# SADDLE POINT LEAST SQUARES ITERATIVE SOLVERS FOR THE TIME HARMONIC MAXWELL EQUATIONS

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*Dedicated to Peter Monk on his 60th birthday*

Abstract. We introduce a least squares method for discretizing the time-harmonic Maxwell equations written as a first order system. The method belongs to the recently introduced least squares approach called the Saddle Point Least Squares (SPLS) method (see [12, 13]). It is related to two known methods: the Bramble and Pasciak least squares method for solving electromagnetic problems [22, 23] and the minimum residual DPG method of Demkowicz and Gopalakrishnan [32, 33]. The proposed discretization is based on reduction to a symmetric saddle point formulation on spaces of piecewise polynomial functions. The edge elements are avoided. A discrete test space is chosen first among standard conforming finite element spaces and the discrete trial space is chosen second as the image of the first order Maxwell differential operator on the discrete test space. For the proposed iterative processes, a nodal basis for the trial space is not required and the main inversion at each step is done only on the test space. The method is efficient for both convex and non-convex domains and for a various range of frequencies as shown by the numerical results.

## 1. Introduction

Efficient approximation of the time-harmonic Maxwell equations is of significant importance to practical applications such as *analog signal packages*. For Maxwell's equations, one needs a robust methodology independent of frequency. Recently, there have been many advances in applying finite element least squares methods to approximate first order systems of PDEs [16, 17, 18, 27, 28, 29, 30, 36, 37, 38]. Our proposed framework is different in the sense that it uses a saddle point formulation of the problem that requires different types of test and trial spaces. The proposed method connects with the theory of discretizing symmetric saddle point systems as presented in [4, 9, 10, 15, 24, 25, 26, 35, 39, 41, 42], leading to robust and fast iterative processes. In this paper, we will focus on constructing efficient iterative processes that are able to approximate simultaneously the electric and magnetic field solutions of the time-harmonic Maxwell equations. Our goal is to use standard finite element spaces and preconditioning techniques to obtain simple to implement, robust iterative solvers. Numerical experiments presented in Section 5 show that the method performs well for a large range of frequencies.

Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain with boundary $\Gamma$. Consider two positive functions for electric permittivity and magnetic permeability,

$$\varepsilon, \mu \in L^\infty(\Omega), \qquad \varepsilon_1 > \varepsilon \geq \varepsilon_0 > 0, \qquad \mu_1 > \mu \geq \mu_0 > 0 \qquad \text{on } \Omega.$$

We look for a solution to the time-harmonic Maxwell problem given by the equations

$$\nabla \times \mathbf{h} - \lambda\, \varepsilon\, \mathbf{e} = \mathbf{j} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla \times \mathbf{e} + \lambda\, \mu\, \mathbf{h} = \mathbf{m} \qquad \text{in } \Omega, \tag{1.1b}$$

$$(\mu\, \mathbf{h}) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{1.1c}$$

$$\mathbf{e} \times \mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma, \tag{1.1d}$$

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where $\mathbf{h}$ and $\mathbf{e}$ are the magnetic and electric vector fields, $\mathbf{j}$ and $\mathbf{m}$ are the electric and magnetic current densities, $\varepsilon$ is the electric permittivity, $\mu$ is the magnetic permeability, and $\lambda = -i\omega$. Here, $\omega \in \mathbb{R}$ is given and represents the frequency of propagation of the electromagnetic waves. The boundary conditions correspond to a region surrounded by a perfect conductor.

There are many methods designed to efficiently approximate (1.1), see [39] and the references therein. Most of these methods use curl-conforming edge elements. In [40], an interior penalty DG method based on a mixed variational formulation was introduced. Many other techniques, including adaptive methods for (1.1) starting with the work in [34], have been investigated in the last two decades. Our approach in discretizing (1.1) is similar with the work of Bramble and Pasciak in [23]. We assume $L^2$ type spaces for the magnetic and electric vector fields and start from a natural weak formulation as presented in [23]. For discretization, we require that the test spaces be $H^1$-conforming with suitable boundary conditions. We depart from the Bramble-Pasciak least squares method in the way the discrete trial spaces are chosen. We build the discrete trial spaces using the action of the continuous first order differential operator $B$ associated with the problem (1.1), see Section 3. The choice of the discrete trial space leads to a saddle point variational formulation with automatic discrete $\inf - \sup$ condition. Assembly of stiffness matrices for the trial spaces is avoided.

The proposed SPLS approach could also be placed in the class of minimum residual type methods. Due to its saddle point mixed formulation, it might be viewed as another version of the DPG family. As presented in [32, 33, 31], for the DPG method a trial space is chosen first and then a test space that provides stability of the pairs is chosen second. Our method is different from DPG because of the order and roles of the test and trial spaces.

In addition to these advantages that are characteristic to the SPLS discretization method, the main contribution of the proposed discretization for the time-harmonic Maxwell Equations resides in investigating the stability of the proposed families of discretization spaces. For the case of a *no projection* choice of discrete trial space, we investigated numerical stability of the proposed family of discrete spaces, see Section 4.1. For the other two *projection* choices of discrete trial spaces, we proved that the stability is at least as good as the stability for the *no projection* case, see Section 3.5.2 and Theorem 4.1. The numerical tests we perform show that the method is efficient for both convex and non-convex domains and is robust with respect to the frequency parameter $\omega$.

