INTERPOLATION BETWEEN SUBSPACES OF HILBERT SPACES AND APPLICATIONS TO SHIFT THEOREMS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS AND FINITE ELEMENT METHODS

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ABSTRACT

Interpolation between Subspaces of Hilbert Spaces and
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In the real method of interpolation one starts with two Hilbert spaces, X and Y, with certain properties and constructs a family of Hilbert spaces called the interpolation spaces. In applications to partial differential equations and finite element methods, the following question often arises: If the interpolation spaces X and Y are known Sobolev spaces, and if X_M and Y_N are closed subspaces of X and Y, respectively, what are the interpolation spaces of X_M and Y_N ? For certain boundary value problems, the answer to this kind of question, together with a complete characterization of the range of the corresponding differential operator, leads to stability estimates for solutions in terms of fractional norms. These types of estimates are known as shift theorems. The thesis is concerned with developing new interpolation results and shift theorems for the special case of polygonal plane domains, and presents some applications of the new results.

TABLE OF CONTENTS

CHAPTER Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Regularity estimates of the solutions of elliptic boundary value problems in terms of Sobolev-fractional norms are known as shift theorems or shift estimates. The shift estimates are significant in finite element theory. Applications of the shift theorems can be found for example in the following.

- Nitsche's duality argument for polygonal domains.
- Multigrid convergence theorems.
- Convergence of "mortar" finite element methods.

The shift estimates for the Laplace operator with Dirichlet boundary conditions on nonsmooth domains are studied in [25], [28] and [32]. On the question of shift theorems for second order elliptic boundary value problems with mixed boundary conditions or biharmonic problem on nonsmooth domains, there seems to be no work answering this question.

One way of proving shift results is by using the real method of interpolation of Lions and Peetre [2], [29] and [30]. The interpolation problems we are led to are of the following type. If X and Y are Sobolev spaces of integer order and X_K is a subspace of finite codimension of X then characterize the interpolation spaces between X_K and Y.

When X_K is of codimension one the problem was studied by Kellogg in some particular cases in [25]. The interpolation results presented in Chapter II give a natural formula connecting the norms on the intermediate subspaces $[X_K, Y]_s$ and $[X, Y]_s$.

The journal model is *Mathematics of Computation.*

The main result of Chapter II is a theorem which provides sufficient conditions (the **(A1)** and **(A2)** conditions) to compare the topologies on $[X_K, Y]_s$ and $[X, Y]_s$ and gives rise to an extension of Kellogg's method in proving shift estimates for more complicated boundary value problems.

Kellogg's approach in solving subspace interpolation problems on sector domains involves reduction of the problem to subspace interpolation on Sobolev spaces defined on all of R^2. This reduction requires construction of "extension" and "restriction" operators connecting Sobolev spaces defined on sectors and Sobolev spaces defined on R^2. Another difficulty in Kellogg's method is finding the asymptotic expansion of the Fourier transform of certain singular functions. The new approach taken in this dissertation in solving subspace interpolation on sector domains, which avoids the Fourier transform and the construction of the extension and restriction operators, is to use a convenient scaling norm for the Sobolev spaces on a sector domain. An example of a convenient scaling norm in our case is a multilevel norm originating from multigrid theory [14], [16] and [34]. Using classical preconditioning techniques ([9]-[15]), a proof of the fact that the multilevel norm on H^1 is equivalent to the standard norm on H^1 is presented in Chapter III. In Chapter IV we use the multilevel norm described in Chapter III in order to solve a codimension one subspace interpolation problem which applies mainly for the special cases of polygonal domains and for sector domains.

A new proof of the main subspace interpolation result presented in [25] and an extension to subspace interpolation of codimension greater than one are given in Chapter V.

In the general context of interpolation between Hilbert subspaces we consider the problem of interpolating between $H^2(\Omega) \cap H^1_D(\Omega)$ and $H^1_D(\Omega)$, where Ω is a polygonal domain and $H^1_D(\Omega)$ is the subspace of functions in $H^1(\Omega)$ which vanish on the Dirichlet part D of the boundary of Ω. This question arose in [7] and [8]. In Chapter VI we exhibit some results concerning this problem and in Chapter IX we deal with applications to a nonconforming finite element problem of the theory presented in Chapter VI.

Shift theorems for the Poisson equation (with mixed boundary conditions) on polygonal domains are considered in Chapter VII. The first step in the proposed approach of this issue is to reduce the original shift estimate problem to a similar problem on simpler domains, for example sector domains. The second step is to characterize the range of the corresponding differential operator of the problem by means of dual singular functions on sector domains [19], [23], [24], [27]. The next step is to use an eigenfunction representation of the norm on H^α_D in order to check the validity of the **(A2)** condition for the subspace of L^2 involved here. The final step is to use the equivalent multilevel norm on H^α_D and the interpolation results of Chapter IV to check that the condition **(A1)** holds.

Shift estimates for biharmonic problems are considered in Chapter VIII. The results of Chapter V combined with a Kellogg type approach are used in order to solve the corresponding subspace interpolation problem.

CHAPTER II

INTERPOLATION RESULTS

In this section we give some basic definitions and results concerning interpolation between Hilbert spaces and subspaces using the real method of interpolation of Lions and Peetre ([1, 29, 30]).

A. Interpolation between Hilbert spaces

Let X, Y be separable Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively, and satisfying for some positive constant c,

$$\begin{cases} X \text{ is a dense subset of Y and} \\ \|u\|_Y \leq c\|u\|_X \quad \text{for all } u \in X, \end{cases} \tag{2.1}$$

where $\|u\|_X^2 = (u,u)_X$ and $\|u\|_Y^2 = (u,u)_Y$.

Let $D(S)$ denote the subset of X consisting of all elements u such that the antilinear form

$$v \to (u,v)_X, \ v \in X \tag{2.2}$$

is continuous in the topology induced by Y. For any u in $D(S)$ the antilinear form (2.2) can be extended to a continuous antilinear form on Y. Then, by Riesz representation theorem , there exists an element Su in Y such that

$$(u,v)_X = (Su,v)_Y \quad \text{for all } v \in X. \tag{2.3}$$

In this way S is a well defined operator in Y, with domain $D(S)$. The next result illustrates the properties of S .

Proposition II.1 *The domain $D(S)$ of the operator S is dense in X and consequently $D(S)$ is dense in Y. The operator $S : D(S) \subset Y \to Y$ is a bijective, self-adjoint and positive definite operator. The inverse operator $S^{-1} : Y \to D(S) \subset Y$ is a bounded symmetric positive definite operator and*

$$(S^{-1}z, u)_X = (z, u)_Y \quad \text{for all } z \in y,\ u \in X \tag{2.4}$$

If in addition X is compactly embedded in Y, then S^{-1} is a compact operator.

The interpolating space $[X, Y]_s$ for $s \in (0, 1)$ is defined using the K function, where for $u \in Y$ and $t > 0$,

$$K(t, u) := \inf_{u_0 \in X} (\|u_0\|_X{}^2 + t^2\|u - u_0\|_Y{}^2)^{1/2}.$$

Then $[X, Y]_s$ consists of all $u \in Y$ such that

$$\int_0^\infty t^{-(2s+1)} K(t, u)^2 \; dt < \infty.$$

The norm on $[X, Y]_s$ is defined by

$$\|u\|^2_{[X,Y]_s} := \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} K(t, u)^2 \; dt,$$

where

$$\mathbf{c}_s := \left(\int_0^\infty \frac{t^{(1-2s)}}{t^2 + 1} dt \right)^{-1/2} = \sqrt{\frac{2}{\pi} \sin(\pi s)}$$

By definition we take

$$[X, Y]_0 := X \quad \text{and} \quad [X, Y]_1 := Y.$$

The next lemma provides the relation between $K(t, u)$ and the connecting operator S.

Lemma II.1 *For all $u \in Y$ and $t > 0$,*

$$K(t,u)^2 = t^2\left((I + t^2 S^{-1})^{-1}u, u\right)_Y.$$

Proof. Using the density of $D(S)$ in X, we have

$$K(t,u)^2 = \inf_{u_0 \in D(S)} (\|u_0\|_X{}^2 + t^2\|u - u_0\|_Y{}^2)$$

Let $v = Su_0$. Then

$$\|u_0\|_X{}^2 = (u_0, u_0)_X = (Su_0, u_0)_Y = (S^{-1}v, v)_Y.$$

This implies that

$$K(t,u)^2 = \inf_{v \in Y} ((S^{-1}v, v)_Y + t^2\|u - S^{-1}v\|_Y{}^2). \tag{2.5}$$

Solving the minimization problem (2.5) we obtain that the element v which gives the optimum satisfies

$$(I + t^2 S^{-1})v = t^2 u,$$

For this v, $(S^{-1}u, v)_Y$ is a real number and consequently,

$$\begin{aligned}
(S^{-1}v, v)_Y + t^2\|u - S^{-1}v\|_Y{}^2 &= (S^{-1}v, v + t^2 S^{-1}v)_Y + t^2\|u\|_Y{}^2 - t^2(u, S^{-1}v)_Y \\
&= t^2((S^{-1}v, u)_Y + \|u\|_Y{}^2 - 2(u, S^{-1}v)_Y = t^2(u, u - S^{-1}v)_Y \\
&= (u, t^2(u - S^{-1}v))_Y = (u, v)_Y \\
&= t^2\left((I + t^2 S^{-1})^{-1}u, u\right)_Y.
\end{aligned}$$

■

Remark II.1 *Lemma II.1 gives a new expression for the norm on $[X,Y]_s$, namely:*

$$\|u\|^2_{[X,Y]_s} := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left((I + t^2 S^{-1})^{-1} u, u\right)_Y dt. \tag{2.6}$$

In addition, by this new expression for the norm (see Definition 2.1 and Theorem 15.1 in [29]), it follows that the intermediate space $[X,Y]_s$ coincides topologically with the domain of the unbounded operator $S^{1/2(1-s)}$ equipped with the norm of the graph of the same operator . As a consequence we have that X is dense in $[X,Y]_s$ for any $s \in [0,1]$.

B. Interpolation between subspaces of Hilbert spaces

Let $\mathcal{K} = span\{\varphi_1, \ldots, \varphi_n\}$ be a n-dimensional subspace of X and let $X_{\mathcal{K}}$ be the orthogonal complement of $\mathcal{K}$ in X in the $(\cdot,\cdot)_X$ inner product. We are interested in determining the interpolation spaces of $X_{\mathcal{K}}$ and Y, where on $X_{\mathcal{K}}$ we consider again the $(\cdot,\cdot)_X$ inner product. For certain spaces $X_{\mathcal{K}}$ and Y and $n = 1$, this problem was studied in [25]. To apply the interpolation results from the previous section we need to check that the density part of the condition (2.1) is satisfied for the pair $(X_{\mathcal{K}}, Y)$.

For $\varphi \in \mathcal{K}$, define the linear functional $\Lambda_\varphi : X \to C$, by

$$\Lambda_\varphi u := (u, \varphi)_X, \ u \in X.$$

Lemma II.2 *(Bacuta Bramble) The space $X_{\mathcal{K}}$ is dense in Y if and only if the following condition is satisfied:*

$$\begin{cases} \Lambda_\varphi \textit{ is not bounded in the topology of } Y \\ \quad \textit{for all } \varphi \in \mathcal{K}, \ \varphi \neq 0. \end{cases} \tag{2.7}$$

Proof. First let us assume that the condition (2.7) does not hold. Then for some $\varphi \in \mathcal{K}$ the functional L_φ is a bounded functional in the topology induced by Y.

Thus, the kernel of L_φ is a closed subspace of X in the topology induced by Y. Since $X_\mathcal{K}$ is contained in $Ker(L_\varphi)$ it follows that

$$\overline{X_\mathcal{K}}^Y \subset \overline{Ker(L_\varphi)}^Y = Ker(L_\varphi).$$

Hence $X_\mathcal{K}$ fails to be dense in Y.

Conversely, assume that $X_\mathcal{K}$ is not dense in Y, then $Y_0 = \overline{X_\mathcal{K}}^Y$ is a proper closed subspace of Y. Let $y_0 \in Y$ be in the orthogonal complement of Y_0, and define the linear functional $\Psi : Y \to C$, by

$$\Psi u := (u, y_0)_Y, \;\; u \in Y.$$

Ψ is a continuous functional on Y. Let ψ be the restriction of Ψ to the space X. Then ψ is a continuous functional on X. By Riesz Representation Theorem, there is $v_0 \in X$ such that

$$(u, v_0)_X = (u, y_0)_Y, \qquad \text{for all } u \in X. \tag{2.8}$$

Let $P_\mathcal{K}$ be the X orthogonal projection onto $\mathcal{K}$ and take $u = (I - P_\mathcal{K})v_0$ in (2.8). Since $(I - P_\mathcal{K})v_0 \in X_\mathcal{K}$ we have $((I - P_\mathcal{K})v_0, y_0)_Y = 0$ and

$$0 = ((I - P_\mathcal{K})v_0, v_0)_X = ((I - P_\mathcal{K})v_0, (I - P_\mathcal{K})v_0)_X.$$

It follows that $v_0 = P_\mathcal{K} v_0 \in \mathcal{K}$ and, via (2.8), that $\psi = \Lambda_{v_0}$ is continuous in the topology of Y. This is exactly the opposite of (2.7) and the proof is completed.

■

Remark II.2 *The result still holds if we replace the finite dimensional subspace $\mathcal{K}$ with any closed subspace of X.*

For the next part of this section we assume that the condition (2.7) holds. By the above lemma, the condition (2.1) is satisfied. It follows from the previous section that the operator $S_{\mathcal{K}} : D(S_{\mathcal{K}}) \subset Y \to Y$ defined by

$$(u, v)_X = (S_{\mathcal{K}} u, v)_Y \qquad \text{for all } v \in X_{\mathcal{K}}, \tag{2.9}$$

has the same properties as S. Consequently, the norm on the intermediate space $[X_{\mathcal{K}}, Y]_s$ is given by:

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left((I + t^2 S_K^{-1})^{-1} u, u\right)_Y \, dt. \tag{2.10}$$

Let $[X, Y]_{s,K}$ denote the closure of $X_{\mathcal{K}}$ in $[X, Y]_s$. Our aim in this section is to determine sufficient conditions for φ_i's such that

$$[X_{\mathcal{K}}, Y]_s = [X, Y]_{s,K}. \tag{2.11}$$

First, we note that the operators $S_{\mathcal{K}}$ and S are related by the following identity:

$$S_{\mathcal{K}}^{-1} = (I - Q_{\mathcal{K}}) S^{-1}, \tag{2.12}$$

where $Q_{\mathcal{K}} : X \to \mathcal{K}$ is the orthogonal projection onto $\mathcal{K}$. The proof of (2.12) follows easily from the definitions of the operators involved.

Next, (2.12) leads to a formula relating the norms on $[X_{\mathcal{K}}, Y]_s$ and $[X, Y]_s$. Before deriving it , we introduce some notation. Let

$$(u, v)_{X,t} := \left((I + t^2 S^{-1})^{-1} u, v\right)_X \qquad \text{for all } u, v \in X. \tag{2.13}$$

and denote by M_t the Gram matrix associated with the set of vectors $\{\varphi_1, \ldots, \varphi_n\}$ in the $(\cdot, \cdot)_{X,t}$ inner product,i.e.,

$$(M_t)_{ij} := (\varphi_j, \varphi_i)_{X,t}, \ i, j \in \{1, \ldots, n\}.$$

Theorem II.1 *Let u be arbitrary in $X_{\mathcal{K}}$. Then,*

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} = \|u\|^2_{[X,Y]_s} + \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} \left\langle M_t^{-1} d, d \right\rangle \; dt, \tag{2.14}$$

where $< \cdot, \cdot >$ is the inner product on $\mathbf{C}^n$ and d is the n-dimensional vector in $\mathbf{C}^n$ whose components are

$$d_i := (u, \varphi_i)_{X,t}, \;\; i = 1, \ldots, n.$$

Proof. Let u be fixed in $X_{\mathcal{K}}$ and denote

$$w := (I + t^2 S^{-1})^{-1} u \;\; \text{and} \;\; w_{\mathcal{K}} := (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u. \tag{2.15}$$

Then the norms on $[X_{\mathcal{K}}, Y]_s$ and $[X, Y]_s$ are given by

$$\|u\|^2_{[X,Y]_s} = \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} (w, y)_Y \; dt \tag{2.16}$$

and

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} = \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} (w_{\mathcal{K}}, y)_Y \; dt \tag{2.17}$$

respectively. For v in Y, using (2.12), we have

$$S_{\mathcal{K}}^{-1} v = S^{-1} v - Q_{\mathcal{K}}(S^{-1} v) = S^{-1} v - \sum_{i=1}^n \alpha_i \varphi_i \tag{2.18}$$

where $\alpha_i = (S^{-1} v, \varphi_i)_X$. From (2.15) it follows that

$$(I + t^2 S_k^{-1}) w_{\mathcal{K}} = u. \tag{2.19}$$

Replacing $S_{\mathcal{K}}^{-1} w_{\mathcal{K}}$ with the expansion given by (2.18), we obtain

$$(I + t^2 S^{-1}) w_{\mathcal{K}} = u + t^2 \sum_{i=1}^n \alpha_i \varphi_i.$$

where $\alpha_i = (S^{-1}w_{\mathcal{K}}, \varphi_i)$. Equivalently, applying $(I + t^2S^{-1})^{-1}$ to both sides,we have

$$w_{\mathcal{K}} = w + t^2 \sum_{i=1}^{n} \alpha_i (I + t^2S^{-1})^{-1}\varphi_i. \tag{2.20}$$

We calculate the coefficients α_i by taking the $(\cdot,\cdot)_X$ inner product with φ_j on both sides of (2.20) for $j = 1, \ldots, n$. From (2.19) one sees that $w_{\mathcal{K}} \in X_{\mathcal{K}}$. Hence

$$\sum_{i=1}^{n} \big((I + t^2S^{-1})^{-1}\varphi_i, \varphi_j\big)_X \, \alpha_i = -t^{-2}(w, \varphi_j)_X \quad j = 1, \ldots, n \; .$$

With the notation adopted in (2.15) and (2.13) the system becomes

$$\sum_{i=1}^{n} (\varphi_i, \varphi_j)_{X,t} \; \alpha_i = -t^{-2}(u, \varphi_j)_{X,t} \quad j = 1, \ldots, n \; .$$

Let α be the n-dimensional vector from $\mathbf{C}^n$ whose components are α_i. Then

$$M_t \alpha = -t^{-2} d.$$

Since the vectors $\varphi_1, \ldots, \varphi_n$ are linearly independent, the matrix M_t is invertible and

$$\alpha = -t^{-2} M_t^{-1} d.$$

Now, going back to (2.20), we get

$$\begin{aligned}
(w_{\mathcal{K}}, u)_Y &= (w, u)_Y + \sum_{i=1}^{n} \alpha_i (t^2 (I + t^2S^{-1})^{-1}\varphi_i, u)_Y \\
&= (w, u)_Y + \sum_{i=1}^{n} \alpha_i (t^2 S^{-1}(I + t^2S^{-1})^{-1}\varphi_i, u)_X \\
&= (w, u)_Y + \sum_{i=1}^{n} \alpha_i ((\varphi_i, u)_X - (I + t^2S^{-1})^{-1}\varphi_i, u)_X \\
&= (w, u)_Y - \sum_{i=1}^{n} \alpha_i \bar{d}_i.
\end{aligned}$$

Thus

$$(w_{\mathcal{K}}, u)_Y = (w, u)_Y + t^{-2} \left\langle M_t^{-1} d, d \right\rangle. \tag{2.21}$$

Combining (2.16) , (2.17) and (2.21) completes the proof.

■

For $n = 1$, let $\mathcal{K} = span\{\varphi\}$ and denote $X_{\mathcal{K}}$ by X_φ. Then, for $u \in X_\varphi$, the formula (2.14) becomes

$$\|u\|^2_{[X_\varphi,Y]_s} = \|u\|^2_{[X,Y]_s} + \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} \frac{|(u,\varphi)_{X,t}|^2}{(\varphi,\varphi)_{X,t}}\, dt. \tag{2.22}$$

Next theorem gives sufficient conditions for (2.11) to be satisfied. Before we state the result we introduce the conditions:

(A.1) $[X_{\varphi_i}, Y]_s = [X,Y]_{s,\varphi_i} \quad \text{for } i = 1,\ldots,n.$

(A.2) There exist $\delta > 0$ and $\gamma > 0$ such that

$$\sum_{i=1}^n |\alpha_i|^2 (\varphi_i,\varphi_i)_{X,t} \le \gamma \langle M_t \alpha, \alpha\rangle \qquad \text{for all } \alpha = (\alpha_1,\ldots,\alpha_n)^{\mathbf{t}} \in \mathbf{C^n},\ \mathbf{t} \in (\delta,\infty).$$

Theorem II.2 *(Bacuta, Bramble, Pasciak) Assume that, for some $s \in (0,1)$, the conditions* **(A.1)** *and* **(A.2)** *hold. Then*

$$[X_{\mathcal{K}}, Y]_s = [X,Y]_{s,\mathcal{K}}.$$

Proof. Let s be fixed in $(0,1)$. Since $X_{\mathcal{K}}$ is dense in both these spaces, in order to prove (2.11) it is enough to find, for a fixed s, positive constants c_1 and c_2 such that

$$c_1\|u\|_{[X,Y]_s} \le \|u\|_{[X_{\mathcal{K}},Y]_s} \le c_2\|u\|_{[X,Y]_s} \quad \text{for all } u \in X_{\mathcal{K}}. \tag{2.23}$$

The function under the integral sign in (2.14) is nonnegative, so the lower inequality of (2.23) is satisfied with $c_1 = 1$. For the upper part, we notice that, for $u \in X_{\mathcal{K}}$ and

$w_{\mathcal{K}}$ as defined in the proof of Theorem II.1,

$$(w_{\mathcal{K}}, u)_Y = \left((I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u\right)_Y = (u, u)_Y - t^2 \left(S_{\mathcal{K}}^{-1}(I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u\right)_Y$$

$$\leq (u, u)_Y \leq c(s) \|u\|^2_{[X,Y]_s}$$

Then, using (2.17), (2.21) and the above estimate, we have that for any positive number δ,

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} \leq c(\delta, s) \|u\|^2_{[X,Y]_s} + \int_\delta^\infty t^{-2s+1} (w_{\mathcal{K}}, u)^2_Y \, dt$$

$$\leq c(\delta, s) \|u\|^2_{[X,Y]_s} + \int_\delta^\infty t^{-2s+1} (w, u)^2_Y \, dt + \int_\delta^\infty t^{-2s+1} (M_t^{-1} d, d) \, dt.$$

Hence the upper inequality of (2.23) is satisfied if one can find a positive δ and $c = c(\delta)$ such that

$$\int_\delta^\infty t^{-2s+1} (M_t^{-1} d, d) dt \leq c \|u\|^2_{[X,Y]_s} \quad \text{for all } u \in X_{\mathcal{K}}. \tag{2.24}$$

From **(A.2)**, there exist $\delta > 0$ and $\gamma > 0$ such that

$$\left\langle M_t^{-1} \alpha, \alpha \right\rangle \leq \gamma \sum_{i=1}^n |\alpha_i|^2 \, (\varphi_i, \varphi_i)^{-1}_{X,t}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)^{\mathbf{t}} \in \mathbf{C}^{\mathbf{n}}$, $t \in (\delta, \infty)$. In particular, for $\alpha_i = (u, \varphi_i)_{X,t}$, $i = 1, \ldots, n$, we obtain

$$\left\langle M_t^{-1} d, d \right\rangle \leq \gamma \sum_{i=1}^n \frac{|(u, \varphi_i)_{X,t}|^2}{(\varphi_i, \varphi_i)_{X,t}} \quad \text{for all } t \in (\delta, \infty), u \in X_{\mathcal{K}},$$

where $d = (d_1, \ldots, d_n)^{\mathbf{t}}$. Thus, using the above estimate, (2.22) and **(A.1)** we have

$$\begin{aligned} \int_\delta^\infty t^{-2s+1} \left\langle M_t^{-1} d, d \right\rangle dt &\leq \gamma \sum_{i=1}^n \int_\delta^\infty t^{-2s+1} \frac{\left|(u, \varphi_i)_{X,t}\right|^2}{(\varphi_i, \varphi_i)_{X,t}} dt \\ &\leq \gamma \sum_{i=1}^n \int_0^\infty t^{-2s+1} \frac{\left|(u, \varphi_i)_{X,t}\right|^2}{(\varphi_i, \varphi_i)_{X,t}} dt \\ &\leq \gamma c_s^{-2} \sum_{i=1}^n \|u\|^2_{[X_{\varphi_i}, Y]_s} \leq \gamma c_s^{-2} n \|u\|^2_{[X,Y]_s} \end{aligned}$$

Finally, (2.24) holds, and the result is proved.

