\mathcal{K}_1^{m-1}(U_k)} &\leq_c \|c_k^{-1}\|_{\mathcal{W}^{m-1,\infty}}\|r_\Omega^2 p_\beta u - Q_k u\|_{\mathcal{K}_1^{m-1}(U_k)} \\ &\leq_c \frac{R(\beta)^{m-1}}{\rho(\beta)}\big(1 + R(\beta)^{2N_{m-1}+2}\big)\|F\|_{V_{m-1}^-} \leq_c \frac{R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_{m-1}^-}.\end{aligned}$$

Substituting back into Equation (44) the estimates of Equations (41), (43), and (48), we obtain

$$\begin{aligned}(49)\quad \|Y_k u\|_{\mathcal{K}_1^m(U_k)} &\leq_c \frac{R(\beta)^{N_{m-1}} + R(\beta)^{2N_{m-1}+1} + R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_m^-} \\ &\leq_c \frac{R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_m^-}.\end{aligned}$$

In a completely analogous manner, substituting back into Equation (40) the estimates of Equations (41), (43), and (49), we obtain

$$\begin{aligned}(50)\quad \|u\|_{\mathcal{K}_1^{m+1}} &\leq_c \|u\|_{\mathcal{K}_1^m} + \sum_{k=1}^N \|X_k u\|_{\mathcal{K}_1^m(U_k)} + \sum_{k=1}^N \|Y_k u\|_{\mathcal{K}_1^m(U_k)} \\ &\leq_c \frac{R(\beta)^{N_{m-1}} + R(\beta)^{2N_{m-1}+1} + R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_m^-} \\ &\leq_c \frac{R(\beta)^{2N_{m-1}+m+1}}{\rho(\beta)}\|F\|_{V_m^-}.\end{aligned}$$

In all the statements above, saying $\|v\|_Z < \infty$ for some Banach space Z means, implicitly, that $v \in Z$. We thus have that $u \in \mathcal{K}_1^{m+1}$ and that it satisfies the required estimate with $N_m = 2N_{m-1} + m + 1$. The proof is complete. □

Remark 4.5. Remark 4.3 gives that there exist parameters γ_1 and γ_2, independent of β, such that $\rho(\beta(a)) \geq \rho(\beta) - \gamma_1|a| - \gamma_2 a^2$. Moreover, an induction argument gives that $N_m = 2^{m+2} - m - 3 \geq 0$ in two dimensions. We ignore if this is true in higher dimensions as well.

As mentioned in the introduction, an example to Theorem 4.4 is the Schroedinger operator $H := -\Delta + c r_\Omega^{-2}$ on Ω with *pure Neumann* boundary conditions. See also Theorem 5.4.

5. Extensions and applications

5.1. **Some direct consequences of Theorem 4.4.** We conclude with a few corollaries. For simplicity, we formulate them only in the case $a = 0$, since Remark 4.3 allows us to reduce to the case $a = 0$. Throughout the rest of this section, we continue to assume that $\beta = (a_{ij}, b_i, c) \in Z_m$ and that Ω is a bounded, curvilinear polygonal domain with $\partial_D\Omega$ nonempty.

Recall that $\mathcal{L}(V; V^*)_c \subset Z$ denotes the set of coefficients that yield a coercive operator.

Corollary 5.1. *Let $U := \mathcal{L}(V; V^*)_c \cap Z_m$. Then U is an open subset of Z_m and the map $U \times V_m^- \ni (\beta, F) \to (P^\beta)^{-1}F \in V_m$ is analytic and*

$$\|(P^\beta)^{-1}F\|_{V_m} \ \leq \ C_m \frac{\|\beta\|_{Z_m}^{N_m}}{\rho(\beta)^{N_m+1}} \|F\|_{V_m^-} .$$

Proof. The inclusion $\mathcal{W}^{m,\infty}(\Omega) \to Z_m$ is continuous and $\mathcal{L}(V; V^*)_c \cap Z_m$ is open in Z_m. Hence U is open in Z_m. Next we proceed as in Lemma 2.6 using that the map $\Phi : U \times V_m^- \to V_m$, $\Phi(\beta, F) := (P^\beta)^{-1}F$ is the composition of the maps

$$U \times V_m^- \ni (\beta, F) \to (P^\beta, F) \in \mathcal{L}_{inv}(V_m; V_m^-) \times V_m^- ,$$
$$\mathcal{L}_{inv}(V_m; V_m^-) \times V_m^- \ni (P, F) \to (P^{-1}, F) \in \mathcal{L}(V_m^-; V_m) \times V_m^- , \quad \text{and}$$
$$\mathcal{L}(V_m^-; V_m) \times V_m^- \ni (P^{-1}, F) \to P^{-1}F \in V_m .$$

The first of these three maps is well defined and linear by Theorem 4.4. The other two maps are analytic by Lemma 2.3. Since the composition of analytic functions is analytic, the result follows. □

The following result is useful in approximating solutions of parametric problems.

