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Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps

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Summary. Let u and $u_V \in V$ be the solution and, respectively, the discrete solution of the non-homogeneous Dirichlet problem $\Delta u = f$ on $\mathbb{P}$, $u|_{\partial\mathbb{P}} = 0$. For any $m \in \mathbb{N}$ and any bounded polygonal domain $\mathbb{P}$, we provide a construction of a new sequence of finite dimensional subspaces V_n such that $\|u - u_{V_n}\|_{H^1} \leq C \dim(V_n)^{-m/2}\|f\|_{H^{m-1}}$, where $f \in H^{m-1}(\mathbb{P})$ is arbitrary and C is a constant that depends only on $\mathbb{P}$ and not on n (we do *not* assume $u \in H^{m+1}(\mathbb{P})$). The existence of such a sequence of subspaces was first proved in a ground–breaking paper by Babuška [8]. Our method is different; it is based on the homogeneity properties of Sobolev spaces with weights and the well–posedness of non-homogeneous Dirichlet problem in suitable Sobolev spaces with weights, for which we provide a new proof, and which is a substitute of the usual "shift theorems" for boundary value problems in domains with smooth boundary. Our results extended right away to domains whose boundaries have conical points. We also indicate some of the changes necessary to deal with domains with cusps. Our numerical computation are in agreement with our theoretical results.

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Introduction

Let $\mathbb{P}$ be a bounded polygonal domain in the plane and $m \in \mathbb{N} = \{1, 2, \dots\}$. Denote by $H^m(\mathbb{P})$ the mth order Sobolev space on $\mathbb{P}$, with norm $\|u\|_{H^m}$. We shall consider the *non-homogeneous Dirichlet problem*

$$\Delta u = f \qquad u|_{\partial\mathbb{P}} = 0, \tag{1}$$

and the convergence properties of the finite element discretizations associated with it. The regularity of the solution of this problem and applications towards efficient algorithms for solving it have received a lot of attention [8, 18, 21, 24, 36, 38, 41, 47].

Let u and $u_V \in V$ be the (weak) solution and the finite element (discrete) solution of the equation (1), respectively. In [8, 38], it was shown that it is possible to obtain an optimal order of convergence for a suitable choice of a sequence of spaces V.

One of the main results of our paper is to provide, for any fixed $m \in \mathbb{N}$ and any polygonal domain $\mathbb{P}$, a construction of a simple, explicit new sequence of finite dimensional subspaces V_n such that

$$\|u - u_{V_n}\|_{H^1} \le C \dim(V_n)^{-m/2} \|f\|_{H^{m-1}}, \tag{2}$$

where $f \in H^{m-1}(\mathbb{P})$ is arbitrary and C is a constant that depends only on $\mathbb{P}$ and not on n (we do *not* assume $u \in H^{m+1}(\mathbb{P})$). That is, we recover the asymptotic order of convergence that is expected when the solution is smooth. Similar results were obtained before in [8, 9, 38]. See also [4, 6, 7, 14, 18, 28, 30] for surveys and related results. Our construction of the sequence of spaces V_n is new and based on different principles than the ones used in the aforementioned works.

Our proof that the asymptotic order of convergence is as stated is based on a generalisation of the Bramble–Hilbert Lemma to Sobolev spaces with weights on $\mathbb{P}$ and on the dilation invariance of these spaces. We provide a new proof of the fact that the Dirichlet's problem with zero boundary conditions (*i.e.,* the non-homogeneous Dirichlet problem) is well posed in suitable Sobolev spaces with weights [26]. We hope that a similar construction in 3D will lead to meshes that are easy to construct explicitly and provide the optimal rates of convergence [31]. Moreover, we briefly indicate some of the changes that allow us to extend our method to domains with cusps. This will be treated in detail in a forthcoming publication.

Let $H_0^1(\mathbb{P})$ be the subspace of distributions in $H^1(\mathbb{P})$ with vanishing trace on $\mathbb{P}$. The inner product of $f_1, f_2 \in H^0(\mathbb{P}) = L^2(\mathbb{P})$ will be denoted by $\langle f_1, f_2 \rangle$. It will be convenient to write the non-homogeneous Dirichlet problem in the form

$$a(u, v) := \langle \nabla u, \nabla v \rangle = \langle f, v \rangle, \quad \forall v \in H_0^1(\mathbb{P}). \tag{3}$$

Let $V \subset H^1_0(\mathbb{P})$ be a subspace, we shall denote by u_V the solution to the problem

$$a(u_V, v) = \langle f, v\rangle, \quad \forall v \in V. \tag{4}$$

It is a basic problem to construct finite dimensional subspaces $V \subset H^1_0(\mathbb{P})$ such that the error $\|u - u_V\|_{H^1}$ is small. We also want to achieve this in an "economic" way, that is, with $\dim(V)$ not too large.

To better explain our results, let us recall the following basic result in approximation theory [4,15,19,20]. Let $\mathcal{T} = (T_j)$ be a triangulation of $\mathbb{P}$ with triangles. Let $V = V(\mathcal{T}, m+1)$ be the finite element space associated to the degree m Lagrange triangle [15]. More precisely, $V = V(\mathcal{T}, m+1)$ consists of of those continuous functions on $\overline{\mathbb{P}}$ that restrict to polynomials of order $\leq m$ on each triangle $T_j \in \mathcal{T}$. For any function $u \in C(\mathbb{P})$, we shall denote by $u_I \in V = V(\mathcal{T}, m+1)$ the *interpolating* function associated to u, the nodes being obtained by taking points with baricentric coordinates in $m^{-1}\mathbb{Z}$. Thus, for any continuous function u on $\overline{\mathbb{P}}$, u_I is uniquely determined by the conditions that $u_I(x) = u(x)$ for any node x and $u_I \in V(\mathcal{T}, m+1)$. Then, if we consider on $H^1(\mathbb{P})$ the norm $|\cdot|_1$ defined by the form $a(\cdot,\cdot)$ (the energy norm), we have the standard inequality

$$|u - u_V|_1 \leq |u - u_I|_1. \tag{5}$$

The equivalence of the $\|\cdot\|_{H^1}$ and $|\cdot|_1$ yields the following well known result [4,15,19,20]:

Theorem 0.1 *Let $V = V(\mathcal{T}, m+1)$. Assume that all triangles T_j of the triangulation $\mathcal{T} = (T_j)$ of $\mathbb{P}$ have angles $\geq \alpha$ and edges of length $\leq h$ and $\geq ah$. Then there exists an constant $C_1 = C_1(\alpha, m)$ such that*

$$c^{-1}\|u - u_V\|_{H^1} \leq \|u - u_I\|_{H^1} \leq C_1 h^m \|u\|_{H^{m+1}} \tag{6}$$

for any $u \in H^{m+1}(\mathbb{P})$. Similarly, there exists an absolute constant $C_2 = C_2(\alpha, a, m)$ such that

$$c^{-1}\|u - u_V\|_{H^1} \leq \|u - u_I\|_{H^1} \leq C_2 \dim(V)^{-m/2} \|u\|_{H^{m+1}} \tag{7}$$

The constants c, C_1, and C_2 in the above theorem do not depend on the triangulation $\mathcal{T}$ or on the function u.