∎

Remark II.3 *By Lemma II.2, the space $X_{\mathcal{K}}$ is dense in $[X, Y]_s$ if and only if the functionals L_φ, $\varphi \in \mathcal{K}, \varphi \neq 0$ are not bounded in the topology induced by $[X, Y]_s$.*

C. Some useful lemmas

Lemma II.3 *Let X, Y be separable Hilbert spaces and let X_0, Y_0 be closed subspaces of X, Y respectively. Let X_0 and Y_0 be equipped with the topology and the geometry induced by X and Y respectively, and assume that the pairs (X, Y) and (X_0, Y_0) satisfy* (2.1). *Then, for $s \in [0, 1]$,*

$$[X_0, Y_0]_s \subset [X, Y]_s \cap Y_0.$$

Proof. For any $u \in Y_0$ we have

$$K(t, u, X, Y) \leq K(t, u, X_0, Y_0).$$

Thus,

$$\|u\|_{[X,Y]_s} \leq \|u\|_{[X_0,Y_0]_s} \quad \text{for all } u \in [X_0, Y_0]_s, \quad s \in [0, 1], \tag{2.25}$$

which proves the lemma.

∎

For any two Banach spaces X and Y satisfying (2.1), the interpolating space $[X, Y]_s$ can be defined as in Section A by using the function $K = K(t, u, X, Y)$.

The next lemma will be used in Chapter VII.

Lemma II.4 *Let X, Y be Banach spaces satisfying (2.1), let X_1 be the space X equipped with an equivalent norm $\|\cdot\|_{X_1}$ and let Y_1 be the space Y equipped with an equivalent norm $\|\cdot\|_{Y_1}$. Then, for $s \in [0,1]$, we have*

$$[X_1, Y_1]_s = [X, Y]_s. \tag{2.26}$$

Proof. The proof follows directly from the definition of the function K and the definition of the interpolating spaces which appear in (2.26).

Lemma II.5 *Let $H^i, \tilde{H}^i$, $(i = 1, 2)$, be separable Hilbert spaces such that the pairs (H^2, H^1) , $(\tilde{H}^2, \tilde{H}^1)$ satisfy (2.1). We assume furthere that there are linear operators E and R such that*

$$E : H^i \to \tilde{H}^i \quad \textit{is a bounded operator}, \;\; i = 1, 2, \tag{2.27}$$

$$R : \tilde{H}^i \to H^i \quad \textit{is a bounded operator}, \;\; i = 1, 2, \tag{2.28}$$

$$REu = u \qquad \textit{for all } u \in H^2. \tag{2.29}$$

Then, for $s \in [0,1]$, an equivalent norm on $[H^2, H^1]_s$ is given by $\|E(\cdot)\|_{[\tilde{H}^2, \tilde{H}^1]_s}$, i.e., there are positive constants c_1 and c_2 such that

$$c_1 \|u\|_{[H^2,H^1]_s} \le \|Eu\|_{[\tilde{H}^2,\tilde{H}^1]_s} \le c_2 \|u\|_{[H^2,H^1]_s} \qquad \textit{for all } u \in [H^2, H^1]_s. \tag{2.30}$$

Proof. From (2.28), by interpolation, we obtain that for some positive constant c_1

$$c_1 \|Rv\|_{[H^2,H^1]_s} \le \|v\|_{[\tilde{H}^2,\tilde{H}^1]_s} \qquad \text{for all } v \in [\tilde{H}^2, \tilde{H}^1]_s\ . \tag{2.31}$$

For $u \in H^2$, let $v = Eu$. Then, using (2.29) and (2.31), we get

$$c_1\|u\|_{[H^2,H^1]_s} \leq \|Eu\|_{[\tilde{H}^2,\tilde{H}^1]_s} \qquad \text{for all } u \in H^2. \tag{2.32}$$

From the hypothesis (2.27), again by interpolation, we have that

$$\|Eu\|_{[\tilde{H}^2,\tilde{H}^1]_s} \leq c_2\|u\|_{[H^2,H^1]_s} \qquad \text{for all } u \in [H^2, H^1]_s, \tag{2.33}$$

for some positive constant c_2. Thus, from (2.32) and (2.33) we obtain that the inequalities involved in (2.30) are satisfied for any $u \in H^2$. By Remark II.1, H^2 is dense in $[H^2, H^1]_s$. Therefore, (2.30) holds and lemma is comletly proved.

■

Let $\Omega \subset \widetilde{\Omega}$ be domains in R^2 and $V^1(\Omega)$, $V^1(\widetilde{\Omega})$ be subspaces of $H^1(\Omega), H^1(\widetilde{\Omega})$, respectively. (On $V^1(\Omega)$, $V^1(\widetilde{\Omega})$ we consider inner products such that the induced norms are equivalent with the standard norms on $H^1(\Omega)$, $H^1(\widetilde{\Omega})$, respectively). In addition, we assume that $V^1(\Omega)$, $V^1(\widetilde{\Omega})$ are dense in $L^2(\Omega), L^2(\widetilde{\Omega})$, respectively. Let's denote the duals of $V^1(\Omega)$, $V^1(\widetilde{\Omega})$ by $V^{-1}(\Omega)$, $V^{-1}(\widetilde{\Omega})$, respectively. We suppose that there are linear operators E and R such that

$$E : L^2(\Omega) \to L^2(\widetilde{\Omega}), \quad E : V^1(\Omega) \to V^1(\widetilde{\Omega}) \ \text{ are bounded operators,} \tag{2.34}$$

$$R : L^2(\widetilde{\Omega}) \to L^2(\Omega), \quad R : V^1(\widetilde{\Omega}) \to V^1(\Omega), \ \text{ are bounded operators,} \tag{2.35}$$

$$REu = u \qquad \text{for all } u \in L^2(\Omega). \tag{2.36}$$

Let $\psi \in L^2(\Omega)$, $\widetilde{\psi} = E\psi \in L^2(\widetilde{\Omega})$ and $\theta \in (0, 1)$ be such that

$$L^2(\Omega)_\psi := \{u \in L^2(\Omega) : (u, \psi) = 0\} \ \text{ is dense in } \ [L^2(\Omega), V^{-1}(\Omega)]_\theta, \tag{2.37}$$

$$L^2(\widetilde{\Omega})_{\widetilde{\psi}} := \{u \in L^2(\widetilde{\Omega}) : (u, \widetilde{\psi}) = 0\} \ \text{ is dense in } \ V^{-1}(\widetilde{\Omega}), \tag{2.38}$$

$$[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta = [L^2(\widetilde{\Omega}), V^{-1}(\widetilde{\Omega})]_\theta. \tag{2.39}$$

Lemma II.6 *Using the above setting, assume that* (2.34)-(2.39) *are satisfied. Then,*

$$[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta = [L^2(\Omega), V^{-1}(\Omega)]_\theta. \tag{2.40}$$

Proof. Using the duality , from (2.34)-(2.36) we obtain linear operators E^*, R^* such that

$$E^* : L^2(\widetilde{\Omega}) \to L^2(\Omega), \quad E^* : V^{-1}(\widetilde{\Omega}) \to V^{-1}(\Omega), \quad \text{are bounded operators,} \tag{2.41}$$

$$R^* : L^2(\Omega) \to L^2(\widetilde{\Omega}), \quad R^* : V^{-1}(\Omega) \to V^{-1}(\widetilde{\Omega}) \quad \text{are bounded operators,} \tag{2.42}$$

$$E^*R^*u = u \qquad \text{for all } u \in L^2(\Omega), \tag{2.43}$$

$$E^* \ \text{ maps } \ L^2(\widetilde{\Omega})_{\widetilde{\psi}} \ \text{ to } \ L^2(\Omega)_\psi, \tag{2.44}$$

$$R^* \ \text{ maps } \ L^2(\Omega)_\psi \ \text{ to } \ L^2(\widetilde{\Omega})_{\widetilde{\psi}}. \tag{2.45}$$

From (2.41) and (2.44), by interpolation, we obtain

$$\|E^*v\|_{[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta} \le c\|v\|_{[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta} \qquad \text{for all } \ v \in L^2(\widetilde{\Omega})_{\widetilde{\psi}}. \tag{2.46}$$

For $u \in L^2(\Omega)_\psi$, let $v := R^*u$. Then, using (2.45), we have that $v \in L^2(\widetilde{\Omega})_{\widetilde{\psi}}$. Taking $v := R^*u$ in (2.46) and using (2.43), we get

$$\|u\|_{[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta} \le c\|R^*u\|_{[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{2.47}$$

Also, from the hypothesis (2.39), we deduce that

$$\|R^*u\|_{[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta} \le c\|R^*u\|_{[L^2(\widetilde{\Omega}), V^{-1}(\widetilde{\Omega}]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{2.48}$$

From (2.42), again by interpolation, we have in particular

$$\|R^*u\|_{[L^2(\widetilde{\Omega}), V^{-1}(\widetilde{\Omega}]_\theta} \le c\|u\|_{[L^2(\Omega), V^{-1}(\Omega)]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{2.49}$$

Combining (2.47)-(2.49), it follows that

$$\|u\|_{[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta} \le c\|u\|_{[L^2(\Omega), V^{-1}(\Omega)]_\theta} \qquad \text{for all} \;\; u \in L^2(\Omega)_\psi. \tag{2.50}$$

The reverse inequality of (2.50) holds because $L^2(\Omega)_\psi$ is a closed subspace of $L^2(\Omega)$. Thus, the two norms in (2.50) are equivalent for $u \in L^2(\Omega)_\psi$. From the assumption (2.37), $L^2(\Omega)_\psi$ is dense in both spaces appearing in (2.40). Therefore, we obtain (2.40).

■

Remark II.4 *The proof does not change if we consider $\Omega \subset \widetilde{\Omega}$ to be domains in R^n and H^1 is replaced by any other Sobolev space of positive integer order k.*

CHAPTER III

MULTILEVEL REPRESENTATIONS OF NORMS

Let Ω be a domain in R^2 and let $M_1 \subset M_2 \subset \ldots$ be a nested sequence of finite dimensional approximating subspaces of $H^1(\Omega)$ (or a subspace of $H^1(\Omega)$). Then, a natural norm can be considered on $H^1(\Omega)$ in terms of $L^2(\Omega)$-orthogonal projections onto M_k, $k = 1, 2, \ldots$. Under some special conditions, it turns out that the new norm is equivalent with the standard norm on $H^1(\Omega)$. These types of results have been studied for the first time in the finite element multilevel theory (see , e.g., [34]) by means of tools from approximation theory. Inspired more by the theory of preconditioning (as presented in [9], [14], [16]), this chapter provides complete proofs of the norm equivalence result for the special cases of polygonal or sector domains.

A. Some results from the multilevel theory

In this section we present some lemmas which are used through the rest of the chapter. A more complete presentation of multilevel theory can be found in [16].

Lemma III.1 *Let $\rho \in (0, 1)$ and let $\{l_{mn}\}$ be a double sequence of nonnegative real numbers satisfying*

$$l_{mn} \leq \rho^{|m-n|} \qquad \text{for all } m, n = 1, 2, \ldots .$$

Then, for any $a = \{a_n\}, b = \{b_n\} \in l_2$ with nonnegative entries, we have

$$\sum_{m,n=1}^{\infty} l_{mn} a_m b_n \leq \frac{1+\rho}{1-\rho} \|a\|_{l_2} \|b\|_{l_2} \ ,$$

where $\|\cdot\|_{l_2}$ denotes the norm on l_2.

Proof. By the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\sum_{m,n=1}^{\infty} l_{mn} a_m b_n &\le \sum_{n=1}^{\infty} a_n b_n + 2 \sum_{m>n=1}^{\infty} \rho^{m-n} a_m b_n \\ &\le \|a\|\|b\| + 2 \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \rho^k a_{n+k} b_n \\ &\le \|a\|\|b\| + 2\|a\|\|b\| \sum_{k=1}^{\infty} \rho^k = \frac{1+\rho}{1-\rho}\|a\|\|b\|.\end{aligned}$$

■

Lemma III.2 *Let M be a Hilbert space with inner product $(\cdot,\cdot)$, and let $\{M_k\}$ be a sequence of nested subspaces of M ($M_k \subset M_{k+1}, k = 1, 2, \ldots$). Denote by Q_k the orthogonal projections onto M_k and for any positive integer J let $B_J : M_J \to M_J$, $B_J := \sum_{k=1}^{J} \lambda_k^{-1} Q_k$, where $\lambda_k > 0$. Then B_J is a symmetric positive definite operator and B_J^{-1} exists and is characterized by*

$$(B_J^{-1}v, v) = \min \left\{ \sum_{k=1}^{J} \lambda_k \|v_k\|^2,\ v = \sum_{k=1}^{J} v_k,\ v_k \in M_k \right\},$$

where $\|\cdot\|$ is the norm induced by the inner product $(\cdot,\cdot)$.

Proof. Obviously, B_J is a symmetric operator. For any $v \in M_J$ we have

$$(B_J v, v) = \sum_{k=1}^{J} \lambda_k^{-1}(Q_k v, v) \ge \lambda_{max}^{-1} \sum_{k=1}^{J} (Q_k v, v) \ge \lambda_{max}^{-1}\|v\|^2.$$

Thus, B_J is a positive definite operator and a straightforward application of the Riesz Representation Theorem shows that B_J^{-1} exists with domain M_J. Now let $v \in M_J$, and let $v = \sum_{k=1}^{J} v_k$ be any decomposition of v such that $v_k \in M_k$. By the Cauchy-

Schwarz inequality we get

$$\begin{aligned}(B_J^{-1}v, v) &= \sum_{k=1}^{J}(B_J^{-1}v, v_k) = \sum_{k=1}^{J}(Q_k B_J^{-1}v, v_k) \le \sum_{k=1}^{J}(Q_k B_J^{-1}v, Q_k B_J^{-1}v)^{1/2}(v_k, v_k)^{1/2} \\ &= \sum_{k=1}^{J}\lambda_k^{-1/2}(Q_k B_J^{-1}v, B_J^{-1}v)^{1/2}\ \lambda_k^{1/2}(v_k, v_k)^{1/2} \\ &\le \left(\sum_{k=1}^{J}\lambda_k^{-1}(Q_k B_J^{-1}v, B_J^{-1}v)\right)^{1/2}\left(\sum_{k=1}^{J}\lambda_k(v_k, v_k)\right)^{1/2} \\ &= (v, B_j^{-1}v)^{1/2}\left(\sum_{k=1}^{J}\lambda_k(v_k, v_k)\right)^{1/2}.\end{aligned}$$

This leads us to the inequality

$$(B_J^{-1}v, v) \le \min\left\{\sum_{k=1}^{J}\lambda_k\|v_k\|^2,\ v = \sum_{k=1}^{J}v_k,\ v_k \in M_k\right\}.$$

Since the equality can be obtained by taking $v_k = \lambda_k^{-1}Q_k B_J^{-1}v$, the proof is complete.

■

Lemma III.3 *Assume that the hypotheses of Lemma III.2 are satisfied and that $\lambda_k < \rho\ \lambda_{k+1}$ for some number $\rho \in (0,1)$. Then, there is a constant c independent of J such that*

$$(B_J^{-1}v, v) \le \sum_{k=1}^{J}\lambda_k\|(Q_k - Q_{k-1})v\|^2 \le c\ (B_J^{-1}v, v), \qquad \textit{for all}\ \ v \in M_J. \tag{3.1}$$

Proof. Let $v \in M_J$ be fixed and define $v_k := (Q_k - Q_{k-1})v$. Clearly, $v_k \in M_k$ and $v = \sum_{k=1}^{J}v_k$. Thus, by Lemma III.2, we obtain the left part of (3.1). For the other part, let $v \in M_J$ and $v = \sum_{k=1}^{J}v_k$ be any decomposition of v such that $v_k \in M_k$. Then,

$$(Q_k - Q_{k-1})v = \sum_{i=k}^{J}(Q_k - Q_{k-1})v_i,$$

and

$$\|(Q_k - Q_{k-1})v\| \leq \sum_{i=k}^{J} \|(Q_k - Q_{k-1})v_i\| \leq \sum_{i=k}^{J} \|v_i\|.$$

Consequently,

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2 \leq \sum_{k=1}^{J} \left(\sum_{i=k}^{J} \lambda_k^{1/2} \|v_i\| \right)^2 = \sum_{k=1}^{J} \left(\sum_{i=k}^{J} \left(\frac{\lambda_k}{\lambda_i} \right)^{1/2} \lambda_i^{1/2} \|v_i\| \right)^2.$$

Defining Λ to be the symmetric $J \times J$ matrix with entries $\Lambda_{ik} = \left(\frac{\lambda_k}{\lambda_i}\right)^{1/2}$ for $i \geq k$, and $\vec{x}$ the $J-$dimensional vector with components $\lambda_i^{1/2}\|v_i\|$, we see that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2 \leq \|\Lambda \vec{x}\|_{l_2}^2 \leq \|\Lambda\|_{l_2}^2 \|\vec{x}\|_{l_2}^2.$$

By an easy application of Lemma III.1, $\|\Lambda\|_{l_2}$ can be bounded uniformly in J, and consequently, there is a constant c independent of J, v and the splitting of v, such that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})v\|^2 \leq c \sum_{k=1}^{J} \lambda_k \|v_k\|^2 \qquad \text{for all } \; v \in M_J.$$

The required inequality follows again by applying Lemma III.2.

■

B. Multilevel norm equivalence on H^1

Let Ω be a domain in R^2 with boundary $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$, where $(\partial\Omega)_D$ is not of measure zero, and $(\partial\Omega)_D$ and $(\partial\Omega)_N$ are essentially disjoint. Let $M = H_D^1(\Omega)$ denote the space of all functions in $H^1(\Omega)$ which vanish on $(\partial\Omega)_D$. Assume that

$$M_1 \subset M_2 \subset ... \subset M_J \subset ... \subset M,$$

is a nested sequence of finite dimensional approximation subspaces of M in the sense that $\cup_{k=1}^{\infty} M_k$ is dense in M. Since $H_D^1(\Omega)$ is dense in $L^2(\Omega)$ it follows that for $u \in L^2(\Omega)$, $\lim_{k\to\infty} Q_k u = u$. Let $(\cdot, \cdot)$ denote the $L^2(\Omega)$ inner product and let $\|\cdot\|$ be the norm on $L^2(\Omega)$ induced by $(\cdot,\cdot)$. For $k = 1, 2, \ldots$, we define the operator $P_k : M \to M_k$ to be the orthogonal projection with respect to the inner product $A(\cdot,\cdot)$, where

$$A(u, v) := \int_\Omega \nabla u \cdot \nabla v \; dx \qquad \text{for all } u, v \in M,$$

and $A_k : M_k \to M_k$ is defined by

$$(A_k u, v) = A(u, v) \quad \text{for all } u, v \in M_k.$$

Let μ_k be the largest eigenvalue of A_k and assume that the sequence $\{\mu_k\}$ is equivalent to $\{4^{k-1}\}$, i.e., there exist positive constants α_1, α_2 such that

$$\alpha_1 4^k \leq \mu_k \leq \alpha_2 4^k, \quad k = 1, 2, \ldots. \tag{3.2}$$

We denote 4^{k-1} by λ_k.

The goal of this section is to show that, under certain conditions on the sequence of subspaces $\{M_k\}$, we have the following.

(ML.0) There exist positive constants c_1 and c_2 such that

$$c_1 A(u, u) \leq \sum_{k=1}^{\infty} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq c_2 A(u, u) \qquad \text{for all } u \in M.$$

All the considerations of this section remain valid if we replace $\{\lambda_k\}$ by an equivalent sequence, for example $\{\mu_k\}$. In order to study the above norm equivalence we start by introducing the following conditions:

(ML.1) There is a positive constant c independent of j and k and a number $\rho \in (0, 1)$ such that

$$|A(u_k, u_j)| \leq c\rho^{|k-j|} A(u_k, u_k)^{1/2} A(u_j, u_j)^{1/2} \qquad \text{for all } u \in M.$$

where $u_k := q_k u := (Q_k - Q_{k-1})u$.

(ML.2) There exists c independent of J such that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq cA(u,u) \qquad \text{for all } u \in M_J.$$

(ML.3) There exists c independent of k and J such that

$$\|(I - P_{k-1})u\| \leq c\lambda_k^{-1/2} A(u,u)^{1/2}, \qquad \text{for all } u \in M_J.$$

Condition **(ML.1)** is known as a Strengthened Cauchy-Schwarz Condition. The next result gives some connection between the above conditions. The result is known in multigrid theory (see [16] or [15]). For completeness we provide the proof here also.

Proposition III.1 *(a) The norm equivalence* **(ML.0)** *holds whenever Condition* **(ML.1)** *and Condition* **(ML.2)** *are satisfied.*

(b) Condition **(ML.2)** *holds whenever Condition* **(ML.3)** *holds.*

Proof of part (a). Let $u \in M$ be fixed. Then $u = \sum_{k=1}^{\infty} u_k$, where $u_k = q_k u \in M_k$. From (3.2) we get that

$$A(u_k, u_k) \leq c\lambda_k \|u_k\|^2, \qquad \text{for all } \ k = 1, 2, \ldots. \tag{3.3}$$

Using Condition **(ML.1)**, we obtain

$$A(u,u) = \sum_{k,j=1}^{\infty} A(u_k, u_j) \leq c \sum_{k,j=1}^{\infty} \rho^{|k-j|} A(u_k, u_k)^{1/2} A(u_j, u_j)^{1/2}.$$

By Lemma III.1 and (3.3), we have

$$A(u,u) \leq c\frac{1+\rho}{1-\rho} \sum_{k=1}^{\infty} A(u_k, u_k) \leq c\frac{1+\rho}{1-\rho} \sum_{k=1}^{\infty} \lambda_k \|u_k\|^2,$$

which gives the lower inequality in **(ML.0)**. For the other inequality consider a sequence $\{u_J\}$ convergent to u in the $H^1_D(\Omega)$ norm, chosen such that $u_J \in M_J$. Then

Condition **(ML.2)** implies that for any positive integer N,

$$\sum_{k=1}^{N} \lambda_k \|(Q_k - Q_{k-1})u_J\|^2 \le cA(u_J, u_J),$$

where c is independent of N, J and u. Letting J to tend to infinity in the above inequality we have

$$\sum_{k=1}^{N} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \le cA(u, u) \qquad \text{for all } u \in M.$$

Since N was arbitrary, this justifies the validity of the upper inequality of **(ML.0)**.

Proof of part (b). First, Condition **(ML.3)** implies

$$\begin{aligned} \|(P_k - P_{k-1})u\|^2 &= \|(I - P_{k-1})(P_k - P_{k-1})u\|^2 \\ &\le c\lambda_k^{-1} A((P_k - P_{k-1})u, (P_k - P_{k-1})u)), \qquad \text{for all } u \in M_J, \end{aligned}$$

with c independent of k and J. Next, by applying Lemma III.2 and Lemma III.3 with $v = u$ and $v_k := (P_k - P_{k-1})u$, we deduce that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \le c \sum_{k=1}^{J} \lambda_k \|(P_k - P_{k-1})u\|^2, \qquad \text{for all } u \in M_J.$$

Combining the last two estimates, we have

$$\begin{aligned} \sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 &\le c \sum_{k=1}^{J} A((P_k - P_{k-1})u, (P_k - P_{k-1})u)) \\ &= c \sum_{k=1}^{J} A((P_k - P_{k-1})u, u) = cA(u, u), \qquad \text{for all } u \in M_J, \end{aligned}$$

with c independent of k and J. Therefore, Condition **(ML.2)** holds.

∎

If the domain Ω is nice enough, for example Ω is a convex polygonal domain and $\partial\Omega = (\partial\Omega)_D$ and $\{M_k\}$ is associated with a sequence of nested meshes on Ω, then the regularity condition **(ML.3)** holds. The above result suggests that, in order to prove Condition **(ML.2)**, we can have an overlapping domain decomposition of Ω such that on each subdomain Condition **(ML.3)** is satisfied. Then, we use the above proposition to obtain Condition **(ML.2)** relative to each subdomain. To get the result on the whole domain, one can use multiplicative Schwarz preconditioning type arguments. Next, we shall make this outline more precise.

Let $M_J = \sum_{i=0}^{n} M_J^i$ be a splitting of M_J associated with an overlapping domain decomposition of Ω

$$\Omega = \bigcup_{i=1}^{n} \Omega_i, \tag{3.4}$$

i.e.,

$$M_J^i = \{u \in M_J : supp(u) \subset \overline{\Omega_i}\}. \tag{3.5}$$

Let $Q_k^i : L^2(\Omega) \to M_k^i$, $P_k^i : M \to M_k^i$ be the orthogonal projections with respect to $(\cdot,\cdot)$ and $A(\cdot,\cdot)$, respectively. A useful condition about the splitting of M_J is

(ML.4) For each $u \in M_J$ there exists a partition $u = \sum_{i=0}^{n} u_i$, with $u_i \in M_J^i$, satisfying

$$\sum_{i=0}^{n} A(u_i, u_i) \leq cA(u, u),$$

where c is independent of J and $u \in M_J$.