Corollary 5.2. *Let Y be a Banach space and let $U \subset Y$ be an open subset. Let $F : U \to V_m^-$ and $\beta : U \to \mathcal{L}(V; V^*)_c \cap \mathcal{W}^{m,\infty}(\Omega)$ be analytic functions. Then $U \ni y \to (P^{\beta(y)})^{-1}F(y) \in V_m$ is analytic and we have*

$$\|(P^{\beta(y)})^{-1}F(y)\|_{V_m} \ \leq \ C_m \frac{\|\beta(y)\|_{Z_m}^{N_m}}{\rho(\beta(y))^{N_m+1}} \|F(y)\|_{V_m^-} .$$

In particular, if the functions $\|\beta(y)\| = \|\beta(y)\|_{\mathcal{W}^{m,\infty}(\Omega)}$ and $\|F(y)\|_{V_m^-}$ are bounded and *there exists $c > 0$ such that $\rho(\beta(y)) > c$, then $(P^{\beta(y)})^{-1}F(y)$ is a bounded analytic function.*

Proof. The composition of two analytic functions is analytic. The first part is therefore an immediate consequence of the first part of Corollary 5.1. The second part follows also from Corollary 5.1. □

The method used to obtain analytic dependence of the solution in terms of coefficients can be extended to other settings.

Remark 5.3. Let us assume the following:

(i) We are given continuously embedded Banach spaces $W_D^{m+1} \subset V \subset H^1(\Omega)$, $\breve{W}^{m-1} \subset V^*$, and $\mathcal{Z} \subset Z_m$ satisfying the following properties:
(ii) For any $\beta \in \mathcal{Z}$, the operator P^β defines continuous maps $V \to V^*$ and $W_D^{m+1} \to \breve{W}^{m-1}$.
(iii) $\|P^\beta\|_{\mathcal{L}(W_D^{m+1}; \breve{W}^{m-1})} \leq_c \|\beta\|_{\mathcal{Z}}$ and $\|P^\beta\|_{\mathcal{L}(V;V^*)} \leq_c \|\beta\|_{\mathcal{Z}}$.

(iv) If $\beta \in \mathcal{Z}$ and $P^\beta : V \to V^*$ is coercive, then the map $(P^\beta)^{-1} : V^* \to V$ maps $\breve{W}^{m-1}$ to W_D^{m+1} continuously and there exists a continuous, increasing function $\mathfrak{N}_m : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that

$$\|(P^\beta)^{-1}\|_{\mathcal{L}(\breve{W}^{m-1};W_D^{m+1})} \leq \mathfrak{N}_m\big(\rho(\beta)^{-1}, \|\beta\|_Z\big).$$

Then our previous results (in particular, Corollaries 5.1 and 5.2) extend to the new setting by replacing $\mathcal{W}^{m,\infty}(\Omega)$ with Z, V_m with W_D^{m+1}, V_m^- with $\breve{W}^{m-1}$, and by using $\mathfrak{N}_m$ in the bounds for the norm. We thank Markus Hansen and Christoph Schwab for their input related to this remark.

5.2. **General domains with conical points.** The same argument as in the proof of Theorem 4.4 gives a proof of a similar result on general domains with conical points. In the neighborhood of a conical point, the domain is of the form $\Omega = \{rx' |\ 0 < r < 1, x' \in \omega$, where $\omega \subset S^{n-1}$ is a smooth domain on the unit sphere S^{n-1}. The main difference is that we will need to additionally straighten the boundary of ω.

5.3. **Dirichlet and Neumann boundary conditions.** We conclude this paper by an application of Theorem 4.4 to estimates for Schroedinger operators. We note that the following result applies to arbitrary mixed boundary conditions (including pure Neumann).

Theorem 5.4. *Let $P^\beta u = -\sum_{ij=1}^d \partial_i a_{ij} \partial_j u + \frac{c}{r_\Omega^2} u$, $c \geq 0$, be a strongly elliptic operator (so $b_i = 0$). In case $p \in \mathcal{V} \subset \partial\Omega$ is a vertex that belongs to two adjacent Neumann edges, we assume that $c(p) > 0$. Then P^β is coercive. Moreover,*

$$P^\beta : V_m := \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial_D\Omega} = 0\} \to V_m^- := \mathcal{K}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\Omega) \tag{51}$$

is an isomorphism and its inverse has norm

$$\|(P^\beta)^{-1}\| \leq C\,\rho(\beta)^{-N_m-1}\Big(\sum_{ij} \|a_{ij}\|_{\mathcal{W}^{m,\infty}(\Omega)} + \|c\|_{\mathcal{W}^{m,\infty}(\Omega)}\Big)^{N_m}, \quad |a| \leq 1,$$

with N_m as in Theorem 4.4 and C independent of β.

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