One can argue, based on the theory of n-widths and Weyl's lemma on the asymptotic of eigenvalues of the Laplace operator with Dirichlet boundary conditions [44], that the estimates obtained by combining Theorem 0.1 and Equation (5) are optimal as far as the asymptotic order of convergence m, if $u \in H^{m+1}(\mathbb{P})$. However, it is *not* true, in general, that $u \in H^{m+1}(\mathbb{P})$, even if $f = \Delta u \in \mathcal{C}^\infty(\mathbb{P})$, because the boundary of $\mathbb{P}$ is not smooth [21,25,43]. On the other hand, Weyl's theorem mentioned above does not prevent similar asymptotic rates of convergence for polygons. However, it is known [9] that the only hope to achieve similar rates of convergence is to choose carefully the triangulation $\mathcal{T}$. It is the purpose of this paper to provide conditions on the triangulation $\mathcal{T}$ under which such higher asymptotic rates of convergence are obtained.

More precisely, we shall construct for any $\mathbb{P}$ a class $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ of partitions $\mathcal{T}$ of $\mathbb{P}$, depending on $m \in \mathbb{N}$ and some parameters $h, \kappa, \alpha, \epsilon, b$, such that the following theorem holds.

Theorem 0.2 *Let $\mathbb{P}$ be a bounded polygon in the plane with vertices at distance $\geq l$. Then there exists a constant $B = B(\kappa, \alpha, l, \epsilon, b)$ such that for any partition $\mathcal{T}$ in $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$, the solutions u and u_V of the Equations* (3) *and* (4) *satisfy*

$$\|u - u_V\|_{H^1} \leq B \dim(V)^{-m/2} \|f\|_{H^{m-1}}, \quad \forall f \in H^{m-1}(\mathbb{P}),$$

where $V = V(\mathcal{T}, m+1)$.

The precise meaning of the constants $h, \kappa, \alpha, \epsilon$, and b is explained in Section 4. Suffices to say now that m is the degree of the polynomials used in the approximation, h is the largest admissible length of the sides of the triangles in the partition, κ controls the decay of the triangles as they approach a vertex, α is the minimum admissible angle of a triangle in the partition, $0 < \epsilon < \pi/\alpha_1$, where α_1 is the largest angle of the polygon, and b controls the ratio of the sizes of close triangles. The constants $h, \kappa, \alpha, \epsilon$, and b must satisfy certain conditions for the class $\mathcal{C}(m, h, \kappa, \alpha, \epsilon, b)$ to be non-empty. The following result is therefore relevant.

Theorem 0.3 *For any polygon $\mathbb{P}$ there exist $0 < \epsilon \leq 1$, $\alpha > 0$, $1 > b > 0$ and a sequence $h_n \to 0$ such that, if $\kappa = 2^{-m/\epsilon}$, the class $\mathcal{C}(m, h_n, \kappa, \alpha, \epsilon, b)$ is not empty.*

The constants α and b can, in principle, be written explicitly in terms of ϵ and the geometry of $\mathbb{P}$. We should point out at this time that our results could be supplemented by a study of best choices of the parameters $\kappa, \alpha, \epsilon, b$ appearing in the definition of the partitions in $\mathcal{C}(m, h_n, \kappa, \alpha, \epsilon, b)$. This is because it is important to choose α and κ large and $b < 1$ close to 1 in order to decrease the constant $B = B(\kappa, \alpha, l, \epsilon, b)$. Too small or too large values for ϵ will increase the error. For this reason, we keep close track of the precise dependence of B on its arguments. Also, it is not clear when the class $\mathcal{C}(m, \kappa, \alpha, \epsilon, b)$ is not empty.

The proofs use some estimates on the Dirichlet problem in Sobolev spaces with weights [17,25,35]. These estimates follow from the results in [35], see Section 2. The same estimates can be used for domains with cusps. An example of such a domain is obtained by changing the metric on the polygon to one of the form $(dr)^2 + r^{2+2\gamma}(d\theta)^2$, $\gamma > 0$, in the neighborhood of each vertex (using polar coordinates at that vertex).

Throughout this paper, "$x := y$" will mean that "x" is defined to be equal to "y," as customary.

1 Sobolev spaces with weights

We now recall the definition of Sobolev spaces with weights [17,22,25] and establish some properties of these spaces needed for our results. Some similar estimates were obtained in [39]. See also [9,33] for additional results on Sobolev spaces with weights.

1.1 Notation

We shall use the standard notation and denote by $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}$, a constant coefficient differential monomial on $\mathbb{R}^2$, for any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. Also, $|\alpha| := \alpha_1 + \alpha_2$. By $L^2(\Omega)$, we shall denote the space of square integrable functions on a subset set $\Omega \subset \mathbb{R}^2$ with respect to the usual Lebesgue measure, with norm $\|u\|_{L^2}^2 := \int_\Omega |u(x)|^2 dx$. Also, by $L^2_{loc}(\Omega)$ we shall denote the space of functions on Ω whose restriction to any compact subset K of Ω is in $L^2(K)$.

First, let us recall [43] that the mth Sobolev space $H^m(\Omega)$, $\Omega \subset \mathbb{R}^2$ open, is defined by

$$H^m(\Omega) := \{u \in L^2(\Omega),\ \partial^\alpha u \in L^2(\Omega),\ \forall |\alpha| \le m\},$$

and is endowed with the norm

$$\|u\|_{H^m}^2 := \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{L^2}^2.$$

(Compare with the definition of the norm on Sobolev spaces with weights below.) We agree that $\|u\|_{H^m} = \infty$ if $u \notin H^m(\Omega)$. Also, by $H^m_{sl}(\Omega)$ we shall denote the space of functions on Ω that restricted to V belong to $H^m(V)$ for any open subset $V \subset \Omega$ such that $\overline{V}$ is compact and it does not contain the vertices of Ω.

1.2 Weighted Sobolev spaces

A crucial role in our proof is played by a certain modification of the usual definition above of Sobolev spaces. For the simplicity of the presentation, *we shall assume from now on that $\mathbb{P}$ is a triangle all of whose angles are acute*. The general case can be treated in a similar way by first dividing the obtuse angles in acute angles, but the notation is more complicated. To make a choice, we shall decide that $\mathbb{P}$ is an open set.