Lemma III.4 *Assume that Condition* **(ML.4)** *is satisfied and that Condition* **(ML.3)** *holds on each subdomain, i.e.,*

$$\|(I - P_{k-1}^i)u_i\| \leq c\lambda_k^{-1/2} A(u_i, u_i)^{1/2}, \qquad \textit{for all } u_i \in M_J^i,\ i = 1, .., n, \tag{3.6}$$

for some constant c independent of J and i. Then Condition **(ML.2)** *is also satisfied.*

Proof. By Proposition III.1(b), there is a constant c independent of J such that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k^i - Q_{k-1}^i)u_i\|^2 \leq cA(u_i, u_i) \qquad \text{for all } u_i \in M_J^i. \tag{3.7}$$

For any $u \in M_J$ we consider the decomposition $u = \sum_{i=0}^{n} u_i$, with $u_i \in M_J^i$ given by Condition **(ML.4)**. Then,

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq n \sum_{k=1}^{J} \lambda_k \sum_{i=1}^{n} \|(Q_k - Q_{k-1})u_i\|^2$$
$$= n \sum_{i=1}^{n} \sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u_i\|^2.$$

Next, for each fixed i and $u_i \in M_J^i \subset M_J$ we have that

$$u_i = \sum_{k=1}^{J} (Q_k^i - Q_{k-1}^i)u_i,$$

where $(Q_k^i - Q_{k-1}^i)u_i \in M_k^i \subset M_k$. Thus, by applying Lemma III.2, Lemma III.3 and (3.7), we obtain that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u_i\|^2 \leq c \sum_{k=1}^{J} \lambda_k \|(Q_k^i - Q_{k-1}^i)u_i\|^2 \leq cA(u_i, u_i).$$

Combining the above estimates with Condition(**ML.4**), we have

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq cnA(u, u),$$

with c independent of J.

Therefore Condition **(ML.2)** is satisfied.

■

The last part of this section is concerned with the Strengthened Cauchy-Schwarz Condition **(ML.2)** for the case of polygonal domains.

Let Ω be a polygonal domain with with boundary $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$, as considered at the beginning of the section, and let $(\mathcal{T}_k)$ be a quasi-uniform sequence of nested triangulations of Ω such that the parameter h_k associated to $(\mathcal{T}_k)$ is $h_k \approx 2^{-k}$. For $k \geq 1$ the space M_k is defined to be the space of all functions which are piecewise linear with respect to $\mathcal{T}_k$, vanish on $(\partial\Omega)_D$ and are continuous on Ω. We denote the space $H^1_D(\Omega)$ by M.

Lemma III.5 *Under the above considerations, the Strengthened Cauchy-Schwarz Condition* **(ML.1)** *holds.*

Proof. Let $1 \leq k < j$ be fixed. For $\tau \in \mathcal{T}_k$, u, $v \in M$ we adopt the notation

$$D_\tau(u,v) := \int_\tau \nabla u \cdot \nabla v \; dx.$$

First, we prove the existence of a constant c, independent of k and j, such that

$$A(u,v) \leq c \left(h_k^{-1} A(u,u)\right)^{1/2} \left(h_j^{-1}\|v\|^2 + h_j A(v,v)\right)^{1/2}, \tag{3.8}$$

for all $u \in M_k$, $v \in M_j$.

Indeed, let $u \in M_k$, $v \in M_j$ and start by writing

$$A(u,v) = \sum_{\tau \in \mathcal{T}_k} D_\tau(u,v). \tag{3.9}$$

Using Green's formula and the fact that u is linear on τ we have

$$D_\tau(u,v) = \int_{\partial\tau} \frac{\partial u}{\partial n} v \; ds. \tag{3.10}$$

Since u is linear on τ we have, in addition,

$$\int_{\partial\tau} \left(\frac{\partial u}{\partial n}\right)^2 ds \leq c_1 h_k^{-1} D_\tau(u,u), \tag{3.11}$$

where c_1 is independent of k, j, τ and u. Further, by the trace inequality we get

$$\int_{\partial\tau} v^2 \; ds \leq c_2 \left(h_j^{-1}\|v\|^2_{L^2(\tau)} + h_j D_\tau(v,v)\right), \tag{3.12}$$

where c_2 is independent of k, j, τ and v. From (3.9)-(3.12) and the Cauchy-Schwarz inequality we obtain

$$A(u,v) \le c \sum_{\tau \in \mathcal{T}_k} \left(h_k^{-1} D_\tau(u,u)\right)^{1/2} \left(h_j^{-1}\|v\|_{L^2(\tau)}^2 + h_j D_\tau(v,v)\right)^{1/2},$$

where $c = \sqrt{c_1 c_2}$. Finally by applying the Cauchy-Schwarz inequality for vectors we conclude that (3.8) holds.

Next, let $u \in M$ and let $u_k = (Q_k - Q_{k-1})u,\ u_j = (Q_j - Q_{j-1})u$. Using standard estimates for the L^2 projections we have

$$\|u_j\|^2 = \|(I - Q_{j-1})u_j\|^2 \le c h_j^2 A(u_j, u_j),$$

where c is another constant independent of j and u. Taking $u = u_k$ and $v = u_j$ in (3.8) and combining with the above estimate we obtain

$$A(u_k, u_j) \le c \left(\frac{h_j}{h_k}\right)^{1/2} A(u_k, u_k) A(u_j, u_j).$$

Therefore, Condition **(ML.1)** holds with $\rho = 1/\sqrt{2}$.

■

C. The case of polygonal-sector domains

In this section we choose a specific sequence of approximating subspaces $\{M_k\}$ and verify that the sufficient conditions used in proving **(ML.0)** are satisfied for a particular type of polygonal domain. Though the considerations which follow might be true for a more complicated domain Ω, we restrict our study to a simple case when Ω is the polygonal-sector domain. We say that Ω is a polygonal-sector domain (see Figure 1) if

$$\overline{\Omega} = \bigcup_{i=1}^{n+1} \overline{\tau_i}, \tag{3.13}$$

where, for $i = 1, \ldots, n+1$, τ_i is a triangular domain with vertices $S_i,\ O,\ S_{i+1}$ and O is taken to be the origin of a Cartesian system of coordinates in the plane.

FIGURE 1. Polygonal-sector domain.

We assume, without loss of generality, that S_1 lies on the positive semi-axis. For $i = 1, \ldots, n+2$, let Γ_i denote the segment $[O, S_i]$, and for $i = 1, \ldots, n+1$, let α_i be the measure of the angle between Γ_i and Γ_{i+1}, and define the angle ω of Ω by

$$\omega := \sum_{i=1}^{n+1} \alpha_i.$$

For our results concerning interpolation, it is enough to consider only the cases $(\partial\Omega)_N = \emptyset$, $(\partial\Omega)_N = \Gamma_{n+2}$ or $(\partial\Omega)_N = \Gamma_1 \cup \Gamma_{n+2}$. Let $\mathcal{T}_1 = \{\tau_1, \ldots, \tau_n\}$ be the initial triangulation of Ω. We define multilevel triangulations recursively. For $k > 1$, the triangulation $\mathcal{T}_k$ is obtained from $\mathcal{T}_{k-1}$ by splitting each triangle in $\mathcal{T}_{k-1}$ into four triangles by connecting the midpoints of the edges. The space M_k is defined to be the space of all functions which are piecewise linear with respect to $\mathcal{T}_k$, vanish on $(\partial\Omega)_D$ and are continuous on Ω. Let Q_k denote the $L^2(\Omega)$ orthogonal projection onto M_k

and $\lambda_k = 4^{k-1}$.

In addition, for $i = 1, \ldots, n$, we define the subdomain Ω_i of Ω to be the domain made up by τ_i and τ_{i+1} ($\overline{\Omega}_i = \bar{\tau}_i \cup \bar{\tau}_{i+1}$), and define the subspaces M_k^i of M_k to be

$$M_k^i = \{u \in M_k : supp(u) \subset \overline{\Omega}_i\}, \quad k = 1, 2, \ldots.$$

Lemma III.6 *Let Ω be a polygonal-sector domain as defined above and assume that $(\partial\Omega)_N = \emptyset$ or $(\partial\Omega)_N = \Gamma_{n+2}$. Then the splitting $M_J = \sum_{i=0}^{n} M_J^i$ satisfies Condition* **(ML.4)**.

Proof. For $i = 2, \ldots, n+1$, let Ω^i be the polygonal-sector domain such that

$$\overline{\Omega^i} = \bigcup_{j=1}^{i-1} \overline{\tau_j}.$$

Then Γ_i is a part of $\partial\Omega^i$ (see Figure 1). We fix J, and for $u \in M_J$, we define $\gamma_i u$ to be the restriction of u to Γ_i. By standard results about traces of functions in H^1, we have

$$\gamma_i u \in H_{00}^{1/2}(\Gamma_i)$$

and

$$\|\gamma_i u\|_{H_{00}^{1/2}(\Gamma_i)} \leq c\|u\|_{H^1(\Omega^i)} \leq cA(u,u) \quad \text{for all} \quad u \in M. \tag{3.14}$$

Throughout the whole proof of this lemma, c is a constant independent on J, i, and it might be different at different occurrences. For $i = 2, \ldots, n$, we extend by zero $\gamma_i u$ to the rest of $\partial\tau_i$ and consider an extension of the new function to τ_i, denoted by $\tilde{u}_i$ and satisfying

$$\begin{cases} \tilde{u}_i \in M_J^{i-1,i} := \{v|_{\tau_i} : v \in M_J^{i-1}\}, \\ \\ |\tilde{u}_i|^2_{H^1(\tau_i)} \leq c\|\gamma_i u\|_{H_{00}^{1/2}(\Gamma_i)} \quad \text{for all} \quad u \in M_J. \end{cases} \tag{3.15}$$

For example, we can take $\tilde{u}_i$ to be the discrete harmonic extension of $\gamma_i u$ to τ_i.

Define $u_i \in M_J^i$ by

$$u_1(x) := \begin{cases} u(x) & \text{if } x \in \tau_1 \\ \tilde{u}_2(x) & \text{if } x \in \tau_2 \\ 0 & \text{if } x \in \Omega \backslash \Omega_1, \end{cases}$$

$$u_i(x) := \begin{cases} u(x) - \tilde{u}_i(x) & \text{if } x \in \tau_i \\ \tilde{u}_{i+1}(x) & \text{if } x \in \tau_{i+1} \\ 0 & \text{if } x \in \Omega \backslash \Omega_i, \end{cases}$$

for $i = 2, \ldots, n-1$ and

$$u_n(x) := \begin{cases} u(x) - \tilde{u}_n(x) & \text{if } x \in \tau_n \\ u(x) & \text{if } x \in \tau_{n+1} \\ 0 & \text{if } x \in \Omega \backslash \Omega_n. \end{cases}$$

Clearly, $u = u_1 + u_2 + \cdots + u_n$. Using (3.14), (3.15) and the Cauchy-Schwartz inequality we obtain that

$$A(u_i, u_i) \leq cA(u, u) \qquad \text{for all } u \in M_J, \quad i = 1, \ldots, n.$$

Therefore,

$$\sum_{i=1}^{n} A(u_i, u_i) \leq cnA(u, u) \qquad \text{for all } u \in M_J,$$

which concludes the validity of Condition **(ML.4)**.

■

Theorem III.1 *Let Ω be a polygonal-sector domain. Assume that $(\partial\Omega)_N = \emptyset$ or $(\partial\Omega)_N = \Gamma_{n+2}$ or $(\partial\Omega)_N = \Gamma_1 \cup \Gamma_{n+2}$. Assume that the angles of the polygon $\partial\Omega$, excepting ω, are not greater than π for those angles contained in $(\partial\Omega)_D$, and not greater than $\pi/2$ for the mixed angles with one edge in $(\partial\Omega)_D$ and the other edge in $(\partial\Omega)_N$. Let the sequence $\{M_k\}$ of subspaces of $H_D^1(\Omega)$ be as described before Lemma*

III.6. Then Condition **(ML.2)** *holds.*

Proof. First we consider the case when $(\partial\Omega)_N = \emptyset$ or $(\partial\Omega)_N = \Gamma_{n+2}$. In this case, by using the assumptions about the angles of $\partial\Omega$, and eventually by increasing the number n of subdomains, we have full regularity for the Laplace operator on each subdomain Ω_i. Thus (3.6) is satisfied (see, e.g., Theorem 2.3.7 in [24] and [16]). On the other hand, from Lemma III.6 the splitting $M_J = \sum_{i=0}^{n} M_J^i$ satisfies Condition **(ML.4)**. Thus, by Lemma III.4, Condition **(ML.2)** holds.

Next, we study the case $(\partial\Omega)_N = \Gamma_1 \cup \Gamma_{n+2}$. If ω is not greater than π, Condition **(ML.3)** is again fulfilled. According to Proposition III.1(b) we obtain that Condition **(ML.2)** holds. Let ω be greater than π and define $\hat{\Omega}$ to be the polygonal domain made up by completing Ω with $\tau_{n+2} := [S_{n+2}, O, S_1]$. Let $\partial\hat{\Omega}$ be the boundary of $\hat{\Omega}$, and define $(\partial\hat{\Omega})_N := \emptyset$ and $(\partial\hat{\Omega})_D := \partial\hat{\Omega}$. Assume, without loss of generality, that $\hat{\Omega}$ is a convex domain. Consider

$$\hat{\mathcal{T}}_1 := \{\tau_0, .., \tau_{n+1}, \tau_{n+2}\}.$$

Then we define the multilevel triangulation $\hat{\mathcal{T}}_k$ recursively in the same manner we defined $\mathcal{T}_k$. For $k = 1, 2, \ldots$, the space $\hat{M}_k$ is defined to be the space of all functions which are piecewise linear with respect to $\hat{\mathcal{T}}_k$, vanish on $(\partial\hat{\Omega})_D$ and are continuous on $\hat{\Omega}$. The $L^2(\hat{\Omega})$ orthogonal projection onto $\hat{M}_k$ is denoted by $\hat{Q}_k$.

We fix J and for $u \in M_J$ we denote by $\gamma_N u$ the restriction of u to $(\partial\Omega)_N$. Thus, we have

$$\gamma_N u \in H_{00}^{1/2}((\partial\Omega)_N)$$

and

$$\|\gamma_N u\|_{H_{00}^{1/2}(\Gamma_N)} \leq c\|u\|_{H^1(\Omega)} \leq cA(u,u) \quad \text{for all } \ u \in M_J, \tag{3.16}$$

where c is a constant independent of J, which might be different at different occurrences. The set $(\partial\Omega)_N$ is part of the boundary of τ_{n+2}. We extend $\gamma_N u$ by zero to the rest of $\partial\tau_{n+2}$ and consider an extension of the new function to τ_{n+2} denoted $\tilde{u}_{n+2}$ and satisfying

$$\begin{cases} \tilde{u}_{n+2} \in \hat{M}_J^{n+2} := \{v|_{\tau_{n+2}} : v \in \hat{M}_J\}, \\ \\ |\tilde{u}_{n+2}|^2_{H^1(\tau_{n+2})} \leq c\|\gamma_N u\|_{H^{1/2}_{00}(\Gamma_N)} \quad \text{for all} \;\; u \in M_J. \end{cases} \tag{3.17}$$

For example, we can take $\tilde{u}_{n+2}$ to be the discrete harmonic extension of $\gamma_N u$ to τ_{n+2}. Define $\hat{u} \in \hat{M}_J$ by

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \; \in \Omega \\ \tilde{u}_{n+2}(x) & \text{if } x \; \in \tau_{n+2}. \end{cases}$$

Using (3.16), (3.17) and the Cauchy-Schwarz inequality we obtain that

$$A(\hat{u}, \hat{u}) \leq cA(u, u) \qquad \text{for all} \;\; u \in M_J.$$

From Lemma III.2 and Lemma III.3, we obtain that

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq c \sum_{k=1}^{J} \lambda_k \|u_k\|^2 \qquad \text{for all} \;\; u \in M_J \text{ and} \tag{3.18}$$

for any partition of u,

$$u = \sum_{k=1}^{J} u_k, \quad \text{with} \;\; u_k \in M_k.$$

On the other hand, we have $\hat{u}|_\Omega = u$ and

$$\hat{u} = \sum_{k=1}^{J} (\hat{Q}_k - \hat{Q}_{k-1})\hat{u}.$$

The restrictions to Ω of functions in $\hat{M}_k$ are in M_k. Hence, we can take $u_k := ((\hat{Q}_k - \hat{Q}_{k-1})\hat{u})|_\Omega$ in (3.18). In addition, since $\hat{\Omega}$ is a convex domain and $(\partial\hat{\Omega})_N := \emptyset$. Condition **(ML.3)** is fulfilled for $\hat{\Omega}$. By Proposition III.1(b), we obtain that Condition **(ML.2)** holds on $\hat{\Omega}$. Then

$$\sum_{k=1}^{J} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq c \sum_{k=1}^{J} \lambda_k \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{u}\|^2_{L^2(\Omega)} \leq c \sum_{k=1}^{J} \lambda_k \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{u}\|^2_{L^2(\hat{\Omega})}$$
$$\leq cA(\hat{u}, \hat{u}) \leq cA(u, u),$$

for all u in M_J. Therefore we have proved that Condition **(ML.2)** also holds in this case and the proof of the theorem is complete .

■

The main result of this section is now easy to prove.

Theorem III.2 *Let Ω be a polygonal-sector domain with $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ satisfying the hypotheses of the previous theorem. Let $\{M_k\}$ be the sequence of subspaces of $H^1_D(\Omega)$ described before Lemma* (III.6). *Then* **(ML.0)** *holds.*

Proof. The Strengthened Cauchy-Schwarz Condition **(ML.1)** holds for our domain and the sequence $\{M_k\}$ of subspaces of $H^1_D(\Omega)$ given here (see Section B). Moreover, by the previous theorem, Condition **(ML.2)** also holds. Thus, the theorem follows from Proposition III.1.

■

D. The case of sector domains

Let Ω be a sector domain subdivided in say $n+1$ smaller sector domains. (See Figure 2). Then, (3.13) holds for τ_i being a triangle with one curved edge, $i = 1, \dots, n$.

Next we preserve the setting from the previous section except that the sequence of subspaces $\{M_k\}$ is differently defined. Again we consider $\mathcal{T}_1 = \{\tau_1, \dots, \tau_n\}$ to be the initial triangulation of Ω and define multilevel triangulations recursively as follows.

FIGURE 2. Triangulating a sector domain.

For $k > 1$, the triangulation $\mathcal{T}_k$ is obtained from $\mathcal{T}_{k-1}$ by splitting each triangle in $\mathcal{T}_{k-1}$ into four triangles by connecting the midpoints of the sides including the curved sides (as suggested in Figure 2 for τ_1). The space M_k is defined to be the space of all functions which are are continuous on Ω, piecewise linear with respect to $\mathcal{T}_k$, vanish on $(\partial\Omega)_D$ and in addition vanish on all the curved triangles of $\mathcal{T}_k$. The sequence of subspaces $\{M_k\}$ defined in this way remains a nested sequence of approximation subspaces of $H^1_D(\Omega)$ (see, e.g., [9]). We note that the parameter h_k associated with the quasi-uniform sequence of triangulations $\{\mathcal{T}_k\}$, satisfies

$$\lim_{k\to\infty} 2^k\ h_k = c > 0.$$

Thus, the assumption (3.2) about the eigenvalue μ_k of the operator A_k is satisfied. The proof of the Strengthened Cauchy-Schwarz Condition **(ML.1)** can be carried out in the same way we presented in Lemma III.5 for the case of polygonal domains.

Finally, the results presented in the previous section can be reproduced for the case of the sector domain and we can conclude that the multilevel norm associated with our sequence M_k provides an equivalent norm on $H^1_D(\Omega)$.

CHAPTER IV

SUBSPACE INTERPOLATION BY MULTILEVEL NORMS

Let Ω be a domain in R^2 with boundary $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$, where $(\partial\Omega)_D$ is not of measure zero, and $(\partial\Omega)_D$ and $(\partial\Omega)_N$ are essentially disjoint. Let $H^1_D(\Omega)$ denote the space of all functions in $H^1(\Omega)$ which vanish on $(\partial\Omega)_D$. Assume that

$$M_1 \subset M_2 \subset, \ldots, \subset M_k \subset \ldots$$

is a sequence of finite dimensional subspaces of $H^1_D(\Omega)$ whose union is dense in $H^1_D(\Omega)$, and assume that an equivalent norm on $H^1_D(\Omega)$ is given by

$$\|u\|_1^2 := \sum_{k=1}^{\infty} \lambda_k \|(Q_k - Q_{k-1})u\|^2, \tag{4.1}$$

where Q_k denotes the $L^2(\Omega)$ orthogonal projection onto M_k, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $Q_0 = 0$, and $\lambda_k = 4^{k-1}$. The goal of this chapter is to solve a codimension one subspace interpolation problem by means of multilevel geometry and topology. From the applications point of view, this is equivalent to providing sufficient conditions for the function factored out from $X = L^2(\Omega)$ in order to satisfy the Condition **(A1)** of Theorem II.2.

A. Scales of multilevel norms

On $H^1_D(\Omega)$ consider the norm given by (4.1) and define $H^{-1}_D(\Omega)$ to be the dual of $H^1_D(\Omega)$. Then, the elements of $L^2(\Omega)$ can be viewed as continuous linear functionals on $H^1_D(\Omega)$ and we have the natural continuous and dense embeddings

$$H^1_D(\Omega) \subset L^2(\Omega) \subset H^{-1}_D(\Omega).$$

One can easily check that

$$\|u\|_{-1}^2 := \sum_{k=1}^{\infty} \lambda_k{}^{-1} \|(Q_k - Q_{k-1})u\|^2 \qquad \text{for all } u \in L^2(\Omega), \tag{4.2}$$

where $\|\cdot\|_{-1}$ denotes the norm on $H_D^{-1}(\Omega)$. Further, we consider the inner product on $H_D^{\alpha}(\Omega)$ to be

$$(u,v)_\alpha := \sum_{k=1}^{\infty} \lambda_k{}^{\alpha} ((Q_k - Q_{k-1})u, v)_{L^2(\Omega)} \qquad \text{for all } u, v \in H_D^{\alpha}(\Omega) \cap L^2(\Omega),\ \alpha = -1, 1.$$

Then the pairs $(H_D^1(\Omega), L^2(\Omega))$ and $(L^2(\Omega), H_D^{-1}(\Omega))$ satisfy the condition (2.1) and the operator S associated with each of these pairs is given (in both cases) by

$$Su = \sum_{k=1}^{\infty} \lambda_k (Q_k - Q_{k-1})u, \qquad \text{for all } u \in D(S). \tag{4.3}$$

For any $\theta \in [0,1]$, let

$$H_D^{\theta}(\Omega) := [H_D^1(\Omega), L^2(\Omega)]_{1-\theta}, \qquad H_D^{-\theta}(\Omega) := [L^2(\Omega), H_D^{-1}(\Omega)]_{\theta},$$

and let $\|\cdot\|_\alpha$ be the norm on $H_D^{\alpha}(\Omega)$ for $\alpha \in [-1,1]$. By using (2.6), one can easily check that

$$\|u\|_\alpha^2 := \sum_{k=1}^{\infty} \lambda_k^{\alpha} \|(Q_k - Q_{k-1})u\|^2, \qquad \text{for all } u \in H_D^{\alpha}(\Omega) \cap L^2(\Omega). \tag{4.4}$$

Consequently, $H_D^{-\theta}(\Omega)$ is the dual of $H_D^{\theta}(\Omega)$ for $\theta \in [0,1]$.

Remark IV.1 *For any $\alpha \in (0,1]$, the norm on $H_D^{\alpha}(\Omega)$ is given by (4.4). On the other hand, for $u \in H_D^{\alpha}(\Omega)$,*

$$\sum_{k=1}^{J} \lambda_k^{\alpha} \|(Q_k - Q_{k-1})u\|^2 = \|u\|^2 + (4^\alpha - 1) \sum_{k=1}^{J-1} \lambda_k^{\alpha} \|(I - Q_k)u\|^2 - \lambda_J^{\alpha} \|(I - Q_J)u\|^2$$

and

$$\lim_{J \to \infty} \lambda_J^{\alpha} \|(I - Q_J)u\|^2 = 0.$$

Thus, we obtain that an equivalent norm on $H_D^\alpha(\Omega)$ is given by

$$|||u|||_\alpha^2 := \|u\|^2 + \sum_{k=1}^{\infty} \lambda_k^\alpha \|(I - Q_k)u\|^2.$$

B. Sufficient conditions for **(A1)**

Let $X = L^2(\Omega)$ and $Y = H_D^{-1}(\Omega)$. For a fixed θ_0 in the interval $(0, 1)$, let $\phi \in L^2(\Omega)$ satisfy the following conditions:

(C.0) $\phi \notin H_D^{\theta_0}(\Omega)$.

(C.1) There exist $c_1 > 0$ and $\delta > 0$ such that

$$(\phi, \phi)_{X,t} = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2} \|(Q_k - Q_{k-1})\phi\|^2 \geq c_1 t^{-2\theta_0}, \ \ for\ t \geq \delta.$$

(C.2) There exist $c_2 > 0$ such that

$$\|(Q_k - Q_{k-1})\phi\|^2 \leq c_2 \lambda_k^{-\theta_0}, \ k = 1, 2, \ldots.$$

Our goal in this section is to characterize the space $[X_\phi, Y]_\theta$ for θ in $(0, 1)$, $\theta \neq \theta_0$.

Remark IV.2 *From* **(C.2)** *it follows that $\phi \in H_D^\theta(\Omega)$ for $\theta < \theta_0$. Thus, from* **(C.0)** *and* **(C.2)***, by applying Lemma II.2 (see the proof of* (4.6)*), we have that X_ϕ is dense in Y. Consequently, the space $[X_\phi, Y]_s$ is well defined.*

Theorem IV.1 *Let $\phi \in L^2(\Omega)$ and satisfy* **(C.0)-(C.2)***. Then*

$$\left[L^2(\Omega)_\phi, H_D^{-1}(\Omega)\right]_\theta = \left[L^2(\Omega), H_D^{-1}(\Omega)\right]_{\theta,\phi}, \ 0 \leq \theta \leq 1, \ \theta \neq \theta_0. \tag{4.5}$$

Furthermore, if $\theta_0 < \theta \leq 1$ then

$$\left[L^2(\Omega)_\phi, H_D^{-1}(\Omega)\right]_\theta = \left[L^2(\Omega), H_D^{-1}(\Omega)\right]_\theta. \tag{4.6}$$

Proof. Let $\theta \neq \theta_0$ be fixed. Following the proof of Theorem II.2 until (2.24), we see that in order to prove (4.5), it is enough to show that, for δ given by **(C.1)**, there is a positive constant $c = c(\theta, \delta, c_1, c_2)$ so that we have

$$I := \int_{\delta}^{\infty} t^{-(2\theta+1)} \frac{|(u,\phi)_{X,t}|^2}{(\phi,\phi)_{X,t}} dt \le c\|u\|_{-\theta}^2 \qquad \text{for all } u \in X_\phi. \tag{4.7}$$

Let $u \in X = L^2(\Omega)$ be fixed. Denote $Q_k - Q_{k-1}$ by q_k, with $Q_0 = 0$, and for $u \in L^2(\Omega)$ denote

$\tilde{u}_k := \lambda_k^{-\theta/2}\|q_k u\|$ and $\tilde{u} := \{u_k\}$. Then we have

$$\|u\|_{-\theta} = \|\tilde{u}\|_{l_2}.$$

Here $(\cdot,\cdot)_X$ is simply the $L^2(\Omega)$ inner product $(\cdot,\cdot)$. Then, we have

$$(u,\phi)_{X,t} = \big((I + t^2 S^{-1})^{-1}u, \phi\big) = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2}(q_k u, q_k \phi).$$

Using the Cauchy-Schwarz inequality and the estimate given by **(C.2)** we obtain

$$|(u,\phi)_{X,t}| \le c_2 \sum_{k=1}^{\infty} \frac{\lambda_k^{1-\theta_0/2}}{\lambda_k + t^2}\|q_k u\|. \tag{4.8}$$

For $u \in X_\phi$ we have $(u,\phi) = 0$. Then

$$\sum_{k=1}^{\infty}(q_k u, \phi) = 0.$$

Thus,

$$(u,\phi)_{X,t} = -t^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k + t^2}(q_k u, q_k \phi),$$

and hence we also have the estimate

$$|(u,\phi)_{X,t}| \le c_2 t^2 \sum_{k=1}^{\infty} \frac{\lambda_k^{-\theta_0/2}}{\lambda_k + t^2}\|q_k u\|. \tag{4.9}$$

Now we are prepared to estimate the integral I. The constant c, to be used next, may have different values at different places in which it appears but depends only on the constants θ, δ, c_1 and c_2 . First we will treat the case $0 < \theta < \theta_0$. Let $\theta_1 = \theta_0 - \theta$.

Then, by **(C.1)** and the estimate (4.8), we have

$$\begin{aligned} I &\le c\int_\delta^\infty t^{-1+2\theta_1}\left(\sum_{k=1}^\infty \frac{\lambda_k^{1-\theta_0/2}}{\lambda_k+t^2}\|q_k u\|\right)^2 dt \\ &\le c\int_\delta^\infty t^{-1+2\theta_1}\left(\sum_{m,n=1}^\infty \frac{(\lambda_m\lambda_n)^{1-\theta_0/2}}{(\lambda_m+t^2)(\lambda_n+t^2)}\|q_m u\|\|q_n u\|\right) dt \\ &= c\sum_{m,n=1}^\infty (\lambda_m\lambda_n)^{1-\theta_0/2}\|q_m u\|\|q_n u\|\int_\delta^\infty \frac{t^{-1+2\theta_1}}{(\lambda_m+t^2)(\lambda_n+t^2)}\, dt. \end{aligned}$$

Next, we use the formula

$$\int_0^\infty \frac{t^{3-2\theta}}{(a+t^2)(b+t^2)}dt = \frac{1}{\mathbf{c}_\theta^2}\frac{a^{1-\theta}-b^{1-\theta}}{a-b}, \quad 0<\theta<2, \quad \theta\neq 1, \quad a,b>0. \tag{4.10}$$

The integral can be calculated by elementary calculus methods. If $a=b$, then the right side of the above identity is replaced by $\frac{1-\theta}{\mathbf{c}_\theta^2}a^{-\theta}$. Thus,

$$\int_\delta^\infty \frac{t^{-1+2\theta_1}}{(\lambda_m+t^2)(\lambda_n+t^2)}\, dt \le \int_0^\infty \frac{t^{-1+2\theta_1}}{(\lambda_m+t^2)(\lambda_n+t^2)}\, dt = \mathbf{c}_{\theta_1}^{-2}(\lambda_m\lambda_n)^{\theta_1-1}\frac{\lambda_m^{1-\theta_1}-\lambda_n^{1-\theta_1}}{\lambda_m-\lambda_n}.$$

Combining the above inequalities, we get

$$I \le c\sum_{m,n=1}^\infty (\lambda_m\lambda_n)^{\theta_1/2}\ \frac{\lambda_m^{1-\theta_1}-\lambda_n^{1-\theta_1}}{\lambda_m-\lambda_n}\lambda_m^{-\theta/2}\|q_m u\|\lambda_n^{-\theta/2}\|q_n u\|.$$

Let

$$l_{mn} = (\lambda_m\lambda_n)^{\theta_1/2}\ \frac{\lambda_m^{1-\theta_1}-\lambda_n^{1-\theta_1}}{\lambda_m-\lambda_n}.$$

Then, the above estimate becomes

$$I \le c\sum_{m,n=1}^\infty l_{mn}\tilde{u}_m\tilde{u}_n.$$

An elementary calculation gives

$$l_{mn} = \frac{2^{(m-n)(1-\theta_1)}-2^{-(m-n)(1-\theta_1)}}{2^{(m-n)}-2^{-(m-n)}} \le\ 2^{-|m-n|\theta_1}, \quad m,n=1,2,\ldots.$$

Now we can apply Lemma III.1 and obtain

$$I \leq c\|\tilde{u}\|_{l_2}^2 = c\| \ u\|_{-\theta}^2,$$

which proves (4.7) in this case.

For the remaining part, i.e., $\theta_0 < \theta < 1$, we set $\theta_1 := \theta - \theta_0$. The estimate (4.7) can be done in the same manner. The only difference here is that we use the inequality (4.9) instead of (4.8). This completes the proof of (4.6).

Now let $\theta_0 < \theta \leq 1$ be fixed. By the previous part, it is enough to show that $L^2(\Omega)_\phi$ is dense in $H_D^{-\theta}(\Omega)$. Using Lemma II.2, this is equivalent to proving that the functional

$$u \to (u, \phi), \quad u \in L^2(\Omega), \tag{4.11}$$

is not continuous in the topology induced by $H_D^{-\theta}(\Omega)$. To see that, let $\{u_n\}$ be the sequence in $L^2(\Omega)$ defined by

$$u_n := \sum_{k=1}^{n} \lambda_k^{\theta_0} \ q_k\phi.$$

From **(C.0)** we have that

$$(u_n, \phi) = \sum_{k=1}^{n} \lambda_k^{\theta_0} \|q_k\phi\|^2 \to \infty,$$

as $n \to \infty$. On the other hand, using **(C.2)**

$$(u_n, u_n)_{-\theta} = \sum_{k=1}^{n} \lambda_k^{-\theta+2\theta_0} \|q_k\phi\|^2$$

is uniformly bounded. Therefore, the functional defined in (4.11) is not continuous and (4.6) is proved.

■

CHAPTER V

INTERPOLATION BETWEEN SUBSPACES OF $H^{\beta}(R^N)$ AND $H^{\alpha}(R^N)$.

In this chapter we give a simplified proof of the main interpolation result presented in [25]. An extension to the case when the subspace of interpolation has finite codimension bigger than one is also considered.

Let $\alpha \in R$ and consider the Sobolev space $H^{\alpha}(R^N)$ defined by means of the Fourier transform. For a smooth function u with compact support in R^N, the Fourier transform $\hat{u}$ is defined by

$$\hat{u}(\xi) = (2\pi)^{-N/2} \int u(x) e^{-ix\xi}\, dx,$$

where the integral is taken over the whole R^N. For u and v smooth functions the α -inner product is defined by

$$(u, v)_{\alpha} = \int (1 + |\xi|^2)^{\alpha}\ \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi.$$

The space $H^{\alpha}(R^N)$ is the closure of smooth functions in the norm induced by the α -inner product. For $\alpha,\ \beta$ real numbers ($\alpha < \beta$), and $s \in [0, 1]$ it is easy to check, using Remark II.1, that

$$\left[H^{\beta}(R^N), H^{\alpha}(R^N)\right]_s = H^{s\alpha+(1-s)\beta}(R^N).$$

For $\varphi \in H^{\beta}(R^N)$, we are interested in determining the validity of the formula

$$\left[H^{\beta}_{\varphi}(R^N), H^{\alpha}(R^N)\right]_s = \left[H^{\beta}(R^N), H^{\alpha}(R^N)\right]_{s,\varphi}. \tag{5.1}$$

For certain functions φ the problem is studied by Kellogg in [25]. The goal of this section is to give a new proof of Kellogg's result concerning (5.1) and to extend it to the case when $H^{\beta}_{\varphi}(R^N)$ is replaced by a subspace of finite codimension. First, we consider the case when $0 = \alpha < \beta$. The operator S, associated with the pair $X = H^\beta(R^N)$, $Y = H^0(R^N) = L^2(R^N)$, is given by

$$Su = \mu^{2\beta} u, \quad u \in D(S) = H^{2\beta}(R^N),$$

where $\mu(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in R^N$. For the remaining part of this chapter, H^β denotes the space $H^\beta(R^N)$ and $\hat{H}^\beta$ is the space $\{\hat{u} \mid u \in H^\beta\}$. For $\hat{u}$, $\hat{v} \in \hat{H}^\beta$, we define the inner product and the norm by

$$(\hat{u}, \hat{v})_\beta = \int \mu^{2\beta} \hat{u}\overline{\hat{v}}\, d\zeta, \quad ||\hat{u}||_\beta = (\hat{u}, \hat{u})_\beta^{1/2}.$$

To simplify the notation, we denote the the inner product $(\cdot,\cdot)_0$ and the norm $||\cdot||_0$ on H^0 or $\hat{H}^0$ simply by $(\cdot,\cdot)$ and $||\cdot||$, respectively.

Let $\phi \in \hat{H}^\beta$ be such that for some constants $\epsilon > 0$ and $c > 0$,

$$\begin{cases} |\phi(\xi) - b(\omega)\rho^{-\frac{N}{2}-2\beta+\alpha_0}| < c\rho^{-\frac{N}{2}-2\beta+\alpha_0-\epsilon} & \text{for all } \rho > 1 \\ 0 < \alpha_0 < \beta, \end{cases} \tag{5.2}$$

where $\rho \geq 0$ and $\omega \in S^{N-1}$ (the unit sphere of R^N) are the spherical coordinates of $\xi \in R^N$, and where $b(\omega)$ is a bounded measurable function on S^{N-1}, which is non zero on a set of positive measure.

Remark V.1 *From the assumption* (5.2) *about* ϕ *we have that*

$$\phi \in H^{2\beta-\alpha} \quad \textit{if and only if} \quad \alpha_0 < \alpha < \beta, \tag{5.3}$$

and by using Lemma II.2, we find that

$$H_\phi^\beta \textit{ is dense in } H^\alpha \quad \textit{if and only if} \quad \alpha \leq \alpha_0. \tag{5.4}$$

Theorem V.1 *(Kellogg) Let* $\varphi \in H^\beta$ *be such that its Fourier transform* ϕ *satisfies*

(5.2), *and let* $\theta_0 = \alpha_0/\beta$. *Then*

$$\left[H^\beta_\varphi, H^0\right]_s = \left[H^\beta_\varphi, H^0\right]_{s,\varphi}, \quad 0 \le s \le 1,\ 1-s \neq \theta_0, \tag{5.5}$$

Proof. Following the proof of Theorem II.2, we see that in order to prove (5.5), it is enough to verify (2.24) for some positive constant $c = c(s)$ and δ. Using (2.22), the problem reduces to

$$\int_\delta^\infty t^{-(2s+1)} \frac{\left|(u,\phi)_{X,t}\right|^2}{(\phi,\phi)_{X,t}} dt \le c\|u\|^2_{[X,Y]_s} \qquad \text{for all } u \in X_\phi\ ,$$

where $X = \hat{H}^\beta$ and $Y = \hat{H}^0 = L^2(R^N)$. Denoting $1 - s = \theta$ and $\Phi(t) = (\phi,\phi)_{X,t}$, this becomes

$$I := \int_\delta^\infty t^{2\theta-3} \frac{\left|\left(\frac{\mu^{4\beta}u}{\mu^{2\beta}+t^2}, \phi\right)\right|^2}{\left(\frac{\mu^{4\beta}\phi}{\mu^{2\beta}+t^2}, \phi\right)} dt \le c\|u\|^2_{\theta\beta} \qquad \text{for all } u \in \hat{H}^\beta_\phi. \tag{5.6}$$

Using (5.2) it is easy to see that, for a large enough $\delta \ge 1$

$$\left(\frac{\mu^{4\beta}\phi}{\mu^{2\beta}+t^2}, \phi\right) \ge ct^{2(\theta_0-1)} \qquad \text{for all } \ t \ge \delta, \tag{5.7}$$

and (5.2) also implies that

$$|\phi(\xi)| < c|\rho|^{-\frac{N}{2}-2\beta+\alpha_0-\epsilon} \quad \text{for} \quad |\xi| \ > \ 1. \tag{5.8}$$

Before we start estimating I, let us observe that by using spherical coordinates

$$\|u\|^2_{\theta\beta} = \int_0^\infty U^2(\rho)\ d\rho, \qquad u \in \hat{H}^\beta_\phi\ , \tag{5.9}$$

where

$$U(\rho) := \mu(\rho)^{\theta\beta} \rho^{\frac{N-1}{2}} \left(\int\limits_{|\xi|=1} |u(\rho,\omega)|^2\ d\omega \right)^{1/2}, \quad \mu(\rho) = (1+\rho^2)^{1/2}.$$

First, we consider the case $0 < \theta < \theta_0$ and set $\theta_1 := \theta_0 - \theta$. For $u \in \hat{H}^\beta_\phi$ we have

$$\left|\left(\frac{\mu^{4\beta} u}{\mu^{2\beta} + t^2}, \phi\right)\right|^2 = t^4 \left|\left(\frac{\mu^{2\beta} u}{\mu^{2\beta} + t^2}, \phi\right)\right|^2.$$

Thus, by this observation and (5.7) we get

$$I \leq c \int_\delta^\infty t^{3-2\theta_1} \left(\int \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta} + t^2} |u(\xi)\phi(\xi)| \, d\xi \right)^2 \, dt.$$

Then,

$$I_1 = \int_\delta^\infty t^{3-2\theta_1} \Bigg(\int\limits_{|\xi|<1} \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta} + t^2} |u(\xi)\phi(\xi)| \, d\xi \Bigg)^2 dt$$

$$\leq c \int_\delta^\infty \frac{t^{3-2\theta_1}}{t^4} \Bigg(\int\limits_{|\xi|<1} |u(\xi)\phi(\xi)| \, d\xi \Bigg)^2 dt \leq c \int_\delta^\infty t^{-(1+2\theta_1)} \, dt \; \|u\|^2 \; \|\phi\|^2 \leq \; c(\theta) \|u\|^2_{\theta\beta}.$$

On the other hand, by Fubini's theorem, we have

$$I_2 = \int_\delta^\infty t^{3-2\theta_1} \Bigg(\int\limits_{|\xi|>1} \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta} + t^2} |u(\xi)\phi(\xi)| \, d\xi \Bigg)^2 dt$$

$$= \int_\delta^\infty t^{3-2\theta_1} \Bigg(\int\limits_{|\xi|>1} \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta} + t^2} |u(\xi)\phi(\xi)| \, d\xi \Bigg) \Bigg(\int\limits_{|\eta|>1} \frac{\mu(\eta)^{2\beta}}{\mu(\eta)^{2\beta} + t^2} |u(\eta)\phi(\eta)| \, d\eta \Bigg) dt$$

$$= \int\limits_{|\xi|>1} \int\limits_{|\eta|>1} |u(\xi)u(\eta)\phi(\xi)\phi(\eta)| \, \big(\mu(\xi)\mu(\eta)\big)^{2\beta} \int_\delta^\infty \frac{t^{3-2\theta_1}}{\big(\mu(\xi)^{2\beta} + t^2\big)\big(\mu(\eta)^{2\beta} + t^2\big)} \, dt \, d\eta \, d\xi,$$

and by using (4.10), the estimate (5.8) and spherical coordinates $\xi = (\rho, \omega)$, $\eta = (r, \rho)$, we obtain

$$I_2 \leq c(\theta) \int_\delta^\infty \int_\delta^\infty (\mu(r)\mu(\rho))^{2\beta-\theta} (r\rho)^{-\frac{1}{2}-2\beta+\alpha_0} R_{1-\theta_1}(\mu(r)^{2\beta}, \mu(\rho)^{2\beta}) U(r)U(\rho) \, d\rho \, dr,$$

where for $\alpha \in (0,1)$, $x > 0$, $y > 0$, we denote

$$R_\alpha(x,y) = \begin{cases} \frac{x^\alpha - y^\alpha}{x-y}, & for\ x \neq y \\ \alpha x^{\alpha-1}, & for\ x = y. \end{cases}$$

The function $x \to R_\alpha(x,y)$ is decreasing on $(0,\infty)$ for each $y \in (0,\infty)$ and it is symmetric with respect to x and y.

Using this observation, we get

$$I_2 \leq c(\theta) \int_\delta^\infty \int_\delta^\infty (r\rho)^{-\frac{1}{2}+\beta\theta_1} R_{1-\theta_1}(r^{2\beta},\ \rho^{2\beta})\ U(\rho)U(r)\ \ dr\ d\rho$$

$$\leq c(\theta) \int_0^\infty \int_0^\infty K(r,\rho) U(r) U(\rho)\ dr\ d\rho,$$

where

$$K(r,\rho) = (r\rho)^{-\frac{1}{2}+\beta\theta_1} R_{1-\theta_1}(r^{2\beta},\ \rho^{2\beta}). \tag{5.10}$$

In order to estimate the last integral, we apply the following lemma.

Lemma V.1 *(Schur) Suppose $K(x,y)$ is nonnegative, symmetric and homogeneous of degree -1, and f, g are nonnegative measurable functions on $(0,\infty)$. Assume that*

$$k = \int_0^\infty K(1,x) x^{-\frac{1}{2}}\ dx\ <\ \infty.$$

Then

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y)\ dx\ dy \leq k \left(\int_0^\infty f(x)^2\ dx \right)^{\frac{1}{2}} \left(\int_0^\infty g(y)^2\ dy \right)^{\frac{1}{2}}. \tag{5.11}$$

We will prove this lemma later. For the moment, we see that the function $K(x,y)$, given by (5.10), is homogeneous of degree -1, and satisfies

$$k = \int_0^\infty K(x,1) x^{-\frac{1}{2}}\ dx\ <\ \infty.$$

Indeed

$$k = \int_0^\infty x^{-1+\beta\theta_1} \frac{x^{2\beta(1-\theta_1)} - 1}{x^{2\beta} - 1}\, dx \overset{x^\beta = t}{=} \beta \int_0^\infty \frac{t^{1-\theta_1} - t^{\theta_1 - 1}}{t^2 - 1}\, dt \ < \ \infty, \ for\ 0 < \theta_1 < 1.$$

By Lemma V.1,

$$I_2 \le c(\theta) \int_0^\infty U^2(\rho)\, d\rho \ \le \ c(\theta)||u||^2_{\beta\theta}$$

and by combining the estimates I_1 and I_2, we obtain (5.6).

Let us consider now the case $\theta_0 < \theta < 1$, and let $\theta_1 = \theta - \theta_0$. Then, by using (5.7), we have

$$I \le c \int_\delta^\infty t^{2\theta_1 - 1} \left(\int \frac{\mu(\xi)^{4\beta}}{\mu(\xi)^{2\beta} + t^2} |u(\xi)\phi(\xi)|\, d\xi \right)^2 dt.$$

The remaining part of the proof is very similar to the proof of the first case. The theorem is proved.

∎

Proof of Lemma V.1. By Fubini's theorem, it follows

$$\begin{aligned}
\int_0^\infty \int_0^\infty K(x,y) f(x) g(y)\, dx\, dy \ &= \ \int_0^\infty f(x) \left(\int_0^\infty K(x,y) g(y)\, dy \right) dx \\
&= \int_0^\infty f(x) \int_0^\infty x K(x, xt) g(xt)\, dt\, dx = \int_0^\infty f(x) \int_0^\infty K(1,t) g(xt)\, dt\, dx \\
&= \int_0^\infty K(1,t) \int_0^\infty f(x) g(xt)\, dx\, dt \\
&\le \int_0^\infty K(1,t) \left(\int_0^\infty f(x)^2\, dx \right)^{\frac{1}{2}} \left(\int_0^\infty g(xt)^2\, dx \right)^{\frac{1}{2}} dt \\
&\le \int_0^\infty K(1,t) t^{-\frac{1}{2}}\, dt \left(\int_0^\infty f(x)^2\, dx \right)^{\frac{1}{2}} \left(\int_0^\infty g(x)^2\, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

∎

Next we prepare for the generalization of the previous result.

Let $\phi_1, \phi_2, \ldots, \phi_n \in \hat{H}^\beta(R^N)$ such that for some constants $\epsilon > 0$ and $c > 0$ we have

$$\begin{cases} |\phi_i(\xi) - \tilde{\phi}_i(\xi)| < c\rho^{-\frac{N}{2}-2\beta+\alpha_i-\epsilon} \; for \; |\xi| > 1 \\ 0 < \alpha_i < \beta, \; i = 1, \ldots, n, \end{cases} \tag{5.12}$$

where

$$\tilde{\phi}_i(\xi) = b_i(\omega)\rho^{-\frac{N}{2}-2\beta+\alpha_i}, \; \xi = (\rho, \omega),$$

and $b_i(\cdot)$ is a bounded measurable function on S^{N-1}, which is non zero on a set of positive measure.

Define

$$\Phi_{ij}(t) = \left(\frac{\mu^{4\beta}\phi_i}{\mu^{2\beta}+t^2}, \phi_j\right), \quad \tilde{\phi}_{ij}(t) = \left(\frac{|\xi|^{4\beta}\tilde{\phi}_i}{|\xi|^{2\beta}+t^2}, \tilde{\phi}_j\right), \quad \theta_i = \frac{\alpha_i}{\beta},$$

$$[\tilde{\Phi}_i, \tilde{\phi}_j] := \frac{1}{\beta}(b_i, b_j)_\sigma \int_0^\infty \frac{x^{\theta_i} x^{\theta_j}}{x(x^2+1)} \; dx, \; i, \; j \; = 1, 2, \ldots, n,$$

where $(\cdot,\cdot)_\sigma$ is the inner product on $L^2(S^{N-1})$.

Clearly, $[\cdot,\cdot]$ is an inner product on $span\{\tilde{\phi}_i \mid i = 1, 2, \ldots, n\}$.

Lemma V.2 *With the above setting we have*

$$\tilde{\Phi}_{ij}(t) = [\tilde{\phi}_i, \tilde{\phi}_j] t^{\theta_i+\theta_j-2} \tag{5.13}$$

$$|\Phi_{ij}(t) - \tilde{\Phi}_{ij}(t)| \leq ct^{\theta_i+\theta_j-2-\eta} \; , \; t > \delta, \tag{5.14}$$

for some constants $c > 0$, $\eta > 0$ *and* $\delta \geq 1$.

Proof. By using spherical coordinates, we have

$$\tilde{\Phi}_{ij}(t) = \int \frac{|\xi|^{4\beta}}{|\xi|^{2\beta}+t^2} \tilde{\phi}_i \overline{\tilde{\phi}_j} \; d\xi = \int_0^\infty \frac{\rho^{\alpha_i+\alpha_j-1}}{\rho^{2\beta}+t^2} \; d\rho \int\limits_{|\xi|=1} b_i(\omega)\overline{b_j(\omega)} \; d\omega.$$

The change of variable $\rho^\beta = tx$ in the first integral completes the proof of (5.13). The proof of (5.14) is straightforward.