We now introduce some notation that will remain fixed throughout the paper. Let l be the length of the shortest edge of $\mathbb{P}$. We denote by Q_1, Q_2, and Q_3 the vertices of $\mathbb{P}$. Let $\mathbb{P}_\delta$ be the union of the three isosceles triangles that have equal sides of length $= \delta$ with the angle between them common with $\mathbb{P}$. The complement $\mathbb{P} \smallsetminus \mathbb{P}_\delta$ is a hexagon precisely when $\delta < l/2$. (In fact, we shall only need this construction when $\delta \leq l/4$.)

Fix in what follows a smooth function $\rho : \overline{\mathbb{P}} \to [0, \infty)$ such that $\rho(x) =$ the distance from x to the closest vertex of $\mathbb{P}$, for $x \in \mathbb{P}_{l/4}$, and $l/4 \leq \rho(x) \leq l$, for $x \in \mathbb{P} \smallsetminus \mathbb{P}_{l/4}$. We are ready now to recall the following definition (see for example [17,25]).

Definition 1.1 *Let $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$. The mth Sobolev space with weight ρ^a on $\Omega \subset \mathbb{P}$, Ω open, is the space $\mathcal{K}^m_a(\Omega)$ defined by*

$$\mathcal{K}^m_a(\Omega) := \{u \in L^2_{loc}(\Omega),\ \rho^{|\alpha|-a-1}\partial^\alpha u \in L^2(\Omega),\ \ \forall|\alpha| \leq m\}, \quad \alpha \in \mathbb{Z}^2_+.$$

The norm on $\mathcal{K}^m_a$ is

$$\|u\|^2_{\mathcal{K}^m_a} := \sum_{|\alpha|\leq m} \|\rho^{|\alpha|-a-1}\partial^\alpha u\|^2_{L^2}.$$

Standard arguments show that $\mathcal{K}^m_a(\mathbb{P})$ is complete (and hence a Hilbert space).

Our notation is slightly different from the one in above mentioned papers, where the value of the weight parameter a is shifted. More precisely, $\mathcal{K}^m_a = V^m_{2,m-a-1}$, where

$$V^m_{2,b} := \{u \in L^2_{loc}(\Omega),\ \rho^{|\alpha|+b-m}\partial^\alpha u \in L^2(\Omega),\ \ \forall|\alpha| \leq m\}$$

is the standard notation for these weighted Sobolev spaces. Our notation is more convenient for studying the homogeneity properties of these Sobolev spaces.

1.3 Some lemmas

We now record some properties of the spaces $\mathcal{K}^m_a(\mathbb{P})$ that will be needed in what follows. All the following properties follow from straightforward calculations and are, for the most part, well known.

Since most of the functions spaces used below are defined on $\mathbb{P}$, we shall often omit $\mathbb{P}$ from the notation. We shall thus write $\mathcal{K}^m_a := \mathcal{K}^m_a(\mathbb{P})$, $L^2 = L^2(\mathbb{P})$, and so on. We shall denote $\rho^a W = \{\rho^a f, f \in W\}$, for any space of functions W. Below, an isomorphism of Banach spaces means a continuous bijection.

Lemma 1.2 *The function $\rho^{|\beta|-a}\partial^\beta \rho^a$ is bounded on $\mathbb{P}$.*

This gives:

Lemma 1.3 *We have* $\mathcal{K}^0_{-1} = L^2$ *and* $\rho^a \mathcal{K}^m_b = \mathcal{K}^m_{a+b}$*. Moreover, multiplication by* ρ^a *gives rise to an isomorphism* $\mathcal{K}^m_b \to \mathcal{K}^m_{a+b}$*.*

Proof The first part is a direct consequence of the definition of $\mathcal{K}^m_a$. Let us prove the second part. Let $u = \rho^a f$ with $f \in \mathcal{K}^m_b$. Then

$$\rho^{|\alpha|-b-1}\partial^\alpha f \in L^2, \ \forall\, |\alpha| \le m.$$

Next, due to the previous lemma, we have that

$$\begin{aligned}\left|\rho^{|\alpha|-b-1}\partial^\alpha u\right| = \left|\rho^{|\alpha|-b-1}\partial^\alpha(\rho^a f)\right| &= \rho^{|\alpha|-b-1}\left|\sum_{0\le\alpha\le\beta}\partial^\beta\rho^a\partial^{\alpha-\beta} f\right| \\ &\le \sum_{0\le\alpha\le\beta}\rho^{|\alpha-\beta|-b-1}|\partial^{\alpha-\beta} f|,\end{aligned}$$

with $\rho^{|\alpha-\beta|-b-1}|\partial^{\alpha-\beta} f \in L^2$. Thus we have that $\rho^a\mathcal{K}^m_b$ is continuously embedded in $\mathcal{K}^m_{a+b}$. Since the embedding holds for any real number a, we can conclude the opposite embedding as follows:

$$\mathcal{K}^m_{a+b} = \rho^a\rho^{-a}\mathcal{K}^m_{a+b} \subset \rho^a\mathcal{K}^m_b.$$

□

From the definitions of the spaces H^m and $\mathcal{K}^m_a$ we have the following two lemmas:

Lemma 1.4 *Let* $m \ge m'$ *and* $a \ge a'$*. We have:*
(a) $\|u\|_{\mathcal{K}^{m'}_{a'}} \le l^{a-a'}\|u\|_{\mathcal{K}^m_a}$*.*
(b) $\|u\|_{\mathcal{K}^{m'}_{a'}} \le \delta^{a-a'}\|u\|_{\mathcal{K}^m_a}$*, if* $u \in \mathcal{K}^m_a(\mathbb{P}_\delta)$*,* $0 < \delta \le l/4$*.*
(c) $\mathcal{K}^m_a \subset \mathcal{K}^{m'}_{a'}$*.*

The proof of the above lemma is a direct verification.

Lemma 1.5 *We have* $\|u\|_{H^m} \le M\|u\|_{\mathcal{K}^m_{m-1}}$ *and* $\|u\|_{\mathcal{K}^m_{-1}} \le M\|u\|_{H^m}$*, where* $M = \max\{1, l^m\}$*.*

From this we obtain.

Corollary 1.6

$$\mathcal{K}^m_{m+a-1} \subset \rho^a H^m \subset \mathcal{K}^m_{a-1}.$$

Proof From Lemma 1.5 we have that

$$\mathcal{K}^m_{m-1} \subset H^m \subset \mathcal{K}^m_{-1}.$$

Then, we apply Lemma 1.3 □

The following lemma asserts that the H^m and $\mathcal{K}_a^m$–norms are equivalent on $H^m(\Omega)$, for any region Ω on which ρ is bounded from below. More precisely, we have.