∎

Theorem V.2 *(Bacuta, Bramble, Pasciak) Let* $\varphi_1, \varphi_2, \ldots, \varphi_n \in H^\beta$ *be such that the corresponding Fourier transforms* $\phi_1, \phi_2, \ldots, \phi_n$ *satisfy* (5.12) *and in addition, the functions* $\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n$ *are linearly independent.*

Let $\mathcal{K} = span\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$*. Then*

$$[H^\beta_{\mathcal{K}}, H^0]_s \ = \ [H^\beta, H^0]_{s,\mathcal{K}} \ , \ (1-s)\beta \neq \alpha_i, \ for \ i = 1, 2, \ldots, n.$$

Proof. We apply the Theorem II.2 for $X = H^\beta$, $Y = H^0$, $\mathcal{K} = span\{\varphi_1, \ldots, \varphi_n\}$ and s such that $(1-s)\beta \neq \alpha_i$, $i = 1, 2, \ldots, n$. By using the hypothesis (5.12) and Theorem V.1, we get

$$[H^\beta_{\varphi_i}, H^0]_s \ = \ [H^\beta, H^0]_{s,\varphi_i}, \ for \ i = 1, 2, \ldots, n.$$

So **(A1)** is satisfied. In order to verify the condition **(A2)**, we first observe that $(M_t)_{ij} = \Phi_{ij}(t)$. By denoting $D_t = diag(M_t)$, the condition **(A2)** can be written as follows:

There are $\delta > 0$ and $\gamma > 0$ such that

$$M_t - \gamma D_t \geq 0, \quad for \ all \ t \in (\delta, \infty),$$

where for a square matrix A, $A \geq 0$ means that A is a nonnegative definite matrix. From the previous lemma we obtain the behavior of $(M_t)_{ij}$ for t large:

$$(M_t)_{ij} = \big([\tilde{\phi}_i, \tilde{\phi}_j] + f_{ij}(t)\big)t^{\theta_i - 1}t^{\theta_j - 1}$$

where $|f_{ij}(t)| < ct^{-\eta}$, for $t > \delta$. Denote $\tilde{M}_t$, $\tilde{M}$ the n x n matrices defined by

$$(\tilde{M}_t)_{ij} = [\tilde{\phi}_i, \tilde{\phi}_j] + f_{ij}(t), \ (\tilde{M})_{ij} = [\tilde{\phi}_i, \tilde{\phi}_j]$$

and let $D_t = diag\tilde{M}_t$, $\tilde{D} = diag\tilde{M}$. Next, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C^n$, we have

$$\big\langle (M_t - \gamma D_t)\alpha, \alpha \big\rangle \;=\; \big\langle (\tilde{M}_t - \gamma \tilde{D}_t)\alpha_t, \alpha_t \big\rangle$$

where $< \cdot, \cdot >$ is the inner product on C^n and $(\alpha_t)_i = \alpha_i \; t^{\theta_i - 1}$, $i = 1, 2, \ldots, n$.

Hence, the condition **(A2)** is satisfied if one can find $\gamma > 0$, $\delta > 0$, such that

$$\tilde{M}_t - \gamma \tilde{D}_t \geq 0, \qquad \text{for all } t \in (\delta, \infty).$$

On the other hand, since $\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n$ are linearly independent, $\tilde{M}$ is a symmetric positive definite matrix on C^n and

$$\lim_{\gamma \searrow 0} \, (\tilde{M}_t - \gamma \tilde{D}_t) = \tilde{M} > 0.$$

Thus, one can find $\gamma > 0$ so that $\tilde{M}_t - \gamma \tilde{D}_t > 0$.

Finally,

$$\lim_{t \to \infty} \, (\tilde{M}_t - \gamma \tilde{D}_t) \;=\; \tilde{M} - \gamma \tilde{D}_t > 0.$$

Therefore, there are $\gamma > 0$, $\delta > 0$ such that $\tilde{M}_t - \gamma \tilde{D}_t > 0$, for all $t \in (\delta, \infty)$, and **(A2)** holds. The result is proved by applying Theorem II.2.

■

The corresponding case of interpolation between subspaces of H^β of finite codimensions and H^α, where α, β are real numbers, $\alpha < \beta$, is a direct consequence of the previous theorem.

Let $\alpha < \beta$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \in H^\beta$ be such that the corresponding Fourier transform $\phi_1, \phi_2, \ldots, \phi_n$ satisfy for some positive constants c and ϵ,

$$\begin{cases} |\phi_i(\xi) - \tilde{\phi}_i(\xi)| < c\rho^{-\frac{N}{2} - 2\beta + \gamma_i - \epsilon} \; for \; |\xi| > 1 \\ \alpha < \gamma_i < \beta, \; i = 1, \ldots, n, \end{cases} \tag{5.15}$$

where

$$\tilde{\phi}_i(\xi) = b_i(\omega)\rho^{-\frac{N}{2}-2\beta+\gamma_i},\ \xi = (\rho, \omega),$$

and $b_i(\cdot)$ is a bounded measurable function on S^{N-1}, which is non zero on a set of positive measure.

Theorem V.3 *Let $\varphi_1, \varphi_2, \ldots, \varphi_n \in H^\beta$ be such that the corresponding Fourier transforms $\phi_1, \phi_2, \ldots, \phi_n$ satisfy (5.15), and in addition, the functions $\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n$ are linearly independent. Let $\mathcal{L} = span\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Then*

$$[H^\beta_{\mathcal{L}}, H^\alpha]_s \ = \ [H^\beta, H^\alpha]_{s,\mathcal{L}}\ ,\ \ s\alpha + (1-s)\beta \neq \gamma_i,\ \ for\ i = 1, 2, \ldots, n. \tag{5.16}$$

Furthermore , if $s\alpha + (1-s)\beta < \min\{\gamma_i,\ i = 1, 2, \ldots, n\}$, then

$$[H^\beta_{\mathcal{L}}, H^\alpha]_s \ = \ H^{s\alpha+(1-s)\beta}. \tag{5.17}$$

Proof. The first part follows from the main theorem V.2 and the fact that $T : H^\alpha \to H^0$ defined by $\hat{Tu} = \mu^\alpha \hat{u},\ u \in H^\alpha$ is an isometry from H^α to $H^{\gamma-\alpha}$ for any $\gamma \in [\alpha, \beta]$.

Now let $s < \min\{\gamma_i,\ i = 1, 2, \ldots, n\}$. By the first part of the theorem, in order to prove (5.17) we need only to prove that $H^\beta_{\mathcal{L}}$ is dense in $H^{s\alpha+(1-s)\beta}$. By Lemma II.2, this is equivalent to proving that

$$\begin{cases} H^\beta \ni u \xrightarrow{\Lambda_\varphi} (u, \varphi)_\beta = (\hat{u}, \hat{\varphi})_\beta, \\ \text{is not bounded in the topology of } H^{s\alpha+(1-s)\beta} \text{ for all } \varphi \in \mathcal{L},\ \varphi \neq 0. \end{cases} \tag{5.18}$$

For a fixed $\varphi \in \mathcal{L}$ we have $\hat{\varphi} = \sum_{i=1}^{n} c_i \phi_i$.

Since $\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n$ are assumed to be linearly independent, φ fails to be a "good" function (better than $\varphi_i,\ i = 1, 2, \ldots, n$). More precisely, the asymptotic expansion at infinity of $\hat{\varphi}$ is of the same type (except maybe a different b-part) with one of the functions $\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n$. Thus, it is enough to check (5.18) for $\phi \in \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$.

Assuming that Λ_{φ_i} is continuous, it implies that

$$(\hat{u}, \phi_i)_\beta = (\hat{u}, f_i)_{s\alpha+(1-s)\beta}, u \in H^\beta,$$

for a function $f_i \in \hat{H}^{s\alpha+(1-s)\beta}$.

Thus, by using the density of H^β in H^s, for $s < \beta$, we get that $f_i = \mu^{2\beta}\mu^{-2(s\alpha+(1-s)\beta)}\phi_i$.

On the other hand,

$$\begin{aligned} \int \mu^{2(s\alpha+(1-s)\beta)}|f_i|^2 \, d\xi \ &= \ \int \mu^{2\beta-2s\alpha+2s\beta}|\phi_i|^2 \, d\xi \\ \geq \ c\int_\delta^\infty \rho^{2\beta-2s\alpha+2s\beta}\rho^{-N-4\beta+2\gamma_i}\rho^{N-1}d\rho \\ = \ c\int_\delta^\infty \rho^{-1+2(\gamma_i-(s\alpha+(1-s)\beta))}d\rho \ = \ \infty \end{aligned}$$

for $s\alpha + (1-s)\beta < \min\{\gamma_i, \ i = 1, 2, \ldots, n\}$. This completes the proof.

■

CHAPTER VI

INTERPOLATION BETWEEN $H^2(\Omega) \cap H^1_D(\Omega)$ AND $H^1_D(\Omega)$

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$, where $(\partial\Omega)_D$ is not of measure zero, and $(\partial\Omega)_D$ and $(\partial\Omega)_N$ are essentially disjoint and consist of a finite number of line segments. Let $H^1_D(\Omega)$ denote the space of all functions in $H^1(\Omega)$ which vanish on $(\partial\Omega)_D$, and let $s \in [0,1]$ be fixed. In this section it will be shown that

$$[H^2(\Omega) \cap H^1_D(\Omega), H^1_D(\Omega)]_s = [H^2(\Omega), H^1(\Omega)]_s \cap H^1_D(\Omega). \tag{6.1}$$

The space $H^2(\Omega) \cap H^1_D(\Omega)$ is dense in $H^1_D(\Omega)$ (see for example Theorem 1.6.1 in [24]). Applying Lemma II.3 with $X = H^2(\Omega)$, $Y = H^1_D(\Omega)$, $X_0 = H^2(\Omega) \cap H^1_D(\Omega)$ and $Y_0 = H^1_D(\Omega)$, we obtain that

$$[H^2(\Omega) \cap H^1_D(\Omega), H^1_D(\Omega)]_s \subset [H^2(\Omega), H^1(\Omega)]_s \cap H^1_D(\Omega). \tag{6.2}$$

In order to prove the opposite inclusion of (6.2), we need to show that for a positive constant c,

$$\|u\|_{[H^2(\Omega) \cap H^1_D(\Omega), H^1_D(\Omega)]_s} \leq c\|u\|_{[H^2(\Omega), H^1(\Omega)]_s}, \tag{6.3}$$

for all $u \in [H^2(\Omega), H^1(\Omega)]_s \cap H^1_D(\Omega)$.

Let $\partial\Omega$ be the polygonal line $P_1P_2 \cdots P_mP_1$. Here we consider that the set $\{P_1, P_2, \ldots, P_m\}$ consists of all vertices of $\partial\Omega$ and all the points of $(\partial\Omega)_D \cap (\partial\Omega)_N$. We will also call the points of $(\partial\Omega)_D \cap (\partial\Omega)_N$ vertices of $\partial\Omega$.

For $j = 1, 2, \ldots, m$, let U_j be an open disk centered at P_j such that U_j contains no vertices other than P_j. Next we add more disks, say U_j, centered at P_j, $j =$

$m+1,\dots M$, such that $P_j \in \partial\Omega$ or $\overline{U}_j \subset \Omega$, and

$$\overline{\Omega} \subset \bigcup_{j=1}^{M} U_j.$$

By increasing the number M of disks and modifying the radii of the disks, we can assume that P_k is not in U_j, for $k \neq j$ and the radii of the disks are equal to some positive number r_0. Then, there is $\{\phi_j\}_{j=1}^{M}$ a partition of unity subordinate to the covering $\bigcup\limits_{j=0}^{M} U_j$ such that

$$\text{supp}(\phi_j) \subset U_j, \ \sum_{j=0}^{M} \phi_j(x) = 1 \qquad \text{for all } x \in \overline{\Omega}. \tag{6.4}$$

Let us denote $U_j \cap \Omega$ by Ω_j and the restriction of ϕ_j to Ω_j by η_j ($j = 0, 1, \dots, M$). We note here that one can find $r > 0$ such that

$$dist(\Omega \backslash \Omega_j, supp\ u_j) \geq r \qquad j = 1, \dots, M. \tag{6.5}$$

Further, for $j = 1, 2, \dots, M$, we define $(\partial\Omega_j)_D$ and $(\partial\Omega_j)_N$ to be

$$(\partial\Omega_j)_N := (\partial\Omega)_N \cap \partial\Omega_j, \quad (\partial\Omega_j)_D := \overline{\partial\Omega_j \backslash (\partial\Omega_j)_N},$$

and denote the space of functions in $H^1(\Omega_j)$ which vanish on $(\partial\Omega_j)_D$ by $H^1_D(\Omega_j)$. Also we introduce the spaces

$$H^2_*(\Omega_j) := \{u \in H^2(\Omega_j) \cap H^1_D(\Omega_j) : \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \overline{\partial\Omega_j \backslash (\partial\Omega)}\}.$$

We reduce the proof of (6.3) to the following result.

Theorem VI.1 *Let Ω_j be one of the domains defined above. Then, there exist a positive constant c such that*

$$\|u\|_{[H^2_*(\Omega_j),H^1_D(\Omega_j)]_s} \le c\|u\|_{[H^2(\Omega_j),H^1(\Omega_j)]_s} \qquad \textit{for all } u \in [H^2_*(\Omega_j), H^1_D(\Omega_j)]_s \cap M_j(r), \tag{6.6}$$

where $M_j(r) := \{u \in H^1(\Omega_j) : dist(\Omega\backslash\Omega_j, supp\ u) \ge r\}$.

In proving the main result of this chapter, we need also the following

Lemma VI.1 *Let* $\Omega_0 \subset \Omega$ *be domains in* R^N *with Lipschitz boundary. Let* m *be a nonnegative integer,* $0 < s < 1$ *and* $r_0 > 0$. *Define*

$$M(r_0) := \{u \in [H^{m+1}(\Omega), H^m(\Omega)]_s : dist(\Omega\backslash\Omega_0, supp\ u) \ge r_0\}.$$

Then there is a positive constant $c = c(s, r_0)$ *such that*

$$\|u\|_{[H^{m+1}(\Omega),H^m(\Omega)]_s} \le c\|u\|_{[H^{m+1}(\Omega_0),H^m(\Omega_0)]_s} \qquad \textit{for all } u \in M(r_0). \tag{6.7}$$

Proof. Since Ω has Lipschitz boundary (see, e.g., [5], [17]), an equivalent norm on $[H^{m+1}(\Omega), H^m(\Omega)]_{1-s} = H^{m+s}(\Omega)$ is the double integral norm

$$\|u\|^2_{m+s,\Omega} := \|u\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega\int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{N+2s}}\, dx\ dy.$$

A similar statement holds for Ω_0. Let $u \in M(r_0)$. Then,

$$\|u\|_{H^m(\Omega)} = \|u\|_{H^m(\Omega_0)}$$

and for a fixed multi index α with $|\alpha| = m$ we have

$$\int_\Omega\int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{N+2s}}\, dx\ dy = I_1 + 2I_2 =$$

$$= \int_{\Omega_0}\int_{\Omega_0} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{N+2s}}\, dx\ dy + 2\int_{\Omega\backslash\Omega_0}\int_{\Omega_0} \frac{|D^\alpha u(x)|^2}{|x-y|^{N+2s}}\, dx\ dy.$$

Next, let $K := \{x \in \Omega_0 : dist(x, \Omega\backslash\Omega_0) \geq r_0\}$. It follows that

$$I_2 = \int\limits_{\Omega\backslash\Omega_0} \int\limits_{K} \frac{|D^\alpha u(x)|^2}{|x-y|^{N+2s}}\, dx\, dy = \int\limits_{K} \int\limits_{\Omega\backslash\Omega_0} \frac{1}{|x-y|^{N+2s}}\, dy\, |D^\alpha u(x)|^2\, dx$$
$$\leq c \int\limits_{K} |D^\alpha u(x)|^2\, dx \leq c\|u\|^2_{H^m(\Omega_0)}.$$

Summing up these estimates we obtain that (6.7) holds.

■

Now we go back to our polygonal domain $\Omega \subset R^2$. Given Theorem VI.1, we can prove the main result of this section.

Theorem VI.2 *(Bacuta, Bramble, Pasciak) If $\Omega \subset R^2$ is a polygonal domain with Lipschitz boundary then* (6.1) *holds.*

Proof. The constant c , we use here, might be different at different occurrences. Let $u \in [H^2(\Omega), H^1_D(\Omega)]_s \cap H^1_D(\Omega)$. For $j = 0, 1, \ldots, M$, let $u_j := \eta_j\ u$. Then, $u = \sum\limits_{j=1}^{M} u_j$ and by applying Lemma II.3, and Theorem VI.1 we obtain

$$\|u\|_{[H^2(\Omega)\cap H^1_D(\Omega), H^1_D(\Omega)]_s} \leq c\sum_{j=1}^{M} \|u_j\|_{[H^2(\Omega)\cap H^1_D(\Omega), H^1_D(\Omega)]_s}$$
$$\leq c\sum_{j=1}^{M} \|u_j\|_{[H^2_*(\Omega_j), H^1_D(\Omega_j)]_s} \leq c\sum_{j=1}^{M} \|u_j\|_{[H^2(\Omega_j), H^1(\Omega_j)]_s}.$$

Next, using the fact that multiplication by a smooth function is continuous on $[H^2(\Omega), H^1(\Omega)]_s$, we have

$$\|u_j\|_{[H^2(\Omega_j), H^1(\Omega_j)]_s} \leq c\|u_j\|_{[H^2(\Omega), H^1(\Omega)]_s} \leq c\|u\|_{[H^2(\Omega), H^1(\Omega)]_s}.$$

Combining the above estimates we see that (6.3) follows. Finally, from (6.2) and (6.3) we conclude the result.

■

A. Proving the reduction theorem

To begin with, we consider the case when $\Omega_j = U_j$, i.e., Ω_j is a disk. We assume, without loss of generality, that Ω_j is the unit disk U centered at the origin of a Cartesian system of coordinates. In this case we have $(\partial\Omega_j)_D := (\partial\Omega_j)$ and

$$H^1_D(\Omega_j) = H^1_0(U), \quad H^2_*(\Omega_j) = H^2_0(U).$$

Let $E : H^1_0(U) \to H^1(R^2)$, be the extension by zero operator, and let
$R : H^1(R^2) \to H^1_0(U)$ defined as follows:
First, we introduce a cutoff function $\eta \in \mathcal{D}(R^2)$ which depends only on the distance r to the origin and satisfies

$$\eta(r) = 1 \quad for \quad 0 < \ r \leq 1 \quad \text{and} \quad \eta(r) = 0 \quad for \quad r \geq 2.$$

Then, for a function $v \in H^1(R^2)$ we define $Rv \in H^1_0(U)$ by

$$(Rv)(r,\theta) := v_1(r,\theta) - 3v_1(1/r,\theta) + 2v_1(1/r^2,\theta), \quad (r,\theta) \in U,$$

where

$$v_1(r,\theta) := v(r,\theta)\eta(r), \quad (r,\theta) \in R^2.$$

The operators E, R satisfy the hypotheses of Lemma II.5 with $H^2 = H^2_0(U)$, $\tilde{H}^2 = H^2(R^2)$ and $H^1 = H^1_0(U)$, $\tilde{H}^1 = H^1(R^2)$. Thus, according to this lemma, we deduce that

$$\|u\|_{[H^2_0(U),H^1_0(U)]_s} \leq c\|Eu\|_{[H^2(R^2),H^1(R^2)]_s} \qquad \text{for all} \ \ u \in [H^2_0(U), H^1_0(U)]_s,$$

for some positive constant c. On the other hand, by Lemma VI.1 (for another c) we have

$$\|Eu\|_{[H^2(R^2),H^1(R^2)]_s} \le c\|u\|_{[H^2(U),H^1(U)]_s} \qquad \text{for all} \quad u \in [H^2_0(U), H^1_0(U)]_s \cap M(r),$$

where

$$M(r) := \{u \in H^1(U) : dist(\partial U, supp\ u) \ge r\}.$$

Using the above two estimates Theorem VII.1 is proved in this special case.

Before we consider the remaining cases, let us introduce some new notation. Let α, β be real numbers such that $\alpha < \beta$ and $\beta - \alpha < 2\pi$. Using polar coordinates (r, θ) we define the sector domain

$$S_{\alpha,\beta} := \{(r,\theta) : 0 < r < 1,\ \alpha < \theta < \beta\}$$

and the following spaces:

$$H^1_*(S_{\alpha,\beta}) := \{u \in H^1(S_{\alpha,\beta}) : u = 0 \ \text{ for } \ r = 1\ \},$$

$$H^2_*(S_{\alpha,\beta}) := \{u \in H^2(S_{\alpha,\beta}) : u = \frac{\partial u}{\partial n} = 0 \ \text{ for } \ r = 1\ \},$$

$$H^i_{*,\gamma}(S_{\alpha,\beta}) := \{u \in H^i_*(S_{\alpha,\beta}) : u = 0 \ \text{ for } \ \theta = \gamma\ \},$$

$$H^i_{*,\alpha,\beta}(S_{\alpha,\beta}) := \{u \in H^i_*(S_{\alpha,\beta}) : u = 0 \ \text{ for } \ \theta = \alpha \text{ and } \theta = \beta\},$$

where $i = 1, 2$, $\gamma = \alpha$ or $\gamma = \beta$ and the functions are zero on line segments or arcs in the trace sense.

All the remaining cases of Theorem VI.1 can be reduced to the following standard ones:

The domain Ω_j coincides with $S_{0,\omega}$ for some real number $\omega \in (0, 2\pi)$ and

- Case 1. "Free-Free": $H^1_D(\Omega_j) = H^1_*(S_{0,\omega})$ and $H^2_*(\Omega_j) = H^2_*(S_{0,\omega})$ or
- Case 2. "Dirichlet-Free": $H^1_D(\Omega_j) = H^1_{*,0}(S_{0,\omega})$ and $H^2_*(\Omega_j) = H^2_{*,0}(S_{0,\omega})$ or

- Case 3. "Dirichlet-Dirichlet": $H^1_D(\Omega_j) = H^1_{*,0,\omega}(S_{0,\omega})$ and $H^2_*(\Omega_j) = H^2_{*,0,\omega}(S_{0,\omega})$.

Next, we prove Theorem VI.1 in Case 1.

We define the infinite sector domain $\tilde{S}_{0,\omega}$ by

$$\tilde{S}_{0,\omega} := \{(r,\theta) : r > 0,\ 0 < \theta < \omega\}.$$

The operators $E : H^1_*(S_{0,\omega}) \to H^1_*(\tilde{S}_{0,\omega})$ and $R : H^1_*(\tilde{S}_{0,\omega}) \to H^1_*(S_{0,\omega})$ defined in the case of the disk, satisfy the hypotheses of Lemma II.5 with $H^i = H^i_*(S_{0,\omega})$, and $\tilde{H}^i = H^i_*(\tilde{S}_{0,\omega})$, i=1,2. Similar arguments used in the case of the disk can be used now to show that

$$\|u\|_{[H^2_*(S_{0,\omega}),H^1_*(S_{0,\omega})]_s} \leq c\|u\|_{[H^2(S_{0,\omega}),H^1(S_{0,\omega})]_s} \tag{6.8}$$

for all $u \in [H^2_*(S_{0,\omega}), H^1_*(S_{0,\omega})]_s \cap M(r)$, where c is a positive constant and

$$M(r) := \{u \in H^1_*(S_{0,\omega}) : dist(\partial U, supp\ u) \geq r\}.$$

Therefore, the proof for Case 1 is complete .

For the Case 2 and Case 3 we will use again Lemma II.5, but we need to construct operators E and R with stronger properties.

In order to prove Theorem VI.1 in Case 2, let us assume for the moment that the following existence result holds.

Theorem VI.3 *Let $\alpha < 0$ be such that $\omega - \alpha < 2\pi$. Then, there are linear operators E and R such that*

$$E : H^i_*(S_{0,\omega}) \to H^i_*(S_{\alpha,\omega}) \quad \textit{is a bounded operator},\ i = 1,2, \tag{6.9}$$

$$R : H^i_*(S_{\alpha,\omega}) \to H^i_{*,0}(S_{0,\omega}) \quad \textit{is a bounded operator},\ i = 1,2, \tag{6.10}$$

$$REu = u \qquad \textit{for all } u \in H^2_{*,0}(S_{0,\omega}). \tag{6.11}$$

First, we observe that from (6.9), we get in particular that

$$E : H^i_{*,0}(S_{0,\omega}) \to H^i_*(S_{\alpha,\omega}) \text{ is a bounded operator, } i = 1, 2.$$

Thus, we can apply Lemma II.5 with $H^i = H^i_{*,0}(S_{0,\omega})$, and $\tilde{H}^i = H^i_*(S_{\alpha,\omega})$, i=1,2 and obtain that for a positive c,

$$\|u\|_{[H^2_{*,0}(S_{0,\omega}),H^1_{*,0}(S_{0,\omega})]_s} \leq c\|Eu\|_{[H^2_*(S_{\alpha,\omega}),H^1_*(S_{\alpha,\omega})]_s}, \tag{6.12}$$

for all $u \in [H^2_{*,0}(S_{0,\omega}), H^1_{*,0}(S_{0,\omega})]_s$.

From (6.9), by interpolation, we have that for another constant c,

$$\|Eu\|_{[H^2_*(S_{\alpha,\omega}),H^1_*(S_{\alpha,\omega})]_s} \leq c\|u\|_{[H^2_*(S_{0,\omega}),H^1_*(S_{0,\omega})]_s}, \tag{6.13}$$

for all $u \in [H^2_*(S_{0,\omega}), H^1_*(S_{0,\omega})]_s$.

Combining (6.12) and (6.13) we obtain

$$\|u\|_{[H^2_{*,0}(S_{0,\omega}),H^1_{*,0}(S_{0,\omega})]_s} \leq c\|u\|_{[H^2_*(S_{0,\omega}),H^1_*(S_{0,\omega})]_s}, \tag{6.14}$$

for all $u \in [H^2_{*,0}(S_{0,\omega}), H^1_{*,0}(S_{0,\omega})]_s$.

Now we can use the proof of Case 1 to finish the proof of Case 2. More precisely, from (6.8) and (6.14), we see that

$$\|u\|_{[H^2_{*,0}(S_{0,\omega}),H^1_{*,0}(S_{0,\omega})]_s} \leq c\|u\|_{[H^2(S_{0,\omega}),H^1(S_{0,\omega})]_s}, \tag{6.15}$$

for all $u \in [H^2_{*,0}(S_{0,\omega}), H^1_{*,0}(S_{0,\omega})]_s \cap M(r)$. Here ,

$$M(r) := \{u \in H^1_{*,0}(S_{0,\omega}) : dist(\partial U, supp\ u) \geq r\}.$$

Therefore, we have proved Theorem VI.1 in this case too.

The Case 3 can be treated in a similar way. We assume that we have the following result.

Theorem VI.4 *Let $\alpha < 0$ be such that $\omega - \alpha < 2\pi$. Then, there are linear operators E and R such that*

$$E : H^i_{*,\omega}(S_{0,\omega}) \to H^i_{*,\alpha,\omega}(S_{\alpha,\omega}) \quad \textit{is a bounded operator}, \; i = 1, 2, \tag{6.16}$$

$$R : H^i_{*,\alpha,\omega}(S_{\alpha,\omega}) \to H^i_{*,0,\omega}(S_{0,\omega}) \quad \textit{is a bounded operator}, \; i = 1, 2, \tag{6.17}$$

$$REu = u \qquad \textit{for all } u \in H^2_{*,0,\omega}(S_{0,\omega}). \tag{6.18}$$

We can reduce the proof of Case 3 to an estimate which follows from the previous case. The arguments are similar to those of Case 2.

B. Proving the existence of the operators E and R

The proofs of Theorem VI.3 and Theorem VI.4 are based on the following extension result.

Lemma VI.2 *Let Ω be a triangular domain in R^2 with boundary $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$, where $(\partial\Omega)_N = \overline{\Gamma}$ is one of the edges of $\partial\Omega$ (Γ is an open interval in R) and $(\partial\Omega)_D$ consists of the union of the other two edges. Then, there exist a linear operator P such that*

$$P : H^{1/2}_{00}(\Gamma) \to H^1_D(\Omega) \textit{ and } P : \tilde{H}^{3/2}(\Gamma) \to H^2_D(\Omega) \quad \textit{are bounded operators.} \tag{6.19}$$

Here, $H^{1/2}_{00}(\Gamma) = [H^1_0(\Gamma), L^2(\Gamma)]_{1/2}$, $H^2_D(\Omega) = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial n} = 0 \;\; on \;\; (\partial\Omega)_D\;\}$, and $\tilde{H}^{3/2}(\Gamma)$ is the space of all functions u defined on Γ such that $\bar{u} \in H^{3/2}(R)$, where $\bar{u}$ is the continuation of u by zero outside Γ.

Proof. For $v \in H^{1/2}_{00}(\Gamma)$ let $\tilde{v}$ denote the extension by zero of v to the rest of

$\partial\Omega$. Then, for some positive constant c we have

$$\|\tilde{v}\|_{H^{1/2}(\partial\Omega)} \leq c\|v\|_{H^{1/2}_{00}(\Gamma)} \qquad \text{for all} \;\; v \in H^{1/2}_{00}(\Gamma). \tag{6.20}$$

For $v \in C_0^\infty(\Gamma)$ we define Pv to be the solution of the problem:

Find $b \in H^2(\Omega)$ such that

$$\begin{cases} \Delta^2 b = 0 \;\; \text{in} \;\; \Omega, \\ \quad b = \tilde{v} \;\; \text{on} \;\; \partial\Omega, \\ \frac{\partial b}{\partial n} = 0 \;\; \text{on} \;\; \partial\Omega. \end{cases} \tag{6.21}$$

It is known that (see, e.g., Proposition 1.3 in [21]) Problem (6.21) has exactly one solution $b \in H^2(\Omega)$ and

$$\|b\|_{H^2(\Omega)} \leq c\|v\|_{H^{3/2}(\Gamma)} \leq c\|v\|_{\tilde{H}^{3/2}(\Gamma)} \qquad \text{for all} \;\; v \in C_0^\infty(\Gamma), \tag{6.22}$$

where c is a positive constant. In addition, since $v \in C_0^\infty(\Gamma)$, we have $b \in H^3(\Omega)$ (see, e.g., Section 3.4.2 in [24]).

Next, in order to estimate $\|b\|_{H^1(\Omega)}$ we consider the following fourth order problem. Find w such that

$$\begin{cases} \Delta^2 w = -\Delta b \;\; \text{in} \;\; \Omega, \\ \quad w \in H_0^2(\Omega). \end{cases} \tag{6.23}$$

The (weak) solution w of the above problem satisfies $w \in H^3(\Omega) \cap H_0^2(\Omega)$ (see, e.g., Corollary 3.4.2 in [24]). Then, using Green's first and second identities, we get

$$(\nabla b, \nabla b) = (-\Delta b, b) = (\Delta^2 w, b) = \langle \frac{\partial(\Delta w)}{\partial n}, b\rangle + (\Delta w, \Delta b),$$

where $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are the inner products on $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Since $w \in H_0^2(\Omega)$ and Δb is harmonic it follows from Green's identity that $(\Delta w, \Delta b) = 0$. Consequently,

$$\|b\|^2_{H^1(\Omega)} \le c \left\| \frac{\partial(\Delta w)}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} \|b\|_{H^{1/2}(\partial\Omega)} \qquad \text{for all } v \in C_0^\infty(\Gamma), \tag{6.24}$$

where c is a positive constant. Next, we have

$$\left\| \frac{\partial(\Delta w)}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} = \sup_{\varphi \in H^{1/2}(\partial\Omega)} \frac{\langle \frac{\partial(\Delta w)}{\partial n}, \varphi \rangle}{\|\varphi\|_{H^{1/2}(\partial\Omega)}}. \tag{6.25}$$

Denoting the harmonic extension of $\varphi \in H^{1/2}(\partial\Omega)$ to Ω by the same symbol φ, and applying again Green's identity, we obtain

$$\langle \frac{\partial(\Delta w)}{\partial n}, \varphi \rangle = \langle \frac{\partial \varphi}{\partial n}, \Delta w \rangle - (\Delta b, \varphi). \tag{6.26}$$

In order to estimate the right hand side of (6.26), on the one hand we have

$$|(\Delta b, \varphi)| = |(\nabla b, \nabla \varphi)| \le \|b\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)} \le c\|b\|_{H^1(\Omega)} \|\varphi\|_{H^{1/2}(\partial\Omega)}, \tag{6.27}$$

and on the other hand we can prove that

$$|\langle \Delta w, \frac{\partial \varphi}{\partial n} \rangle| \le c\|b\|_{H^1(\Omega)} \|\varphi\|_{H^{1/2}(\partial\Omega)}. \tag{6.28}$$

Indeed, using trace inequalities we have

$$|\langle \Delta w, \frac{\partial \varphi}{\partial n} \rangle| \le c\|\Delta w\|_{H^{1/2}(\partial\Omega)} \|\partial\varphi/\partial n\|_{H^{-1/2}(\partial\Omega)} \le c\|\Delta w\|_{H^1(\Omega)} \ \|\partial\varphi/\partial n\|_{H^{-1/2}(\partial\Omega)}$$

where

$$\left\| \frac{\partial \varphi}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} = \sup_{\theta \in H^{1/2}(\partial\Omega)} \frac{\langle \frac{\partial \varphi}{\partial n}, \theta \rangle}{\|\theta\|_{H^{1/2}(\partial\Omega)}}.$$

Let us denote the harmonic extension of $\theta \in H^{1/2}(\partial\Omega)$ to Ω by the same symbol θ. By applying Green's identity and the fact that φ is a harmonic function, we obtain

$$\langle \frac{\partial \varphi}{\partial n}, \theta \rangle = (\nabla\varphi, \nabla\theta) \le \|\varphi\|_{H^1(\Omega)} \|\theta\|_{H^1(\Omega)} \le \|\varphi\|_{H^{1/2}(\partial\Omega)} \|\theta\|_{H^{1/2}(\partial\Omega)}.$$

Next, since Ω is convex, the operator Δ^2 defines an isomorphism from $H^3(\Omega)\cap H_0^2(\Omega)$ onto $H^{-1}(\Omega)$ (Corollary 3.4.2 in [24]). Thus, we get

$$\|\Delta w\|_{H^1(\Omega)} \leq c\|w\|_{H^3(\Omega)} \leq c\|\Delta^2 w\|_{H^{-1}(\Omega)}. \leq c\|\Delta b\|_{H^{-1}(\Omega)}.$$

Since $\frac{\partial b}{\partial n} = 0$, the Green's identity and the definition of the negative norm gives

$$\|\Delta b\|_{H^{-1}(\Omega)} \leq \|b\|_{H^1(\Omega)}.$$

Finally, from the above estimates we conclude that (6.28) is proved.

Combining (6.24)-(6.28) we deduce that

$$\|b\|_{H^1(\Omega)} \leq c\|b\|_{H^{1/2}(\partial\Omega)} \qquad \text{for all } v \in C_0^\infty(\Gamma), \tag{6.29}$$

where c is a constant independent of the function $v \in C_0^\infty(\Gamma)$. From (6.20), (6.21) and (6.29) we have

$$\|b\|_{H^1(\Omega)} \leq c\|v\|_{H_{00}^{1/2}(\Gamma)} \qquad \text{for all } v \in C_0^\infty(\Gamma), \tag{6.30}$$

Using (6.22), (6.30) and the density of $C_0^\infty(\Gamma)$ in both $\tilde{H}^{3/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$, we can extend the definition of P so that (6.19) is satisfied.

■

Proof of Theorem VI.3. Let O denote the origin of a polar coordinate system used to describe the sector domain $S_{\alpha,\omega}$. Let $\epsilon > 0$ be fixed, and let A, B, C, D denote the points with polar coordinates $(1, 0)$, $(1, \omega)$, $(1, \alpha)$ and (ϵ, π), respectively (see Figure 3). Let $I := (O, A) \equiv (0, 1)$, $I_1 := (D, A) \equiv (-\epsilon, 1)$ and denote by T, T_1 the triangular domains O, A, C and D, A, C, respectively. For $u \in: H_*^1(S_{0,\omega})$,

define γu to be the trace of u to the interval I and $\widetilde{\gamma u}$ an extension of γu to the whole

FIGURE 3. Sector domain.

interval I_1 such that $\widetilde{\gamma u} \equiv 0$ on the interval (on the real line) $(-\epsilon, -\epsilon/2)$ and

$$\|\widetilde{\gamma u}\|_{H^{1/2}_{00}(I_1)} \leq c\|u\|_{H^1(S_{0,\omega})} \qquad \text{for all } u \in H^1_*(S_{0,\omega}), \tag{6.31}$$

$$\|\widetilde{\gamma u}\|_{\tilde{H}^{3/2}(I_1)} \leq c\|u\|_{H^2(S_{0,\omega})} \qquad \text{for all } u \in H^2_*(S_{0,\omega}), \tag{6.32}$$

where c is a positive constant. By Lemma VI.2, we can extend $\widetilde{\gamma u}$ to a function $b = P(\widetilde{\gamma u})$ defined on the whole triangular domain T_1 and such that (6.19) is satisfied for $\Omega = T_1$ and $\Gamma = I_1$. Next, we consider the restriction of b to the triangular domain T and the extension by zero of the new function to the sector domain $S_{\alpha,0}$. Let $\tilde{b}$ be the function obtained by this process. Then, define an extension operator denoted E_b mapping functions defined on $S_{0,\omega}$ into functions defined on $S_{\alpha,\omega}$ by

$$(E_b u)(x) = \begin{cases} u, \text{if } \ x \in S_{0,\omega} \\ \tilde{b}, \ \text{if } \ x \in S_{\alpha,0}. \end{cases}$$

Combining (6.31) and (6.32) with the fact that the operators involved in defining $\tilde{b}$

are continuous, we get that

$$E_b : H^i_*(S_{0,\omega}) \to H^i_*(S_{\alpha,\omega}) \ \text{ is a bounded operator, } \ i = 1, 2. \tag{6.33}$$

Now we introduce another extension operator denoted E_o, which coincides with the classical odd extension operator when $\omega = -\alpha$, mapping functions defined on $S_{0,\omega}$ into functions defined on $S_{\alpha,\omega}$ by

$$(E_o u)(r, \theta) = \begin{cases} u(r, \theta), & \text{if } \ (r, \theta) \in S_{0,\omega} \\ \\ u(r, \frac{\omega}{\alpha}\theta), & \text{if } \ (r, \theta) \in S_{\alpha,0}. \end{cases}$$

Finally, we define the required operators E and R, by

$$(Eu)(r, \theta) := \frac{\alpha}{\omega}(E_o u)(r, \theta) + (1 - \frac{\alpha}{\omega})E_b(r, \theta), \ (r, \theta) \in S_{\alpha,\omega},$$

and

$$(Rv)(r, \theta) := \frac{\omega}{\alpha - \omega}(v(r, \theta) - v(r, \frac{\alpha}{\omega}\theta)), \ (r, \theta) \in S_{0,\omega}.$$

Simple computations show that E and R have the desired properties.

The proof of Theorem VI.4 is similar. The only difference is that we can avoid using the triangular domain T_1 in constructing the extension operator P and apply Lemma VI.2 directly for $\Omega = T$ and $\Gamma = I$.

CHAPTER VII

SHIFT THEOREM FOR THE LAPLACE OPERATOR ON POLYGONAL DOMAINS

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$, where $(\partial\Omega)_D$ is not of measure zero, and $(\partial\Omega)_D$ and $(\partial\Omega)_N$ are essentially disjoint and consist of a finite number of closed line segments. Let $\partial\Omega$ be the polygonal arc $P_1P_2\cdots P_mP_1$. Here we consider that the set $\{P_1, P_2, \dots, P_m\}$ consists of all vertices of $\partial\Omega$ and all the points of $(\partial\Omega)_D \cap (\partial\Omega)_N$. We will also call the points of $(\partial\Omega)_D \cap (\partial\Omega)_N$ vertices of $\partial\Omega$. At each point P_j, we denote the measure of the angle at P_j (measured from inside Ω) by ω_j. Set $P_{m+1} = P_1$ and $P_0 = P_m$. For $j = 1, 2, \dots, m$, let us define $\gamma_j := \max\{\omega_j/\pi, 1\}$ if both edges $[P_{j-1}, P_j]$ and $[P_j, P_{j+1}]$ belong to the same set $(\partial\Omega)_D$ or $(\partial\Omega)_N$, and $\gamma_j := \max\{2\omega_j/\pi, 1\}$ if one edge belongs to $(\partial\Omega)_D$ and the other edge belongs to $(\partial\Omega)_N$. Let $\gamma := \max\{\gamma_j : j = 1, 2, \dots, m\}$. We consider the boundary value problem for the Poisson equation on Ω.

Given $f \in L^2(\Omega)$, find u such that

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ \quad u = 0 \quad \text{on} \quad (\partial\Omega)_D, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega)_N. \end{cases} \tag{7.1}$$

The variational formulation of (7.1) is :

Find $u \in H^1_D(\Omega)$ such that

$$a(u, v) = \int_\Omega fv\; dx \qquad \text{for all } v \in H^1_D(\Omega), \tag{7.2}$$

where $H^1_D(\Omega)$ denotes the space of all functions in $H^1(\Omega)$ which vanish on $(\partial\Omega)_D$. It is known that for $f \in L^2(\Omega)$ the variational problem has a unique solution

$u \in H^1_D(\Omega)$ and

$$\|u\|_{H^1(\Omega)} \leq c\|f\|_{H_D^{-1}(\Omega)} \quad \text{for all } f \in L^2(\Omega), \tag{7.3}$$

where $H_D^{-1}(\Omega)$ is the dual of $H^1_D(\Omega)$.

Let u be the solution of (7.2). By taking v in $\mathcal{D}(\Omega)$, the space of all infinitely differentiable functions with compact support in Ω, one has

$$-\Delta u = f$$

in the sense of distributions in Ω, so the equality is satisfied pointwise, almost everywhere in Ω. Also, the solution u of (7.2) satisfies the boundary conditions of (7.1) (see [24], Chapter 2 therein). In addition, if $\gamma = 1$ then u belongs to $H^2(\Omega) \cap H^1_D(\Omega)$ (see, e.g., [23]), and

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)} \quad \text{for all } f \in L^2(\Omega). \tag{7.4}$$

If we define $T : H_D^{-1}(\Omega) \to H^1_D(\Omega)$ by $Tf := u$, where u is the solution of (7.2), then T is a bounded operator. Moreover, if $\gamma = 1$, T is a bounded operator from $L^2(\Omega)$ to $H^2(\Omega)$. Thus, by interpolation, we have for any $s \in [0, 1]$,

$$\|u\|_{H^{1+s}(\Omega)} \leq c\|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega). \tag{7.5}$$

Here, $H^{1+s}(\Omega) := [H^2(\Omega), H^1(\Omega)]_{1-s}$ and $H_D^{-1+s}(\Omega) := [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}$. We will prove in this section that for $\gamma > 1$, the shift estimate (7.5) still holds for any $s < 1/\gamma$.

A. Reduction to sector domains

For $j = 1, 2, \ldots, m$, let U_j be an open disk centered at P_j such that U_j contains no vertices other than P_j. Next we add more disks with centers in $\partial\Omega$, say U_j, centered at P_j, $j =$

$m+1,\ldots,M,$ such that U_j contains no vertices other than P_j, and

$$\partial\Omega \subset \bigcup_{j=1}^{M} U_j.$$

By increasing the number M of disks, we can assume that for some positive numbers r_0 and ϵ we have

$$\begin{aligned} U_j \cap \Omega &= \{(r_j,\theta_j) : 0 < r_j < r_0,\ 0 < \theta_j < \omega_j\ \} \\ &\subset \{(r_j,\theta_j) : 0 < r_j < (1+\epsilon)r_0,\ 0 < \theta_j < \omega_j\} := \Omega_j \subset \Omega \quad , \quad j = 1,2,\ldots,M, \end{aligned}$$

where (r_j,θ_j) are the polar coordinates with origin at P_j, $\omega_j = \pi$ for $j = m+1,\ldots,M$ and P_k is not in Ω_j, for $k \neq j$. Let U_0 and Ω_0 be two domains with smooth boundaries such that $\overline{U_0} \subset \Omega_0$ and $\overline{\Omega_0} \subset \Omega$ and such that

$$\overline{\Omega} \subset \bigcup_{j=0}^{M} U_j.$$

Then, there is $\{\phi_j\}_{j=0}^{M}$ a partition of unity subordinate to the covering $\overset{M}{\underset{j=0}{\cup}} U_j$. Let us denote the restriction of ϕ_j to Ω_j by η_j ($j = 0,1,\ldots,M$). Further, we define $(\partial\Omega_j)_D$ and $(\partial\Omega_j)_N$ to be

$$(\partial\Omega_j)_N := (\partial\Omega)_N \cap \partial\Omega_j, \qquad (\partial\Omega_j)_D := \overline{\partial\Omega_j \backslash (\partial\Omega_j)_N},$$

and denote the space of functions in $H^1(\Omega_j)$ which vanish on $(\partial\Omega_j)_D$ by $H^1_D(\Omega_j)$, for $j = 1,2,\ldots,M$. Also $(\partial\Omega_0)_D = \partial\Omega_0$.

We reduce the proof of (7.5) to the case when Ω is a sector domain. Let's assume for the moment that, for $j = 1,2,\ldots,M$, the following holds.

Theorem VII.1 *The variational solution u_j of (7.2) relative to Ω_j satisfies*

$$\|u_j\|_{H^{1+s}(\Omega_j)} \leq c\|f\|_{H_D^{-1+s}(\Omega_j)} \quad \textit{for all } f \in L^2(\Omega_j), \quad 0 < s < \gamma_j^{-1}, \tag{7.6}$$

where we take $\gamma_j = 1$ *for* $j = m+1, \dots, M$.

Given this result, we can prove that (7.5) holds for $\gamma > 1$ and $s < 1/\gamma$. Indeed, let $f \in L^2(\Omega)$ and let u be the solution of (7.2). For $j = 0, 1, \dots, M$, let $u_j := \eta_j\, u$. Then, in the sense of distributions in Ω_j, we obtain

$$-\Delta u_j = f\eta_j - u\Delta\eta_j - 2\nabla u \cdot \nabla \eta_j.$$

Since the boundary conditions of (7.1) are satisfied on $(\partial\Omega_j)_D$ and $(\partial\Omega_j)_N$ for $u = u_j$, we have (see [24], Theorem 2.1.1 therein) that u_j is the unique variational solution of the problem:

Find $u_j \in H^1_D(\Omega_j)$ such that

$$A_j(u_j, v) = \int_{\Omega_j} f_j v \; dx \quad \text{ for all } v \in H^1_D(\Omega_j), \tag{7.7}$$

where $f_j = f\eta_j - u\Delta\eta_j - 2\nabla u \cdot \nabla\eta_j$ and

$$A_j(u_j, v) := \int_{\Omega_j} \nabla u_j \cdot \nabla v \; dx.$$

Now, f_j is a function in $L^2(\Omega_j)$ and by Theorem VII.1, we get

$$\|u_j\|_{H^{1+s}(\Omega_j)} \leq c\|f_j\|_{H_D^{-1+s}(\Omega_j)}, \quad j = 1, 2, \dots, M. \tag{7.8}$$

For $j = 0$ the estimate (7.8) holds for any $s \in [0, 1]$, because the boundary of Ω_0 is smooth and we can apply the regularity result for domains with smooth boundaries. From the way we have defined the domains Ω_j one can find $r > 0$ such that

$$dist(\Omega \backslash \Omega_j, supp\ u_j) \geq r \qquad j = 0, 1, \dots, M.$$

Thus, by applying Lemma 6.7, we have

$$\|u_j\|_{H^{1+s}(\Omega)} \leq c\|u_j\|_{H^{1+s}(\Omega_j)}.$$

Here c is independent of f and j. Since $u = \sum_{j=0}^{M} u_j$, using the triangle inequality, the estimate (7.8) and the above observation, we obtain

$$\|u\|_{H^{1+s}(\Omega)} \;\leq\; c \sum_{j=0}^{M} \|f_j\|_{H_D^{-1+s}(\Omega_j)}. \tag{7.9}$$

The estimate of $\|f_j\|_{H_D^{-1+s}(\Omega_j)}$ is as follows. First, $L^2(\Omega_j)$ is continuously embedded in $H_D^{-1+s}(\Omega_j)$, and multiplication by a smooth function is continuous on $H_D^{-1+s}(\Omega_j)$. Thus,

$$\begin{aligned}\|f_j\|_{H_D^{-1+s}(\Omega_j)} &\leq \|f\eta_j\|_{H_D^{-1+s}(\Omega_j)} + c\;\|u\Delta\eta_j + 2\nabla u \cdot \nabla\eta_j\|_{L^2(\Omega_j)} \\ &\leq c(\|f\|_{H_D^{-1+s}(\Omega_j)} + \|u\|_{H^1(\Omega_j)}).\end{aligned}$$

Second, the extension by zero operator $E : H_D^1(\Omega_j) \to H_D^1(\Omega)$ is continuous. It follows that

$$\|f\|_{H_D^{-1}(\Omega_j)} \leq c\|f\|_{H_D^{-1}(\Omega)} \qquad \text{for all } f \in H_D^{-1}(\Omega).$$

Also,

$$\|f\|_{L^2(\Omega_j)} \leq \|f\|_{L^2(\Omega)} \qquad \text{for all } f \in L^2(\Omega).$$

By interpolation, we get

$$\|f\|_{H_D^{-1+s}(\Omega_j)} \leq c\|f\|_{H_D^{-1+s}(\Omega)} \qquad \text{for all } f \in H_D^{-1+s}(\Omega).$$

Third, we have

$$\|u\|_{H^1(\Omega_j)} \leq \|u\|_{H^1(\Omega)} \leq c\|f\|_{H_D^{-1}(\Omega)} \leq c\|f\|_{H_D^{-1+s}(\Omega)} \qquad \text{for all } f \in H_D^{-1+s}(\Omega).$$

Finally, from these inequalities we deduce

$$\|f_j\|_{H_D^{-1+s}(\Omega_j)} \leq c\|f\|_{H_D^{-1+s}(\Omega)} \qquad \text{for all } f \in H_D^{-1+s}(\Omega). \tag{7.10}$$

Thus, from (7.9) and (7.10), since $L^2(\Omega)$ is dense in $H_D^{-1+s}(\Omega)$, we obtain that

$$\|u\|_{H^{1+s}(\Omega)} \leq c\|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega).$$

Therefore we see that (7.5) holds for $\gamma > 1$ and all $s < 1/\gamma$.