Lemma 1.7 *Let* $0 < \delta < l/4$ *and let* $\Omega \subset \mathbb{P}$ *be an open subset such that* $\rho \geq \delta$ *on* Ω. *Then* $\|u\|_{H^m} \leq M_1 \|u\|_{\mathcal{K}_a^m}$ *and* $\|u\|_{\mathcal{K}_a^m} \leq M_2 \|u\|_{H^m}$, *for any* $u \in H^m(\Omega)$, *where* $M_1 := \max\{l^{a+1}, l^{-m+a+1}, \delta^{a+1}, \delta^{-m+a+1}\}$ *and, similarly,* $M_2 := \max\{l^{-a-1}, l^{-m+a+1}, \delta^{-a-1}, \delta^{m-a-1}\}$.

The following lemma compares the weighted Sobolev spaces to the usual Sobolev spaces close to the vertices.

Lemma 1.8 *Let* $0 < \delta < \min\{l/4, 1\}$ *and* $\Omega \subset \mathbb{P}_\delta$ *be an open subset. Then* $\|u\|_{H^m} \leq \delta^{a-m+1} \|u\|_{\mathcal{K}_a^m}$, *if* $a \geq m-1$, *and* $\|u\|_{\mathcal{K}_a^m} \leq \delta^{-a-1} \|u\|_{H^m}$, *if* $a \leq -1$.

The proofs of the above two lemmas are based only on the definitions of the norms involved.

One of the main reasons for using the weighted Sobolev spaces is the homogeneity of their norms. We first need to introduce dilations for certain functions defined on $\mathbb{P}_\delta$. Assume for a moment that $Q_1 = O := (0, 0)$, the origin of the coordinate system. Let $\lambda > 0$ and let $\Omega \subset \mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$ be completely contained in the triangle closest to Q_1. Also, let v be a function defined on Ω. Then we define $v_\lambda(x) := v(\lambda x)$ for any $x \in \lambda^{-1}\Omega$. (The conditions on Ω are formulated so that this definition makes sense.) In general, if $\Omega \subset \mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$, but is not necessarily contained in a single connected component of $\mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$, we define $v_\lambda(x)$ by translating each Q_j, $j = 1, 2, 3$ to the origin first.

Lemma 1.9 *Let* $\lambda > 0$ *and let* $\Omega \subset \mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$ *be an open subset. Then* $\|u_\lambda\|_{\mathcal{K}_a^m} = \lambda^a \|u\|_{\mathcal{K}_a^m}$ *for any* $u \in \mathcal{K}_a^m(\Omega)$.

Proof The proof is based on the change of variable $y = \lambda x$ and the fact that $\rho = r$ on our particular domain. Note that $r(x) = \lambda^{-1} r(y)$. Thus, we have

$$
\begin{aligned}
\|v_\lambda\|^2_{\mathcal{K}_a^m} &= \sum_{|\alpha| \leq m} \|\rho^{|\alpha|-a-1} \partial^\alpha v_\lambda\|^2_{L^2} = \sum_{|\alpha| \leq m} \int_{\frac{1}{\lambda}\Omega} r^{2|\alpha|-2a-2} |\partial^\alpha [v(\lambda x)]|^2 \, dx \\
&= \sum_{|\alpha| \leq m} \int_{\frac{1}{\lambda}\Omega} r(x)^{2|\alpha|-2a-2} \lambda^{2|\alpha|} |\partial^\alpha (v)(\lambda x)|^2 \, dx \\
&= \sum_{|\alpha| \leq m} \int_{\Omega} r(y/\lambda)^{2|\alpha|-2a-2} \lambda^{2|\alpha|-2} |\partial^\alpha (v)(y)|^2 \, dy \\
&= \lambda^{2a} \sum_{|\alpha| \leq m} \int_{\Omega} r(y)^{2|\alpha|-2a-2} |\partial^\alpha (v)(y)|^2 \, dy
\end{aligned}
$$

$$= \lambda^{2a} \sum_{|\alpha| \le m} \|\rho^{|\alpha|-a-1} \partial^\alpha v\|_{L^2}^2$$
$$= \lambda^{2a} \|v\|_{\mathcal{K}_a^m}^2 .$$

□

The above lemma also explains why we are choosing a different normalisation for the weight factor ρ^a. (See the comment after our definition of weighted Sobolev spaces.)

We shall need a well known alternative definition of the Sobolev spaces with weights. Assume again, for a moment, that $Q_1 = O := (0, 0)$, the origin of the coordinate system. Let $\Omega \subset \mathbb{P}_{l/4}$ be completely contained in the triangle closest to Q_1. Then the vector fields ∂_r and ∂_θ are defined using polar coordinates on Ω. For general $\Omega \subset \mathbb{P}_{l/4}$, we define these vector fields by translations (or, which is the same thing, by considering polar coordinates centered at either of the vertices Q_j). Then $\rho = r$, $\partial_r \rho = 1$ and $\partial_\theta \rho = 0$ on $\mathbb{P}_{l/4}$, by definition.

Lemma 1.10 *We have*

$$\mathcal{K}_a^m(\mathbb{P}) = \{u \in H_{sl}^m(\mathbb{P}),\ r^{-a-1}(r\partial_r)^i \partial_\theta^j u \in L^2(\mathbb{P}_{l/4}),\ \forall i + j \le m\}.$$

Let us observe here that the "right" Sobolev spaces needed to treat the case of domains with cusps is similar to the one in the above lemma, but replacing $r^{1+\gamma}\partial_r$ in place of $r\partial_r$.

We conclude our list of lemmas on the weighted Sobolev spaces with the following result, which is also well known [37].