B. Solving the problem on sector domains

Let $\Omega = S_\omega$ be the sector domain defined by

$$S_\omega := \{(r,\theta) : 0 < r < r_0,\ 0 < \theta < \omega\}, \tag{7.11}$$

and let $(\partial\Omega)_N$ be in one of the possibilities listed below (Case 1, Case 2 or Case 3). We assume, without loss of generality, that $r_0 = 1$. Let $V^2(\Omega)$ be defined by

$$V^2(\Omega) := \{u \in H^2(\Omega) :\ u = 0 \text{ on } (\partial\Omega)_D \text{ and } \partial u/\partial n = 0 \text{ on } (\partial\Omega)_N\}.$$

Then, (see, e.g., Theorem 2.2.3 in [24]) the Laplace operator $\Delta : V^2(\Omega) \to L^2(\Omega)$ is an injective Fredholm operator. Consequently,

$$\|u\|_{H^2(\Omega)} \leq c\|\Delta u\|_{L^2(\Omega)} \qquad \text{for all } u \in V^2(\Omega), \tag{7.12}$$

and the range of the operator has finite codimension. Grisvard characterized the orthogonal complement $\mathcal{N}$ of the range of the Laplace operator for the case of a polygonal domain in [23] and [24]. In particular, for our sector domain $\Omega = S_\omega$ the subspace $\mathcal{N}$ is described as follows:

- Case 1. "Dirichlet corner"; $(\partial\Omega)_N = \emptyset$.

    (i) $0 < \omega \leq \pi$; $\mathcal{N} = \{0\}$.

(ii) $\pi < \omega < 2\pi$; $\mathcal{N} = span\{\psi\}$, where

$$\psi(r,\theta) = (r^{-\frac{\pi}{\omega}} - r^{\frac{\pi}{\omega}}) \sin \frac{\pi}{\omega}\theta.$$

- Case 2. "Neuman corner"; $(\partial\Omega)_N = \{(r,\theta) \in \partial\Omega : \theta = 0 \text{ or } \theta = \omega\}$.

(i) $0 < \omega \leq \pi$; $\mathcal{N} = \{0\}$.

(ii) $\pi < \omega < 2\pi$; $\mathcal{N} = span\{\psi\}$, where

$$\psi(r,\theta) = (r^{-\frac{\pi}{\omega}} - r^{\frac{\pi}{\omega}}) \cos \frac{\pi}{\omega}\theta.$$

- Case 3. "Mixed corner"; $(\partial\Omega)_N = \{(r,\theta) \in \partial\Omega : \theta = \omega\}$.

(i) $0 < \omega \leq \pi/2$; $\mathcal{N} = (0)$.

(ii) $\pi/2 < \omega \leq 3\pi/2$; $\mathcal{N} = span\{\psi_1\}$.

(iii) $3\pi/2 < \omega < 2\pi$; $\mathcal{N} = span\{\psi_1, \psi_2\}$, where

$$\psi_k(r,\theta) = (r^{-\nu_k} - r^{\nu_k}) \sin(\nu_k\theta),\ \ \nu_k = (k - 1/2)\frac{\pi}{\omega}.$$

For the (i)-cases, the estimate (7.6) holds for any $s \in [0,1]$. For the remaining cases we will use the interpolation results in Chapter II.

According to ours previous notation, $L^2(\Omega)_{\mathcal{N}}$ denotes the orthogonal complement of the the subspace $\mathcal{N}$ in $L^2(\Omega)$. The Laplace operator is bounded with a bounded inverse from $V^2(\Omega)$ to $L^2(\Omega)_{\mathcal{N}}$. Thus, the operator $T : H_D^{-1}(\Omega) \to H^1(\Omega)$ defined at the beginning of Chapter VII is a bounded operator from $L^2(\Omega)_{\mathcal{N}}$ to $H^2(\Omega)$. By interpolation, we obtain

$$\|u\|_{[H^2(\Omega),H^1(\Omega)]_{1-s}} \leq c\|f\|_{[L^2(\Omega)_{\mathcal{N}},H_D^{-1}(\Omega)]_{1-s}} \qquad \text{for all } f \in [L^2(\Omega)_{\mathcal{N}}, H_D^{-1}(\Omega)]_{1-s}. \tag{7.13}$$

Since $[H^2(\Omega), H^1(\Omega)]_{1-s} = H^{1+s}(\Omega)$ and $[L^2(\Omega), H_D^{-1}(\Omega)]_{1-s} = H_D^{-1+s}(\Omega)$ the only thing which remains to be proved in order to obtain the estimate (7.5) for $s < 1/\gamma$ (Theorem VII.1 as well) is that

$$[L^2(\Omega)_{\mathcal{N}}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s} \quad \text{for} \quad s < 1/\gamma, \tag{7.14}$$

where $\gamma = \omega/\pi$ in Case 1 and Case 2, and $\gamma = 2\omega/\pi$ in Case 3.

Let $\psi = (r^{-\nu} - r^{\nu})g(\theta)$ be one of the functions which defines the subspace $\mathcal{N}$. (Note that $\nu \in (0, 1)$). The next result is of crucial importance in proving (7.14).

Theorem VII.2 *If* $0 < s < \nu$ *, then*

$$[L^2(\Omega)_{\psi}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}. \tag{7.15}$$

Our proof of the above theorem involves reduction of the problem, via the interpolation result of Section II C, to a similar interpolation problem where the domain Ω is replaced by a polygonal-sector domain, containing Ω. We will give the proof of this main result later.

When $dim(\mathcal{N}) = 1$, i.e., we are in one of the (ii)-cases listed above, (7.14) is Theorem VII.2. Let us consider now the case in which $dim(\mathcal{N}) = 2$, i.e., Case 3 (iii). In order to prove (7.14) we apply Theorem II.2. The condition **(A.1)** of Theorem II.2 follows easily from the Theorem VII.2. To verify **(A.2)** for $X = L^2(\Omega)$, $Y = H_D^{-1}(\Omega)$ and $\mathcal{K} = \mathcal{N} = span\{\psi_1, \psi_2\}$, we start by deriving an eigenfunction representation of the norm on $H_D^{\alpha}(\Omega)$. To do this, we consider the following eigenvalue problem.

Find real numbers λ and functions $u \in H^1(\Omega)$, $u \neq 0$ such that

$$\begin{cases} -\Delta u = \lambda u \quad \text{in} \quad \Omega \\ \quad u = 0 \quad \text{on} \quad (\partial\Omega)_D, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega)_N. \end{cases} \tag{7.16}$$

Let J_ν be the Bessel's function of the first kind, of index ν. For $n = 1, 2, \ldots$, let

$$\nu_n := (n - 1/2)\frac{\pi}{\omega} \quad \text{and} \quad \varphi_n(\theta) := \sqrt{2/\omega}\,\sin(\nu_n \theta), \quad \theta \in (0, \omega).$$

For each fixed n and $k = 1, 2, \ldots$, let $\beta_{k,n}$ be the k-th positive zero of $J_{\nu_n}(r) = 0$, and let $f_{k,n}(r) := c_{k,n} J_{\nu_n}(\beta_{k,n} r)$, where $c_{k,n}^{-2}$ is the positive constant given by

$$c_{k,n}^{-2} := \int_0^1 r J_{\nu_n}(\beta_{k,n} r)^2 \, dr.$$

Using separation of variables and polar coordinates for the Laplace operator, we find the following set of (eigenvalue, eigenvector) pairs:

$$(\lambda_{k,n}, \varphi_{k,n}) = \left(\beta_{k,n}^2,\ f_{k,n}(r)\ \varphi_n(\theta)\right), \quad k, n = 1, 2, \ldots.$$

Since $\{\varphi_n\}_{n \geq 1}$ is an orthonormal basis for $L^2([0, \omega])$ and for each fixed n, $\{f_{k,n}\}_{k \geq 1}$ is an orthonormal basis for $L^2([0, 1])$ with respect to the inner product with the weight function $w(r) = r$ (see, e.g., [35]), we obtain that $\{\varphi_{k,n}\}_{k,n \geq 1}$ is an orthonormal basis for $L^2(\Omega)$. Furthermore, each pair $(\lambda_{k,n}, \varphi_{k,n})$ is a solution of (7.16), and by Green's formula we have that

$$\int_\Omega \nabla \varphi_{k,n} \cdot \nabla v = \lambda_{k,n} \int_\Omega \varphi_{k,n}\ v \qquad \text{for all } v \in H_D^1(\Omega).$$

Thus, if $H_D^1(\Omega)$ is provided with the inner product

$$(u, v)_1 = \int_\Omega \nabla u \cdot \nabla v = A(u, v),$$

then $\{\lambda_{k,n}^{-1/2}\ \varphi_{k,n}\}_{k,n \geq 1}$ is an orthonormal basis for $H_D^1(\Omega)$ (see, e.g., [26]). Therefore, the norm on $H_D^1(\Omega)$ is given by

$$\|u\|_1^2 = \sum_{k,n=1}^{\infty} \lambda_{k,n} (u, \varphi_{k,n})^2.$$

Next, the norm on $H^\alpha_D(\Omega)$ for $\alpha \in [-1,1]$ is given by

$$\|u\|^2_\alpha = \sum_{k,n=1}^{\infty} \lambda^\alpha_{k,n}(u,\varphi_{k,n})^2 \qquad \text{for all } u \in H^\alpha_D(\Omega)\cap L^2(\Omega). \tag{7.17}$$

With the notation adopted in Section II B, taking $X = L^2(\Omega)$ and $Y = H^{-1}_D(\Omega)$ we have

$$(u,v)_{X,t} = \sum_{k,n=1}^{\infty} \frac{\lambda_{k,n}}{\lambda_{k,n}+t^2}(u,\varphi_{k,n})(v,\varphi_{k,n}) \qquad \text{for all } u,v \in X. \tag{7.18}$$

Theorem VII.3 *If $dim(\mathcal{N}) = 2$ and $s < 1/\gamma$, then* (7.14) *holds.*

Proof. Let $s < 1/\gamma = \nu_1$ be fixed. First, we verify the conditions **(A.1)** and **(A.2)** of the Theorem II.2 for $n = 2$, $X = L^2(\Omega)$, $Y = H^{-1}_D(\Omega)$ and $\mathcal{K} = \mathcal{N}$. Since $\psi_k \notin H^{1-\nu_k}_D$, by Remark II.3, we have that $L^2(\Omega)_{\psi_k}$ is dense in $[L^2(\Omega), H^{-1}_D(\Omega)]_{1-s}$, for $k = 1,2$. Thus, **(A.1)** is

$$[L^2(\Omega)_{\psi_k}, H^{-1}_D(\Omega)]_{1-s} = [L^2(\Omega), H^{-1}_D(\Omega)]_{1-s}, \quad \text{for} \quad k = 1,2 \tag{7.19}$$

which follows from Theorem VII.2.

Checking the condition **(A.2)** is easy in this case. From (7.18) we have

$$(\psi_1,\psi_2)_{X,t} = \sum_{k,n=1}^{\infty} \frac{\lambda_{k,n}}{\lambda_{k,n}+t^2}(\psi_1,\varphi_{k,n})(\psi_2,\varphi_{k,n}).$$

Since $(\psi_1,\varphi_{k,n}) = 0$ for $n \neq 1$ and $(\psi_2,\varphi_{k,n}) = 0$ for $n \neq 2$, we obtain that $(\psi_1,\psi_2)_{X,t} = 0$ for all $t > 0$. Thus, **(A.2)** is trivially satisfied. By Theorem II.2 we obtain that

$$[L^2(\Omega)_{\mathcal{N}}, H^{-1}_D(\Omega)]_{1-s} = [L^2(\Omega), H^{-1}_D(\Omega)]_{1-s,\mathcal{N}}.$$

Using Remark II.3 again, one sees that $L^2(\Omega)_{\mathcal{N}}$ is dense in $[L^2(\Omega), H^{-1}_D(\Omega)]_{1-s}$. It follows that

$$[L^2(\Omega), H^{-1}_D(\Omega)]_{1-s,\mathcal{N}} = [L^2(\Omega), H^{-1}_D(\Omega)]_{1-s}.$$

Therefore (7.14) holds, and the proof is complete.

■

It remains to prove Theorem VII.2.

C. Proving the subspace interpolation theorem, Theorem VII.2

Let S_ω be a sector domain defined by (7.11) and consider an extension of S_ω to a polygonal-sector domain Ω with the same angle ω and such that

$$S_\omega \subset \{(r,\theta) : 0 < r < 2r_0,\ 0 < \theta < \omega\} \subset \Omega.$$

For Ω we use the notation given in Figure 1 and for simplicity we take $r_0 = 1$.

Assume that the free part $(\partial\Omega)_N$ of $\partial\Omega$ is defined as follows:

$(\partial\Omega)_N = \emptyset$ if S_ω is in Case 1,

$(\partial\Omega)_N = \Gamma_1 \cup \Gamma_{n+2}$ if S_ω is in Case 2 and

$(\partial\Omega)_N = \Gamma_{n+2}$ if S_ω is in Case 3.

Let $\zeta \in \mathcal{D}(\Omega)$ be a cutoff function which depends only on the distance r to the origin and satisfies

$$\zeta(r,\theta) = 1 \quad for \quad 0 < \ r \leq 1/2 \ \text{ and } \ 0 < \theta < \omega,$$

$$\zeta(r,\theta) = 0 \quad for \quad r \geq 1 \ \text{ and } \ (r,\theta) \in \Omega.$$

Let ψ be one of the functions which defines the subspace $\mathcal{N}$, and let $\tilde{\psi}$ be the extension by zero of ψ to Ω. Then $\tilde{\psi} = \phi + u^R$ where $\phi(r,\theta) = \zeta\, r^{-\nu} g(\theta)$ and $u^R \in H^1_D(\Omega)$. The next result is a version of Theorem VII.2 for polygonal-sector domains.

Theorem VII.4 *If* $\nu \in (0,1)$ *and* $0 < s < 1/\gamma$, *then*

$$[L^2(\Omega)_{\tilde{\psi}}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s},$$

for any function $\tilde{\psi} = \phi + u^R$ *with* $\phi(r,\theta) = \zeta\ r^{-\nu} g(\theta)$ *and* u^R *being arbitrary function in* $H^1_D(\Omega)$.

Proof. Let $\{M_k\}$ be the sequence of approximating subspaces of $H^1_D(\Omega)$ introduced in Section III C. From the results of Section IV B, we have that an equivalent norm on $H^\alpha_D(\Omega)$ is given by the multilevel norm (4.4). By Theorem IV.1, it is enough to verify that the function $\tilde{\psi}$ satisfies the conditions **(C.0)**-**(C.2)** defined in Section IV B with $\theta_0 = 1 - \nu$.

To begin with, we will prove that the function ϕ satisfies **(C.0)**-**(C.2)**. Let $\overline{M}_k$ be the space of piecewise linear functions with respect to $\mathcal{T}_k$ defined on Ω, and let $\overline{Q}_k$ be the $L^2(\Omega)$ orthogonal projection onto $\overline{M}_k$. First step in verifying **(C.0)** and **(C.1)** is to prove that there exists a positive constant c such that

$$\|(I - \overline{Q}_k)\phi\|^2 \geq c\lambda_k^{-\theta_0},\ \ k = 1, 2, \ldots. \tag{7.20}$$

We define τ_1^k to be the triangle in $\mathcal{T}_k$ which is the the image of $\tau_1 \in \mathcal{T}_1$ via the map $\hat{x} \to h_k \hat{x}$. Here, without lost of generality, we assume that $h_k^2 = \lambda_k^{-1} = 4^{-k+1}$. Then

$$\|(I - \overline{Q}_k)\phi\|^2_{L^2(\Omega)} \geq \|(I - \overline{Q}_k)\phi\|^2_{L^2(\tau_1^k)} = \|\phi\|^2_{L^2(\tau_1^k)} - \|\overline{Q}_k\phi\|^2_{L^2(\tau_1^k)}.$$

The projection $\overline{Q}_k\phi$ can be estimated on τ_1^k in terms of the three nodal functions $\varphi_1^k, \varphi_2^k, \varphi_3^k$ associated with the three vertices of τ_1^k. If M^k is the 3×3 Gram matrix associated with the set $\{\varphi_1^k, \varphi_2^k, \varphi_3^k\}$, and $S^k := (S^k_{ij}),\ i, j = 1, 2, 3$ is the inverse of M^k, then

$$\|\phi\|^2_{L^2(\tau_1^k)} - \|\overline{Q}_k\phi\|^2_{L^2(\tau_1^k)} = \int_{\tau_1^k} \phi^2\ dx - \sum_{i,j=1}^{3} S^k_{ij} \int_{\tau_1^k} \phi\varphi_i^k\ dx \int_{\tau_1^k} \phi\varphi_j^k\ dx.$$

Further, by doing the change of variable $x = h_k\hat{x}$ in the above integrals, a simple computation shows that

$$\|\phi\|^2_{L^2(\tau_1^k)} - \|\overline{Q}_k\phi\|^2_{L^2(\tau_1^k)} = h_k^{2-2\nu}\left(\int_{\tau_1}\phi^2\ d\hat{x} - \sum_{i,j=1}^{3} S^1_{ij}\int_{\tau_1}\phi\varphi^1_i\ d\hat{x}\int_{\tau_1}\phi\varphi^1_j\ d\hat{x}\right)$$
$$= \lambda_k^{-\theta_0}\left(\|\phi\|^2_{L^2(\tau_1)} - \|\overline{Q}_1\phi\|^2_{L^2(\tau_1)}\right).$$

Since ϕ is not linear on τ_1, the constant $\|\phi\|^2_{L^2(\tau_1)} - \|\overline{Q}_1\phi\|^2_{L^2(\tau_1)}$ is strictly positive. Combining the above estimates, we have proven that (7.20) holds. The second step is to use (7.20) and the fact that M_k is a subspace of $\overline{M}_k$, in order to obtain

$$\|(I - Q_k)\phi\|^2 \geq c\lambda_k^{-\theta_0}\ ,\ \ k = 1, 2, \ldots\ . \tag{7.21}$$

From (7.21) we see that $|||\phi|||_{\theta_0}$, defined in Remark IV.1, is not finite. Hence $\phi \notin H_D^{\theta_0}(\Omega)$. Using again (7.21) and the identity

$$\|(Q_k - Q_{k-1})u\|^2 = \|(I - Q_{k-1})u\|^2 - \|(I - Q_k)u\|^2\ ,\ \ \ u \in L^2(\Omega),$$

we have

$$(\phi, \phi)_{X,t} = \frac{\lambda_1\|\phi\|^2}{\lambda_1 + t^2} + t^2\sum_{k=1}^{\infty}\frac{\lambda_{k+1} - \lambda_k}{(\lambda_{k+1} + t^2)(\lambda_k + t^2)}\|(I - Q_k)\phi\|^2$$

$$\geq ct^2\sum_{k=1}^{\infty}\frac{(4^k)^{1-\theta_0}}{(4^k + t^2)^2} = t^{-2\theta_0}\sum_{k=1}^{\infty}\frac{(4^k/t^2)^{1-\theta_0}}{(4^k/t^2 + 1)^2}.$$

Finally, the last sum can be bounded below by a positive constant independent of t as follows. Let us fix $t \geq 4$ and let k_0 be the integer such that $4^{k_0} \leq t^2 < 4^{k_0+1}$. Then

$$\sum_{k=1}^{\infty}\frac{(4^k/t^2)^{1-\theta_0}}{(4^k/t^2 + 1)^2} > \frac{(4^{k_0}/t^2)^{1-\theta_0}}{(4^{k_0}/t^2 + 1)^2} \geq \inf_{x\in[1/4,1]}\frac{x^{1-\theta_0}}{(x + 1)^2} > 0.$$

Thus, **(C.0)** and **(C.1)** hold for the function ϕ.

To verify **(C.2)** we first observe that

$$\|(Q_k - Q_{k-1})\phi\|^2 = \|Q_k(I - Q_{k-1})\phi\|^2 \leq \|(I - Q_{k-1})\phi\|^2.$$

Hence, it is enough to prove that there exists a positive constant c such that

$$\|(I - Q_k)\phi\|^2 \leq c\lambda_k^{-\theta_0}, \ \ k = 1, 2, \dots . \tag{7.22}$$

Let η_k be a cutoff function which depends only on r and satisfies

$$\eta_k(r) = 0 \ \text{ for } r \leq h_k, \ \eta_k(r) = 1 \ \text{ for } r \geq \ 2h_k,$$

$$|\eta_k'(r)| \leq c/h_k, \ \ |\eta_k''(r)| \leq c/h_k^2 \qquad \text{for all } \ h_k \leq r \leq 2h_k, \ k = 1, 2, \dots,$$

for some positive constant c. For example, we can take

$$\eta_k(r) = 1/2 + 1/2 \sin\left((r - 3h_k/2)\frac{\pi}{h_k}\right) \ \text{ on } \ [h_k, 2h_k].$$

Then, $\phi = (1 - \eta_k)\phi + \eta_k\phi$ and $\eta_k\phi \in H^2(\Omega)$. Let $\Pi_k : H^2(\Omega) \to M_k$ be the interpolant associated with $\mathcal{T}_k$. By applying standard approximation properties and (7.12) we obtain

$$\|(I - Q_k)\phi\| \leq \|(I - Q_k)(1 - \eta_k)\phi\| + \|(I - Q_k)\eta_k\phi\| \leq \|(1 - \eta_k)\phi\| + \|(I - \Pi_k)\eta_k\phi\|$$

$$\leq \|(1 - \eta_k)\phi\| + ch_k^2\|\eta_k\phi\|_{H^2(\Omega)} \leq \|(1 - \eta_k)\phi\| + ch_k^2\|\Delta(\eta_k\phi)\|_{L^2(\Omega)}.$$

Using a simple computation in polar coordinates, and the estimates for the derivative of η_k, we get

$$\|(1 - \eta_k)\phi\|^2 \leq ch_k^{2\theta_0} \qquad \text{for all } k = 1, 2, ...$$

and

$$h_k^2\|\Delta(\eta_k\phi)\|_{L^2(\Omega)} \leq ch_k^{2\theta_0} \qquad \text{for all } k = 1, 2, ... \, .$$

Combining the above inequalities, we conclude that (7.22) is valid. Thus,**(C.3)** holds for the function ϕ.

Verifying **(C.0)**-**(C.3)** for the function ϕ is mainly based on finding some positive constants c_1, c_2 such that

$$c_1\lambda_k^{-\theta_0} \leq \|(I-Q_k)\phi\|^2 \leq c_2\lambda_k^{-\theta_0},\ k=1,2,\dots . \tag{7.23}$$

Since the function u_R belongs to $H^1_D(\Omega)$, we have

$$\|(I-Q_k)u_R\|^2 \leq c\lambda_k^{-1}\ ,\ k=1,2,\dots .$$

Therefore, the function $\widetilde{\psi}$ satisfies an estimate of type (7.23) and **(C.0)**-**(C.3)** hold for the function $\widetilde{\psi}$ too. The result is now a direct consequence of Theorem IV.1.

■

Proof of Theorem VII.2. Let $E : L^2(S_\omega) \to L^2(\Omega)$, be the extension by zero operator, and let $R : L^2(\Omega) \to L^2(S_\omega)$ be defined as follows: First, we introduce a cutoff function $\eta \in \mathcal{D}(\Omega)$ which depends only on the distance r to the origin and satisfies

$$\eta(r,\theta)=1 \ \ for \ \ 0< \ r\leq 1 \ \text{ and } \ \ 0<\theta<\omega,$$

$$\eta(r,\theta)=0 \ \ for \ \ r\geq 2 \ \text{ and } \ \ (r,\theta)\in\Omega.$$

Then, for a function $v \in L^2(\Omega)$ we define $Rv \in L^2(S_\omega)$ by

$$(Rv)(r,\theta) := v_2(r,\theta), \ \ (r,\theta)\in L^2(S_\omega),$$

where,

$$v_1(r,\theta) := v(r,\theta)\eta(r,\theta), \quad (r,\theta)\in\Omega,$$

$$v_2(r,\theta) := v_1(r,\theta)-v_1(2-r,\theta), \quad (r,\theta)\in L^2_D(S_\omega).$$

Let $\widetilde{\psi}$ denote the function $E(\psi)$. According to Theorem VII.4

$$[L^2(\Omega)_{\tilde{\psi}}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}.$$

It follows that the function ψ and the operators E, R satisfy the hypotheses of Lemma II.6 with $\theta = 1 - s$, $V^1(\Omega) = H_D^1(S_\omega)$ and $V^1(\widetilde{\Omega}) = H_D^1(\Omega)$. Thus, (7.15) holds for the sector domain S_ω and the proof is complete.

Remark VII.1 *Theorem VII.2 can be proved directly without using the transfer to the polygonal sector domain. In order to do this, we just need to adapt all of the considerations made through the proof of Theorem VII.4 to the case of sector domain. The sequence of approximating subspaces on $H_D^1(S_\omega)$ which we can use is the one defined in Section III D. In this different approach, we apply Lemma II.4 instead of Lemma II.6. Consequently, we do not need to consider the extension and restriction operators connecting the sector domain and the corresponding polygonal sector domain.*

CHAPTER VIII

APPLICATION TO THE BIHARMONIC DIRICHLET PROBLEM

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$. Let $\partial\Omega$ be the polygonal arc $P_1P_2\cdots P_mP_1$. At each point P_j, we denote the measure of the angle P_j (measured from inside Ω) by ω_j. Let $\omega := \max\{\omega_j : j = 1, 2, \ldots, m\}$.

We consider the biharmonic problem Given $f \in L^2(\Omega)$, find u such that

$$\begin{cases} \Delta^2 u = f \ \text{ in } \ \Omega, \\ \quad u = 0 \ \text{ on } \ \partial\Omega, \\ \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{8.1}$$

Let $V = H_0^2(\Omega)$ and

$$a(u, v) := \sum_{1\le i,j\le 2} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \ \frac{\partial^2 v}{\partial x_i \partial x_j} \ dx, \ \ u, v, \in V.$$

The bilinear form a defines a scalar product on V and the induced norm is equivalent to the standard norm on $H_0^2(\Omega)$. The variational form of (8.1) is : Find $u \in V$ such that

$$a(u, v) = \int_\Omega fv \ dx \qquad \text{for all } v \in V. \tag{8.2}$$

Clearly, if u is a variational solution of (8.2), then one has $\Delta^2 u = f$ in the sense of distributions and because $u \in H_0^2(\Omega)$, the homogeneous boundary conditions are automatically fulfilled. As done in Chapter VII, the problem of deriving the shift estimate on Ω can be localized by a partition of unity so that only sectors domains or domains with smooth boundaries need to be considered. If Ω is a smooth domain, then it is known that the solution u of (8.2) satisfies

$$\|u\|_{H^4(\Omega)} \le c\|f\|, \qquad \text{for all } f \in L^2(\Omega),$$

and

$$\|u\|_{H^2(\Omega)} \le c\|f\|_{H^{-2}(\Omega)}, \qquad \text{for all } f \in H^{-2}(\Omega).$$

Interpolating these two inequalities yields

$$\|u\|_{2+2s} \le c\|f\|_{-2+2s}, \qquad \text{for all } f \in H^{-2+2s}(\Omega), \quad 0 \le s \le 1.$$

So we have the shift theorem for all $s \in [0, 1]$. Let us consider the case of a sector domain. The threshold, s_0, below which the shift estimate for a polygonal domain holds is given, as in the Poisson problem, by the largest internal angle ω of the polygon. Thus, it is enough to consider the domain $S\omega$ defined by

$$S_\omega = \{(r, \theta),\ 0 < r < 1, -\omega/2 < \theta < \omega/2\}.$$

We associate to (8.1) and $\Omega = S_\omega$, the characteristic equation

$$\sin^2(z\omega) = z^2 \sin^2 \omega. \tag{8.3}$$

In order to simplify the exposition of the proof, we assume that

$$\sin\sqrt{\frac{\omega^2}{\sin\omega^2} - 1} \neq \sqrt{1 - \frac{\sin\omega^2}{\omega^2}} \tag{8.4}$$

and

$$Re z \neq 2 \text{ for any solution } z \text{ of } (8.3).$$

The restriction (8.4) assures that the equation (8.3) has only simple roots. Let $z_1, z_2, \ldots, z_n$ be all the roots of (8.3) such that $0 < Re(z_j) < 2$. It is known (see [19], [23], [27], [31]) that the solution u of (8.2) can be written as

$$u = u_R + \sum_{j=1}^{n} k_j S_j, \tag{8.5}$$

where $u_R \in H^4(\Omega)$ and for $j = 1, 2, \ldots, n$, we have $S_j(r, \theta) = r^{1+z_j} u_j(\theta)$,

u_j is smooth function on $[-\omega/2, \omega/2]$ such that $u_j(-\omega/2) = u_j(\omega/2) = u_j'(-\omega/2) = u_j'(\omega/2) = 0$, $k_j = c_j \int_\Omega f\varphi_j \, dx$ and c_j is nonzero and depends only on ω. The function φ_j is called the dual singular function of the singular function S_j and $\varphi_j(r,\theta) = \eta(r)\ r^{1-z_j} u_j(\theta) - w_j$, where $w_j \in V$ is defined for a smooth truncation function η to be the solution of (8.2) with $f = \Delta^2(\eta(r)\ r^{1-z_j} u_j(\theta))$. In addition,

$$\|u_R\|_{H^4(\Omega)} \leq c\|f\|, \qquad \text{for all } f \in L^2(\Omega). \tag{8.6}$$

Next, we define $\mathcal{K} = span\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. As a consequence of the expansion (8.5) and the estimate (8.6) we have

$$\|u\|_{H^4(\Omega)} \leq c\|f\|, \qquad \text{for all } f \in L^2(\Omega)_{\mathcal{K}}. \tag{8.7}$$

Combining (8.7) with the standard estimate

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{H^{-2}(\Omega)}, \qquad \text{for all } f \in H^{-2}(\Omega),$$

we obtain, via interpolation

$$\|u\|_{[H^4(\Omega), H^2(\Omega)]_{1-s}} \leq c\|f\|_{[L^2(\Omega)_{\mathcal{K}}, H^{-2}(\Omega)]_{1-s}}, \ s \in [0,1]. \tag{8.8}$$

Let $s_0 = \min\{Re(z_j) \mid j = 1, 2, \ldots, n\}$. Then, we have

Theorem VIII.1 *If* $0 < 2s < s_0$ *and* $\Omega = S_\omega$, *then*

$$[L^2(\Omega)_{\mathcal{K}}, H^{-2}(\Omega)]_{1-s} = [L^2(\Omega), H^{-2}(\Omega)]_{1-s}. \tag{8.9}$$

Proof. First we prove that there are operators E and R such that

$$E : L^2(\Omega) \longrightarrow L^2(R), \ E : H_0^2(\Omega) \longrightarrow H^2(R^2),$$

$$R : L^2(R^2) \longrightarrow L^2(\Omega), \ R : H^2(R^2) \longrightarrow H_0^2(\Omega)$$

are bounded operators, and $REu = u, \quad$ for all $u \in L^2(\Omega)$.

Indeed, E can be taken to be the extension by zero operator.

To define R, let $\eta = \eta(r)$ be a smooth function on $(0, \infty)$ such that $\eta(r) \equiv 1$ for $0 < r \leq 1$ and $\eta(r) \equiv 0$ for $r > 2$. Define $\alpha = \frac{\omega}{2}$, $a = \frac{\alpha}{\pi - \alpha}$ and

$$g_1(\theta) = \frac{\alpha - \pi}{\alpha}\theta + \pi, \quad g_2(\theta) = \frac{\pi - \alpha}{\alpha^2}(\alpha - \theta)^2 + \alpha, \quad \theta \in [0, \alpha].$$

Note that $g_i(0) = \pi$ and $g_i(\alpha) = \alpha$, $i = 1, 2$. For a smooth function u defined on R^2 we define $Ru := u_3$, where

Step 1. $u_1 = \eta u$.

Step 2. $u_2(r, \theta) = u_1(r, \theta) + 3u_1(1/r, \theta) - 4u_1(1/2 + 1/(2r), \theta),\ r < 1,\ \theta \in [0, 2\pi)$.

Step 3. For $0 < r < 1$

$$u_3(r, \theta) = \begin{cases} u_2(r, \theta) + au_2(r, g_1(\theta)) - (1 + a)u_2(r, g_2(\theta)),\ 0 \leq \theta < \omega/2, \\ u_2(r, \theta) + au_2(r, -g_1(-\theta)) - (1 + a)u_2(r, -g_2(-\theta)),\ -\omega/2 < \theta < 0. \end{cases}$$

One can check that, for $u \in H_0^2(R^2)$, $u_3 \in H_0^2(\Omega)$ and $REu = u$. The operator R can be extended by density to $L^2(R^2)$. The extended operator R satisfies all the desired properties.

Next, let ϕ_j be the Fourier transform of $E\varphi_j$, $j = 1, \ldots, n$. Using asymptotic expansion of integrals theory ([3], [20], [33]), we have that the functions $\{E\varphi_j,\ j = 1, \ldots, n\}$ satisfy for some positive constants c and ϵ,

$$\begin{cases} |\phi_j(\xi) - \tilde{\phi}_j(\xi)| < c\rho^{-1+(-2+s_j)-\epsilon}\ for\ |\xi| > 1 \\ -2 < -2 + s_i < 0,\ i = 1, \ldots, n, \end{cases} \tag{8.10}$$

where $s_j = Re(z_j)$ and

$$\tilde{\phi}_j(\xi) = b_i(\omega)\rho^{-1+(-2+s_j)},\ \xi = (\rho, \omega) \text{ in polar coordinates,}$$

and $b_j(\cdot)$ is a bounded measurable function on the unit circle, which is non zero on a set of positive measure. A method to find the asymptotic form of the function ϕ_j given in (8.10) can be found in [25].

Thus, we have that the functions $\{E\varphi_j,\ j = 1, \ldots, n\}$ satisfy the hypothesis (5.15) of Theorem V.3 with $N = 2$, $\beta = 0$, $\alpha = -2$ and $\gamma_j = -2 + s_j$, $j = 1, \ldots, n$. Denoting $\mathcal{L} := span\{E\varphi_j,\ j = 1, \ldots, n\}$, by Theorem V.3 applied with $1 - s$ instead of s, we have that

$$[L^2(R^2)_{\mathcal{L}}, H^{-2}(R^2)]_{1-s} = [L^2(R^2), H^{-2}(R^2)]_{1-s} = H^{-2+2s}(R^2), \tag{8.11}$$

for $2s < s_0 := \min\{Re(z_j),\ j = 1, 2, \ldots, n\}$.

Finally, using (8.11), the operators E , R and Lemma II.6 (adapted to the case when we work with subspaces of codimension $n > 1$), we conclude that (8.9) holds for $2s < s_0$.

■

From the estimate (8.8) and the interpolation result (8.9) we obtain

$$\|u\|_{2+2s} \leq c\|f\|_{-2+2s}, \qquad \text{for all } f \in H^{-2+s}(\Omega), \quad 0 \leq 2s < s_0.$$

The above estimate still holds for the case when Ω is a polygonal domain and s_0 corresponds to the largest inner angle ω of the polygon. Figure 4 below gives the graph of the function $\omega \to 2 + s_0(\omega)$ which represents the regularity for the biharmonic problem. On the same graph we represent the the number of singular (dual singular) functions as function of $\omega \in (0, \pi)$. Note that if ω is bigger than 1.43π, which is an approximation for the solution in $(0, 2\pi)$ of the equation $\tan\omega = \omega$, the space $\mathcal{K}$ has the dimension six.

FIGURE 4. Regularity for the biharmonic problem.

CHAPTER IX

AN APPLICATION TO A NONCONFORMING FINITE ELEMENT PROBLEM

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$. The $L^2(\Omega)$-inner product and the $L^2(\Omega)$-norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We consider the Dirichlet problem

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ \quad u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{9.1}$$

The variational formulation of (9.1) is :

Find $u \in V := H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{9.2}$$

where $F \in V' := H^{-1}(\Omega)$ and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx \qquad \text{for all } u, v \in H_0^1(\Omega).$$

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω and let $h = \max_{\tau \in \mathcal{T}_h} diam(\tau)$.

Next, we consider the Crouzeix-Raviart finite element nonconforming space

$$\begin{aligned} V_h := \{v| \; & v \text{ is linear on all } \tau \in \mathcal{T}, \\ & v \text{ is "continuous" at the midpoints of the edges} \\ & v = 0 \text{ at the midpoints situated on } \partial\Omega\}, \end{aligned}$$

and define on $V + V_h$ the bilinear form

$$a_h(u,v) := \sum_{\tau \in \mathcal{T}_h} D_\tau(u,v), \text{ where } D_\tau(u,v) = \int_\tau \nabla u \cdot \nabla v \; dx$$

and the associated norm

$$\|u\|_h := \sqrt{a_h(u,u)}.$$

The form $a_h(\cdot,\cdot)$ is positive definite on V_h because $v \in V_h$ and $a_h(v,v) = 0$ implies $v \equiv 0$. Assuming that $F \in V'$ has a linear extension to $V + V_h$, which will still be denoted by F, we consider a first type discretized problem associated with the variational problem (9.2):

Find $u_h \in V_h$ such that

$$a_h(u_h, v) = F(v) \qquad \text{for all } v \in V_h. \tag{9.3}$$

The next statement is a version of Strang's Lemma [4], [17].

Proposition IX.1 *Let $u \in V$ and $w \in V_h$ be completely arbitrary. Then*

$$\|u - w\|_h \leq \inf_{v \in V_h} \|u - v\|_h + \sup_{v \in V_h} \frac{a_h(u - w, v)}{\|v\|_h} \tag{9.4}$$

Proof. Let $\tilde{u} \in V_h$ satisfy

$$a_h(\tilde{u}, v) = a_h(u, v) \qquad \text{for all } v \in V_h.$$

Then, $a_h(\tilde{u} - u, v) = 0$ for all $v \in V_h$ and consequently,

$$\|u - \tilde{u}\|_h = \inf_{v \in V_h} \|u - v\|_h.$$

Thus,

$$\|u - w\|_h \leq \|u - \tilde{u}\|_h + \|\tilde{u} - w\|_h = \|u - \tilde{u}\|_h + \sup_{v \in V_h} \frac{a_h(\tilde{u} - w, v)}{\|v\|_h}.$$

Moreover,

$$a_h(\tilde{u} - w, v) = a_h(\tilde{u} - u + u - w, v) = a_h(u - w, v).$$

Combining the above estimate and equalities we obtain (9.4).

∎

In particular, when u is the solution of (9.2) and $w = u_h$ is the solution of (9.3) we obtain the estimate

$$\|u - u_h\|_h \leq \inf_{v \in V_h} \|u - v\|_h + \sup_{v \in V_h} \frac{a_h(u - u_h, v)}{\|v\|_h}. \tag{9.5}$$

If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, the first term of the right-hand side of (9.5) can be estimated, using standard approximation properties, by $ch|u|_{H^2}$ (with c independent of u and h). For the second term we can use the following known result (see, e.g., [4], [18]).

Lemma IX.1 *Let* $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *be the solution of* (9.2), *where* $f = -\Delta u$ *and* $F(v) = (f, v)$ *for all* $v \in V + V_h$. *Let* u_h *be the solution of the discrete problem* (9.3). *Then, for some positive constant* c

$$\frac{a_h(u - u_h, v)}{\|v\|_h} \leq ch|u|_{H^2}, \quad \text{for all } v \in V_h,\ u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{9.6}$$

Consequently,

$$\|u - u_h\|_h \leq ch|u|_{H^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{9.7}$$

For completeness we include a proof.

Proof. By Green's Formula we have

$$a_h(u - u_h, v) = a_h(u, v) - (f, v) = \sum_{\tau \in \mathcal{T}_h} D_h(u, v) - (f, v) = \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} \frac{\partial u}{\partial n} v \, ds.$$

Let e be one of the three edges of an arbitrary $\tau \in \mathcal{T}_h$. Denote by $I^e u$ the nodal interpolant of the trace of u on e (where the nodes are just the ends of e).

For $v \in V$, let $\bar{v}_e$ denote the average of the trace of v on e. Because v is linear on each $\tau \in \mathcal{T}_h$ and "continuous" at midpoints, $\bar{v}_e$ does not depend on the triangle τ such that $e \in \partial \tau$. Next, we need the following

Proposition IX.2 *Let* $\tau \in \mathcal{T}_h$ *be a triangle and* e *be one of its edges. Then*

$$||v - \bar{v}_e||_{L^2(e)} \leq ch^{\frac{1}{2}} |v|_{H^1(\tau)}, \quad \text{for all } v \in H^1(\tau), \tag{9.8}$$

with c independent of $\tau \in \mathcal{T}_h$.

The proof is based on trace estimate and the Bramble-Hilbert lemma [6], [22]. Next, using the above proposition, we have

$$\begin{aligned}
\sum_{\tau\in\mathcal{T}_h} \int_{\partial\tau} \frac{\partial u}{\partial n}\, v\, ds &= \sum_{e\in\tau} \frac{\partial u}{\partial n}\,(v-\bar{v}_e)\, ds = \sum_{\tau\in\mathcal{T}_h}\sum_{e\in\partial\tau} \int_e \left(\frac{\partial u}{\partial n} - \frac{\partial(I^e u)}{\partial n}\right)(v-\bar{v}_e)\, ds \\
&\le \sum_{\tau\in\mathcal{T}_h}\sum_{e\in\partial\tau} \Big(\int_e |\nabla(u-I^e u)|^2\, ds\Big)^{\frac{1}{2}} \Big(\int_e |v-\bar{v}_e|^2\, ds\Big)^{\frac{1}{2}} \\
&\le c\sum_{\tau\in\mathcal{T}_h} h^{1/2}|u|_{H^2(\tau)}\; h^{1/2}\|v\|_h \le c\sum_{\tau\in\mathcal{T}_h} h|u|_{H^2(\tau)}\; \|v\|_h.
\end{aligned}$$

This ends the proof of Lemma IX.1.

■

The method given by the discretized problem (9.3) has the disadvantage of not being stable on $H^1(\Omega)$. A modified method was shown to the author by Joseph Pasciak. This modified method is as follows:

First, we define $\mathcal{T}_{\frac{h}{2}}$ to be the triangulation obtained from $\mathcal{T}_h$ by joining the midpoints of the edges of the triangles in $\mathcal{T}_h$. Let $S_{\frac{h}{2}}$ be the standard conforming finite element space of all functions in $H_0^1(\Omega)$ which are linear on each triangle $\tau \in \mathcal{T}_{\frac{h}{2}}$. Note that $S_{\frac{h}{2}} \subset V$.

Next, we define the operator $T : V_h \longrightarrow S_{\frac{h}{2}}$ by $Tv = w$, where

1. $w(x) = v(x)$ when x is a midpoint of an edge in $\mathcal{T}_h$,
2. $w(x) = 0$ when x is a vertex of $\partial\Omega$,
3. $w(x) = \frac{1}{n_x}\sum_{j=1}^{n_x} v(y_j)$ when x is an interior vertex of $\mathcal{T}_h$, where $y_1, y_2, \ldots, y_{n_x}$ are the midpoints of those edges in $\mathcal{T}_h$, that are adjacent to x.

Clearly, n_x is bounded above by a fixed natural number. Let M_h be the set of all midpoints of the edges in $\mathcal{T}_h$. Let E_h be the set of all line segments connecting in each triangles in $\mathcal{T}_h$ the mid points of the edges. Finally, let $E_{h/2}$ be the set of all edges in $\mathcal{T}_{\frac{h}{2}}$. Then,

$$\|v\|^2 \approx \; h^2 \sum_{y_i \in M_h} v^2(y_i), \; v \in V_h,$$

$$\|v\|_h^2 \approx \sum_{(y_i,y_j)\in E_h} (v(y_i) - v(y_j))^2, \; v \in V_h,$$

and

$$|w|_{H^1(\Omega)}^2 \approx \sum_{(x_i,x_j)\in E_{h/2}} (w(x_i) - w(x_j))^2, \; w \in S_{h/2}.$$

From the way we defined T and by using the above equivalences, it is easy to verify that

$$|Tv|_{H^1(\Omega)}^2 \leq c\; a_h(v,v), \qquad \text{for all } v \in V_h, \tag{9.9}$$

$$\|Tv - v\|^2 \leq ch^2\; a_h(v,v), \qquad \text{for all } v \in V_h, \tag{9.10}$$

for some positive constant c. Consider the following modified version of problem 9.3: Given $F \in V'$ find $\tilde{u}_h \in V_h$ such that

$$a_h(\tilde{u}_h, v) = F(Tv), \qquad \text{for all } v \in V_h, \tag{9.11}$$

Using the interpolation results of Chapter VII, we deduce an error estimate for the new method.

Theorem IX.1 *Let u be the solution of* (9.2) *and let $\tilde{u}_h$ be the solution of* (9.11)*. Then, for $s \in [0,1]$ we have the following error estimate:*

$$\|u - \tilde{u}_h\|_h \leq ch^s \|u\|_{H^{1+s}(\Omega)}, \qquad \textit{for all } u \in H^{1+s}(\Omega) \cap H_0^1(\Omega). \tag{9.12}$$

Proof. By taking $v = \tilde{u}_h$ in (9.11) and by using (9.9) we get

$$\|\tilde{u}_h\|_h^2 = F(T\tilde{u}_h) \leq \|F\|_{V'} |T\tilde{u}_h|_{H^1(\Omega)} \leq c\|u\|_{H^1(\Omega)} \|\tilde{u}_h\|_h.$$

Hence

$$\|\tilde{u}_h\|_h \le c\|u\|_{H^1(\Omega)}, \qquad \text{for all } u \in H_0^1(\Omega). \tag{9.13}$$

Let $P_h u = \tilde{u}_h$, $u \in H_0^1(\Omega)$. Then (9.13) implies

$$\|(I - P_h)u\|_h \le c\|u\|_{H^1(\Omega)}, \qquad \text{for all } u \in H_0^1(\Omega). \tag{9.14}$$

Next, for $u \in H^2 \cap H_0^1$, from Proposition IX.1, we obtain

$$\|u - \tilde{u}_h\|_h \le \inf_{v \in V_h} \|v - u\|_h + \sup_{v \in V_h} \frac{a_h(u - \tilde{u}_h, v)}{\|v\|_h}.$$

Using standard approximation properties, we have

$$\inf_{v \in V_h} \|u - v\|_h \le \inf_{v \in S_h} \|u - v\|_h \le ch|u|_{H^2(\Omega)}.$$

To estimate the second term in the right-hand side of the above inequality, we proceed as follows:

$$a_h(u - \tilde{u}_h, v) = a_h(u - u_h, v) + a_h(u_h - \tilde{u}_h, v), \; v \in V_h.$$

From Lemma IX.1,

$$a_h(u - u_h, v) \le ch|u|_{H^2(\Omega)}\|v\|_h.$$

On the other hand, with the help of (9.10),

$$\begin{aligned} a_h(u_h - \tilde{u}_h, v) &= a_h(u_h, v) - a_h(\tilde{u}_h, v) = (f, v) - (f, Tv) \\ &\le \|f\| \; \|v - Tv\| \le ch|u|_{H^2(\Omega)}\|v\|_h. \end{aligned}$$

Combining the above estimates, we have

$$\|(I - P_h)u\|_h \le ch\|u\|_{H^2(\Omega)}, \qquad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{9.15}$$

Finally, from (9.14) and (9.15) by using interpolation and the result of Chapter VI we obtain

$$\|(I - P_h)u\|_h \leq ch^s \|u\|_{[H^2 \cap H_0^1, H_0^1]_{1-s}} = ch^s \|u\|_{H^{1+s}(\Omega)} \quad \text{for all } u \in H^{1+s}(\Omega) \cap H_0^1(\Omega),$$

which proves the theorem.

∎

The interpolation result of Chapter VI allows us to adapt the above result with no difficulties to the Dirichlet problem with mixed boundary conditions.

CHAPTER X

CONCLUSIONS

Our studies on subspace interpolation have contributed to new results concerning shift theorems for boundary value problems on nonsmooth domains. The theorems and the lemmas presented in Chapter II are proved on abstract Hilbert spaces and one can use them for particular subspace interpolation problems where the subspaces involved are of finite codimension. The choice of the inner product which provides a norm equivalent to the original norm on a Hilbert space is important in solving subspace interpolation. In studying the shift theorem of Chapter VII, for example, the multilevel inner product we have chosen, leads to a simple way of dealing with the subspace interpolation problem.

The multilevel representation of norms presented in Chapter III is self-contained and may be used in numerical methods for solving certain partial differential equations. A multilevel norm equivalent to the standard norm on H_0^2 (for polygonal or sector domains) is needed in order to prove shift theorems for the Biharmonic Dirichlet problem without involving extension and restriction operators and asymptotic expansions of Fourier transforms. This is an area of further research.

Other alternative for a future research comes from the subspace interpolation theory presented in Chapter V. The results concerning interpolation between Sobolev spaces on R^N, could be extended to the case when the functions to be factored out from the space H^β have a more complicated asymptotic expansion. If the orthogonal complement of the range of the operator associated with a certain boundary value problem can be characterized in terms of functions with asymptotic expansion (for the Fourier transform) different from the form we have presented through Chapter V, then the issue is worth addressing.

The positive answer obtained in Chapter VI concerning interpolation between $H^2 \cap H^1_D$ and H^1_D and the error estimate for finite element problem of Chapter IX gives hope for new methods and new ways to prove error estimates for conforming and nonconforming finite element problems.

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