Lemma 1.11 *Let P be a constant coefficient differential operator of order k on $\mathbb{R}^d$. Then P defines a continuous map $P : \mathcal{K}_a^m(\mathbb{P}) \to \mathcal{K}_{a-k}^{m-k}(\mathbb{P})$, $m \ge k$.*

2 Estimates for Dirichlet's problem

We shall need the following estimates on the solutions of non-homogeneous Dirichlet problem. First, let us notice that $\mathcal{K}_a^m(\mathbb{P}) \subset H_{sl}^m(\mathbb{P})$. (Recall that by $H_{sl}^m(\Omega)$ we denote the space of functions on Ω that restrict to an element of $H^m(V)$ for any open subset $V \subset \Omega$ such that $\overline{V}$ is compact and it does not contain the vertices of $\mathbb{P}$.) Thus, if $m \ge 1$, the trace

$$u|_{\partial\mathbb{P}} \in H_{sl}^{m-1/2}(\partial\mathbb{P}), \quad u \in \mathcal{K}_a^m \tag{8}$$

is defined. One can give a more precise description of the range of this trace, or restriction, map as follows [1,13,35]. Let

$$\mathcal{K}_a^m(\partial\Omega) := \{u \in L^2_{loc}(\partial\Omega),\ \rho^{|\alpha|-a-1/2}\partial^\alpha u \in L^2(\partial\Omega),\ \forall |\alpha| \le m\}, \quad \alpha \in \mathbb{Z}_+^2.$$

We define $\mathcal{K}_a^s(\partial\Omega)$, $s \in \mathbb{R}_+$, by interpolation.

Theorem 2.1 *Let $\mathbb{P}$ be a polygon in the plane. Then the map*

$$\Delta : \mathcal{K}_0^{m+2}(\mathbb{P}) \to \mathcal{K}_{-2}^{m}(\mathbb{P}) \oplus \mathcal{K}_0^{m+3/2}(\partial\mathbb{P}), \quad m \geq 0, \tag{9}$$

is an isomorphism.

The above theorem, except maybe the determination of the space $\mathcal{K}_0^{m+3/2}(\partial\mathbb{P})$ is from [26] (Theorem 6.6.1). This reference was pointed out to us by a referee, and we thank him for that. See also [13,27,32,37]. In particular,

$$\Delta : \mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u \in H_{sl}^1(\mathbb{P}),\ u|_{\partial\mathbb{P}} = 0\} \to \mathcal{K}_{-2}^{m}(\mathbb{P}), \quad m \geq 0, \tag{10}$$

is an isomorphism as well.

We shall write $\mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} := \mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u \in H_{sl}^1(\mathbb{P}),\ u|_{\partial\mathbb{P}} = 0\}$ in what follows, for simplicity.

Proof We can reduce to the case when $u|_{\partial\mathbb{P}} = 0$ as in [13]. Let Δ_C be the Laplace operator associated to the metric $g_C := \rho^{-2} g_E$, where g_E is the Euclidean metric on $\mathbb{R}^2$. Then the main result of [35] asserts that Δ_C defines an isomorphism

$$\Delta_C : \mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \to \mathcal{K}_0^{m}(\mathbb{P}).$$

The result then follows from $\Delta = \rho^{-2}\Delta_C$ and Lemma 1.3. □

This gives right away the following corollary.

Corollary 2.2 *For any polygon $\mathbb{P}$, there exists a constant $\eta > 0$, depending on $\mathbb{P}$, such that*

$$\Delta : \mathcal{K}_\epsilon^{m+2}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \to \mathcal{K}_{\epsilon-2}^{m}(\mathbb{P}), \quad m \geq 0, \tag{11}$$

is an isomorphism for any $|\epsilon| < \eta$.

Proof Fix ϵ and denote by Δ_ϵ the operator defined by Δ but with domain $\mathcal{K}_\epsilon^{m+2}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\}$. Then Δ_ϵ is an isomorphism if, and only if,

$$P_\epsilon := \rho^{-\epsilon}\Delta\rho^{\epsilon} : \mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u|_{\partial\mathbb{P}} = 0\} \to \mathcal{K}_{-2}^{m} \tag{12}$$

is an isomorphism. The result then follows from the fact that P_ϵ is invertible for $\epsilon = 0$ and depends continuously in norm on ϵ. □

2.1 Additional comments

It is possible to show that $\eta = \pi/\alpha_1$, where α_1 is the largest angle of $\mathbb{P}$ [26,37]. We include here a short argument for the interested reader. The eigenvalues of ∂_θ on $L^2([0,\alpha])$ with Dirichlet boundary value conditions are $-(k\pi/\alpha)^2$. The indicial family of $P_\epsilon := \rho^{-\epsilon}\Delta\rho^\epsilon$ is $(\imath\tau + \epsilon)^2 + \partial_\theta^2$ acting on $H^2([0,\alpha])$ with Dirichlet boundary conditions. By the results of [25], the operator $P_\epsilon := \rho^{-\epsilon}\Delta\rho^\epsilon$, with domain and range as in Equation (12) is Fredholm if, and only if, its indicial family is invertible for all $\tau \in \mathbb{R}$. This is seen to be the case, unless $\epsilon = \pm k\pi/\alpha$, with $k \in \mathbb{N} = \{1, 2, \dots\}$. Thus, for $|\epsilon| < \pi/\alpha_1$, where α_1 is the largest value of the angles of our polygon $\mathbb{P}$ (this discussion is valid for arbitrary polygons), the operator $\rho^{-\epsilon}\Delta\rho^\epsilon$ (with Dirichlet boundary conditions) is Fredholm of index zero. Since the kernels of the operators $\rho^{-\epsilon}\Delta\rho^\epsilon$ are decreasing as ϵ is decreasing, we obtain that they are invertible for $0 \geq \epsilon > -\pi/\alpha_1$. By taking adjoints, we obtain the same for $0 \leq \epsilon < \pi/\alpha_1$.

For values of ϵ outside the range studied above, the operator P_ϵ will no longer be invertible. In fact, it will have a non-zero index that can be computed using the results of [29]. Their theorem is very closely related to the Atiyah-Bott index theorem [3]. See also [23,34,42]

Let us also mention here that in the case of cusps there is no restriction on ϵ: we obtain an isomorphism for any $\epsilon > 0$.

3 An approximation result

As we have explained in the introduction, we are looking for extensions of the well known Theorem 0.1.

Let $M(l,\delta,\alpha) := C(\alpha)M_1M_2$, where $C(\alpha)$ is as in Theorem 0.1 and M_1 and M_2 are as in Lemma 1.7. Also, recall that u_I is the interpolant associated to the degree m linear triangle (in the terminology of [15]), whose definition was also recalled in the Introduction in the paragraph right before the statement of Theorem 0.1.

Theorem 3.1 *Fix $\alpha > 0$ and $0 < \delta < l/4$. Let $\mathbb{P}$ be a triangle with the shortest edge $= l$ and $\Omega \subset \mathbb{P}$ be a polygonal domain such that $\rho \geq \delta$ on Ω. Let $\mathcal{T} = (T_j)$ be a triangulation of Ω with triangles with angles $\geq \alpha$ and sides $\leq h$. Then*

$$\|u - u_I\|_{\mathcal{K}_0^1} \leq M(l,\delta,\alpha)h^m\|u\|_{\mathcal{K}_\epsilon^{m+1}} \tag{13}$$

for any $u \in \mathcal{K}_\epsilon^{m+1}(\Omega)$. (All norms above refer to functions defined on Ω.)

Proof This follows from Theorem 0.1 and the equivalence of the H^m and $\mathcal{K}_a^m$–norms on Ω (Lemma 1.7). □

We now extend Theorem 3.1 to trapezoids of the form $\mathbb{P}_\delta \setminus \mathbb{P}_{\kappa\delta}$. Let $C_1(\kappa) = M(l, \kappa l/8, \alpha)(l/4)^m$.

Theorem 3.2 *Let $\kappa, \alpha > 0$, and $0 < \delta < l/4$. Let $\mathcal{T} = (T_j)$ be a triangulation of $\Omega := \mathbb{P}_\delta \setminus \mathbb{P}_{\kappa\delta}$ with triangles with angles $\geq \alpha$ and edges $\leq h$. Then*

$$\|(u - u_I)|_\Omega\|_{\mathcal{K}_0^1} \leq C_1(\kappa)\delta^\epsilon (h/\delta)^m \|u\|_{\mathcal{K}_\epsilon^{m+1}} \tag{14}$$

for any $u \in \mathcal{K}_\epsilon^{m+1}(\Omega)$.

Proof We use Lemma 1.9 with $\lambda = 4\delta/l$ to conclude that $\|u - u_I\|_{\mathcal{K}_0^m} = \|u_\lambda - u_{I\lambda}\|_{\mathcal{K}_0^m}$. Then we notice that $u_{I\lambda} = u_{\lambda I}$ (that is, dilation commutes with interpolation. Therefore, we can apply Theorem 3.1 to the region $\lambda^{-1}\Omega = \mathbb{P}_{l/4} \setminus \mathbb{P}_{\kappa l/4}$, the triangulation $\lambda^{-1}\mathcal{T}$, and the function u_λ to obtain that

$$\|u_\lambda - u_{\lambda I}\|_{\mathcal{K}_0^1} \leq M(l, \kappa l/8, \alpha) \left(\frac{hl}{4\delta}\right)^m \|u_\lambda\|_{\mathcal{K}_0^{m+1}}$$

This then gives

$$\begin{aligned} \|u - u_I\|_{\mathcal{K}_0^1} = \|u_\lambda - u_{\lambda I}\|_{\mathcal{K}_0^1} &\leq C_1(\kappa)(h/\delta)^m \|u_\lambda\|_{\mathcal{K}_0^{m+1}} \\ &\leq C_1(\kappa)\delta^\epsilon (h/\delta)^m \|u_\lambda\|_{\mathcal{K}_\epsilon^{m+1}}, \end{aligned}$$

where the last inequality is provided by Lemma 1.4b. □

4 The main results

We now introduce the class of triangulations for which we will prove our main results, namely the class $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$. Then we prove our main approximations results, including the Theorems 0.2 and 0.3 announced in the introduction.

4.1 The class $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$

The following definition is formulated for the case $\mathbb{P}$ an acute angle triangle, for simplicity. The general case of a polygon possibly with reentrant corners is completely similar and is discussed in Section 5.

We continue to denote by l the shortest edge of the triangle $\mathbb{P}$. Also, recall the constant $M(l, \delta, \alpha)$ introduced in Theorem 3.1.

Definition 4.1 *Fix* $m \in \mathbb{N} = \{1, 2 \dots\}$ *and let* $\epsilon \in (0, 1]$, $h > 0$, $\kappa, b \in (0, 1)$, *and* $\alpha \in (0, \pi/2)$ *be parameters. We define* $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ *to be the set of triangulations* $\mathcal{T}$ *defined as follows. Choose n such that*

$$\kappa^{n\epsilon} \leq M(l, \kappa l/8, \alpha) h^m.$$

We decompose $\mathbb{P}$ *as the union of* $\Omega_0 := \mathbb{P} \setminus \mathbb{P}_{l/4}$, $\Omega_1 := \mathbb{P}_{l/4} \setminus \mathbb{P}_{\kappa l/4}$, ... , $\Omega_n := \mathbb{P}_{\kappa^{n-1} l/4} \setminus \mathbb{P}_{\kappa^n l/4}$, *and* $\widetilde{\Omega}_{n+1} := \mathbb{P}_{\kappa^n l/4}$. *For each* $j = 0, \dots, n$, *we triangulate* Ω_j *with triangles with all angles* $\geq \alpha$, *and edges of length at most*

$$h_{n,j} = h\kappa^{(1-\epsilon/m)j} \tag{15}$$

and at least $bh_{n,j}$. *Then* $\mathcal{T}$ *is the union of the triangles appearing in the triangulations of* Ω_j, $j \leq n$, *and of the three triangles forming* $\widetilde{\Omega}_{n+1}$.

We begin with the following "h^m"–approximation result for the triangulations in $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$.

Theorem 4.2 *There exists a constant* $B_0 = B_0(l, \kappa, \alpha)$ *such that*

$$\|u - u_I\|_{\mathcal{K}_0^1} \leq B_0 (2h)^m \|u\|_{\mathcal{K}_\epsilon^{m+1}}, \tag{16}$$

for any triangulation $\mathcal{T} \in \mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ *and any* $u \in \mathcal{K}_\epsilon^{m+1} \cap \{u|_{\mathbb{P}} = 0\}$.

Proof It is enough to establish the corresponding estimate for $\|(u-u_I)|_{\Omega_j}\|_{\mathcal{K}_0^1}$, for $j = 0, 1, \dots, n+1$ and $B_0 = \max\{1, l\} M(l, \kappa l/8, \alpha)$.

For $j = 0$, we use Theorem 3.1 for $\Omega = \Omega_0$. For $j = 1, 2, \dots, n$, we use Theorem 3.2 for $\Omega = \Omega_j$. Then we notice that $u_I = 0$ on $\widetilde{\Omega}_{n+1}$ and hence

$$\|(u - u_I)|_{\widetilde{\Omega}_{n+1}}\|_{\mathcal{K}_0^1} = \|u|_{\widetilde{\Omega}_{n+1}}\|_{\mathcal{K}_0^1} \leq \delta^\epsilon \|u|_{\widetilde{\Omega}_{n+1}}\|_{\mathcal{K}_\epsilon^{m+1}}, \tag{17}$$

by Lemma 1.4(c), with $\delta = \kappa^n l/4$. The result then follows by adding the squares of all these estimates, using also Equation (15). □

From this we obtain the following estimate on the error $\|u - u_V\|_{H^1}$ for the discrete solution u_V. (Recall that u and u_V were defined in Equations (3) and (4) in the Introduction). From now on we shall assume that $0 < \epsilon \leq 1$ is chosen such that

$$\Delta : \mathcal{K}_\epsilon^{m+1} \cap \{u|_{\mathbb{P}} = 0\} \to \mathcal{K}_\epsilon^{m-1}$$

be an isomorphism. This is possible due to Corollary 2.2.

Theorem 4.3 *There exists a constant* $B_0' = B_0'(l, \kappa, \alpha)$ *such that*

$$\|u - u_V\|_{H^1} \leq B_0' (2h)^m \|f\|_{H^{m-1}}. \tag{18}$$

for any $\mathcal{T} \in \mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ *and any* $f \in \mathcal{K}_\epsilon^{m-1}$, *where* $u \in \mathcal{K}_\epsilon^{m+1} \cap \{u|_{\mathbb{P}} = 0\}$ *is the unique solution of* $\Delta u = f$.

Proof Let ν_ϵ be the norm of $\Delta^{-1} : \mathcal{K}_\epsilon^{m-1} \to \mathcal{K}_\epsilon^{m+1} \cap \{u|_{\mathbb{P}} = 0\}$. We have

$$\begin{aligned}\|u - u_V\|_{H^1} &\leq c\|u - u_I\|_{H^1} \\ &\leq Mc\|u - u_I\|_{\mathcal{K}_0^1} \\ &\leq McB_0(2h)^m\|u\|_{\mathcal{K}_\epsilon^{m+1}} \\ &\leq \nu_\epsilon McB_0(2h)^m\|f\|_{\mathcal{K}_{\epsilon-1}^{m-1}} \\ &\leq \nu_\epsilon M^2cB_0(2h)^m\|f\|_{H^{m-1}}.\end{aligned}$$

For the first inequality we first replace the norm $\|u\|_{H^1}$ with the equivalent norm $|u|_{H^1} = \|\nabla u\|_{L^2}$ (use Poincaré's inequality) and use that u_V is the projection of u onto V in the inner product defined by $|\cdot|_{H^1}$. The second and fifth inequalities are obtained using Lemma 1.5 (use also $\epsilon \leq 1$). The third inequality is obtained from Theorem 4.2. The fourth inequality is obtained from the invertibility of Δ on the corresponding spaces. □

The above theorems are satisfactory, except for one feature, namely that they do not give a bound for the dimension of the finite element spaces $V := V(\mathcal{T}, m+1)$. This is remedied by the following result.

Theorem 4.4 *There exists a constant $B_1 = B_1(l, \epsilon, \kappa, \alpha, b)$ such that*

(19) $$\|u - u_I\|_{\mathcal{K}_0^1} \leq B_1 \dim(V)^{-m/2}\|u\|_{\mathcal{K}_\epsilon^{m+1}}, \;\; V = V(\mathcal{T}, m+1),$$

for any partition $\mathcal{T} \in \mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ and any $u \in \mathcal{K}_\epsilon^{m+1}$.

Proof The area of each triangle in the triangulation of Ω_j is bounded from below because it has all sides of length $\geq bh_{n,j}$ and all angles $\geq \alpha$. The dimension of V is then bounded from above by estimating the minimum area of the triangles in the partition, which must be $\leq area(\mathbb{P})$. □

4.2 The two main theorems

We now prove the two main theorems stated in the introduction. We begin with Theorem 0.2.

Proof The proof of Theorem 0.2 is similar to that of Theorem 4.3, but using Theorem 4.4 instead of Theorem 4.2. □

It remains to prove Theorem 0.3. This is achieved through the following example.

Proof Fix $\mathbb{P}$ and $m \in \mathbb{N} = \{1, 2, \dots\}$ arbitrary. Also, choose $0 < \epsilon \leq 1$ such that

$$\Delta : \mathcal{K}_\epsilon^{m+1} \cap \{u|_{\mathbb{P}} = 0\} \to \mathcal{K}_{\epsilon-2}^{m-1}$$

be an isomorphism. Then choose $\kappa = 2^{-m/\epsilon}$.

For any triangulation $\mathcal{T} = (T_j)$ of a polygonal domain Ω, we shall denote by $2^{-k}\mathcal{T}$ the triangulation of Ω obtained by dividing each triangle T_j of $\mathcal{T} = (T_j)$ into 2^{2k} equal triangles, all similar to T_j. Also, if $\mathcal{T} = (T_j)$ is a triangulation of $\mathbb{P}_\delta \smallsetminus \mathbb{P}_{\kappa\delta}$, then we shall denote by $(\mathcal{T})_\lambda$ the triangulation of $\mathbb{P}_{\lambda\delta} \smallsetminus \mathbb{P}_{\kappa\lambda\delta}$ obtained by applying a suitable similarity of ratio λ to each of the triangles of $\mathcal{T}$ (the center of each similarity is the closest vertex to the triangle that is transformed).

We shall use the notation of Definition 4.1, that is, $\widetilde{\Omega}_{j+1} := \mathbb{P}_{\kappa^j l/4}$, $j = 0, 1, \ldots$, and $\Omega_j := \widetilde{\Omega}_j \smallsetminus \widetilde{\Omega}_{j+1}$. Let $\mathcal{T}_0$ be the triangulation of Ω_0 obtained by joining the baricenter of Ω_0 to each of the six vertices of Ω_0. Next, we write $\Omega_1 = U_1 \cup U_2 \cup U_3$, the union of its connected components. Then, we triangulate each trapezoid U_j into three triangles by joining the midpoint of its long basis to the two opposite vertices. Let $\mathcal{T}_1$ denote the resulting triangulation of Ω_1. We shall denote by $\widetilde{\Omega}^t_{n+1}$ the triangulation of $\widetilde{\Omega}_{n+1}$ into its three connected components.

Next, we define α to be the least of the angles appearing in $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \widetilde{\Omega}_{n+1}$. Let h' be the shortest edge of $\mathcal{T}_0 \cup \mathcal{T}_1$ and $h = \kappa^{\epsilon/m-1}h'$. Let $b_0 < 1$ be the ratio of the shortest and the longest edges in $\mathcal{T}_0$. Define b_1 similarly, as the ratio of the shortest and the longest edges in $\mathcal{T}_0$. We complete our set of choices by taking $b = \min\{b_0, b_1\}\kappa^{\epsilon/m-1}$.

We define

$$\mathcal{T} = 2^{-n}\mathcal{T}_0 \cup 2^{-n+1}\mathcal{T}_1 \cup_{j=2}^{n} 2^{-n+j}(\mathcal{T}_1)_{\kappa^{j-1}} \cup \widetilde{\Omega}^t_{n+1}.$$

Then $\mathcal{T} \in \mathcal{C} := \mathcal{C}(m, 2^{-n}h, \epsilon, 2^{-n/\epsilon}, \alpha, b)$, and hence $\mathcal{C}$ is not empty. The statement of the theorem is obtained by taking $h_n = 2^{-n}h$. □

We note that for domains with cusps, the geometric sequence κ^n is replaced by sequences of the form $n^{-\beta}$, $\beta > 0$. The cusp case will be treated in detail in a forthcoming paper.

5 Numerical tests

We now present a few numerical tests of our theoretical results. We consider the model problem

$$\begin{cases} -\Delta u = 1, & x \in \mathbb{P} \\ u = 0, & x \in \partial\mathbb{P}, \end{cases} \tag{20}$$

where the domain $\mathbb{P}$ is the quadrilateral shown (together with an initial finite element mesh) on the left in Fig. 5.1. We have chosen $m = 1$, because the implementation is then much simpler, while the result is still relevant, because our domain has a reentrant corner, and hence the solution is not automatically

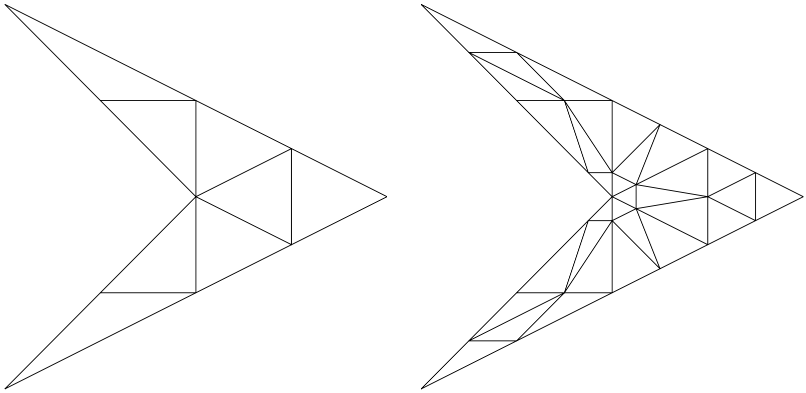

Fig. 5.1. Initial mesh (left) and the mesh after one refinement for $\kappa = \frac{1}{4}$ (right)

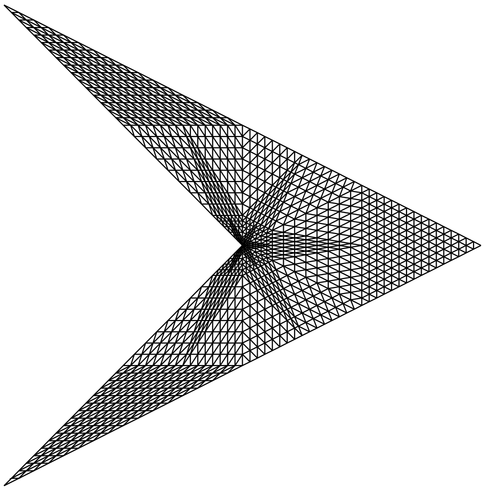

Fig. 5.2. The mesh after 4 levels of refinement for $\kappa = \frac{1}{4}$

in $H^2(\mathbb{P})$. In fact, the apriori estimates in the usual Sobolev spaces only give $u \in H^s(\mathbb{P})$, with $s < 5/3$,[11, 12, 17, 18, 21, 22].

For our quadrilateral, we can take $\eta = \pi/\alpha_1 = 2/3$ (see the discussion in the second half of Section 2). Hence the range of ϵ is $0 < \epsilon < \eta = 2/3$, which gives $0 < \kappa := 2^{-m/\epsilon} < 2^{-3/2} = 1/\sqrt{8} \approx 0.35$.

The coarsest mesh we have used has 32 elements, is shown at the right of Fig. 5.1 and is just one refinement of the initial mesh shown at the left of Fig. 5.1. This is the starting point for our refinement. The method of refinement is the one explained in the proof of Theorem 0.3 and is shown in Fig. 5.1 for $\kappa = 1/4$ (the proof of Theorem 0.3 is given in Section 4). For $\kappa = 1/2$, we obtain a quasi-uniform family of meshes. For other values of $\kappa < 1/2$, this is

Table 5.1. Laplace equations results – estimated convergence rate

j	$\delta\colon \kappa = \frac{1}{2}$	$\delta\colon \kappa = \frac{2}{5}$	$\delta\colon \kappa = \frac{1}{3}$	$\delta\colon k = \frac{1}{4}$	$\delta\colon k = \frac{1}{5}$
2	0.92	0.94	0.95	0.95	0.95
3	0.90	0.95	0.97	0.98	0.98
4	0.86	0.95	0.97	0.99	0.99
5	0.83	0.94	0.98	0.99	0.99
6	0.79	0.94	0.98	0.99	1.00
7	0.76	0.93	0.98	1.00	1.00
8	0.73	0.93	0.98	1.00	1.00

not, however, the case. The meshes are nevertheless topologically equivalent for all values of κ.

The finest mesh used in our numerical experiments is obtained after 8 successive refinements of the coarsest mesh and has $2^{21} \approx 2 \times 10^6$ elements. To solve the resulting system of algebraic equations, we have used a simple algebraic multigrid method developed and implemented by the third author.

We have tested several values of the parameter κ, and the results are summarized in Table 5.1. These results convincingly show that the correctly graded refinement improves the convergence rate with "a factor of about $h^{0.27}$."

In fact, the improvement may be even greater, as the theoretical prediction for the convergence rate in the quasi-uniform case ($\kappa = 1/2$) is $h^{2/3}$ and our theoretical prediction for the convergence rate in the case $\kappa = 1/4$ is h^1. Thus our numerical results completely agree with the theory we have presented in this paper. (In fact, it is rather imprecise here to say that the order of convergence is h^α, for some α, since we are not dealing with quasi-uniform family of meshes. The correct, but less intuitive statements are obtained by replacing h with $N^{-1/2}$, where N is the number of elements in the mesh.)

The left most column in Table 5.1 shows the refinement level number. If we denote the finite element solution on a j-th grid with u_j (j corresponds to a grid with 32×4^j elements), the quantity printed in the last five columns in the table is δ, where

$$\delta = \log_2 \left[\frac{|u_j - u_{j-1}|_{1,\Omega}}{|u_{j+1} - u_j|_{1,\Omega}} \right].$$

The quantity δ is not an exact convergence rate, but turns out to be a quite reasonable approximation to it. One can see from our numerical results, that, for a correctly graded refinement, the difference between two consecutive solutions is decreasing as $N^{-1/2}$, where N is the number of triangles in the mesh.

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