The rest of the paper is organized as follows. In Section 2, we review the weak variational formulation of (1.1) and the connection between the weak formulation and the original formulation. The main results and steps of the SPLS discretization are presented in Section 3. We review two different choices of discrete spaces that automatically satisfy an $\inf - \sup$ condition and that have good approximability properties. We also describe the iterative processes that are associated with the SPLS method. In Section 4, we apply the abstract discretization theory of Section 3 to the operator $B$ associated with the Maxwell equations. Starting with a common test space, we propose three ways to choose trial spaces and investigate stability and approximability of the corresponding pairs of discrete spaces. Numerical results to support our Maxwell discretization method are presented in Section 5.

## 2. Variational formulation of the problem

By writing all the complex functions that appear in (1.1) using the real and imaginary parts, one may conclude that both the real and imaginary parts satisfy a related real problem to the (1.1) system, see Remark 2.2 of [23]. More specifically, $(\mathbf{h}, \mathbf{e}, \mathbf{j}, \mathbf{m})$ satisfies the (1.1) system with

$\lambda = -i\omega$ if and only if $(\Re(\mathbf{h}), \Im(\mathbf{e}), \Re(\mathbf{j}), \Im(\mathbf{m}))$ and $(-\Im(\mathbf{h}), \Re(\mathbf{e}), -\Im(\mathbf{j}), \Re(\mathbf{m}))$ satisfy

$$\nabla \times \mathbf{h} - \omega\,\varepsilon\,\mathbf{e} = \mathbf{j} \qquad \text{in } \Omega, \tag{2.1a}$$

$$\nabla \times \mathbf{e} - \omega\,\mu\,\mathbf{h} = \mathbf{m} \qquad \text{in } \Omega, \tag{2.1b}$$

$$(\mu\,\mathbf{h}) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{2.1c}$$

$$\mathbf{e} \times \mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma. \tag{2.1d}$$

Thus, we can restrict our considerations only to real functions and a real parameter $\omega$. Without loss of generality, we can assume that all Hilbert spaces in this paper are real Hilbert spaces.

2.1. **Notation and background.** In this section, we present some basic concepts about the Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $\mathbf{H}(\mathrm{curl}, \Omega)$, $\mathbf{H}_0(\mathrm{curl}, \Omega)$, $\mathbf{H}(\mathrm{div}, \Omega)$, and $\mathbf{H}_0(\mathrm{div}, \Omega)$ and use them to describe the weak formulation of (2.1) and its connection to the original formulation. For more details on these Sobolev spaces, see [35].

If $V$ is a space of scalar valued functions, then $\mathbf{V} := V^3$ will be the vector-valued product space endowed with the product topology. The representation of the dual space of $H^1_0(\Omega)$ as a space of distributions will be denoted $H^{-1}(\Omega)$. Similarly, we will need the space $\widetilde{H}^{-1}(\Omega) := H^1(\Omega)'$ which is not a space of distributions. We will freely use the following Gelfand triples

$$H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega), \qquad \text{and} \qquad H^1(\Omega) \subset L^2(\Omega) \subset \widetilde{H}^{-1}(\Omega). \tag{2.2}$$

Duality products $V' \times V$ will be denoted using angled brackets, unsubscripted unless otherwise needed. The usual $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ inner products will be denoted with the symbol $(\cdot,\cdot)_\Omega$. Strong solutions of (2.1) are thought of in the following spaces:

$$\begin{aligned} \mathbf{h} \in \mathbf{X}_1(\mu) &:= \mathbf{H}(\mathrm{curl}, \Omega) \cap \{\mathbf{h} \in \mathbf{L}^2(\Omega) \,:\, \nabla \cdot (\mu\,\mathbf{h}) \in L^2(\Omega),\ (\mu\,\mathbf{h}) \cdot \mathbf{n} = 0\}, \\ \mathbf{e} \in \mathbf{X}_2(\varepsilon) &:= \mathbf{H}_0(\mathrm{curl}, \Omega) \cap \{\mathbf{e} \in \mathbf{L}^2(\Omega) \,:\, \nabla \cdot (\varepsilon\,\mathbf{e}) \in L^2(\Omega)\}. \end{aligned}$$

Note that if the equations (2.1) have a solution, the data functions have to satisfy

$$\mathbf{j} \in \mathbf{H}(\mathrm{div}, \Omega), \qquad \mathbf{m} \in \mathbf{H}_0(\mathrm{div}, \Omega). \tag{2.3}$$

We also notice that if $(\mathbf{h}, \mathbf{e})$ is a strong solution to (2.1), we necessarily have

$$\nabla \cdot (\varepsilon\,\mathbf{e}) = -\omega^{-1} \nabla \cdot \mathbf{j}, \qquad \nabla \cdot (\mu\,\mathbf{h}) = -\omega^{-1} \nabla \cdot \mathbf{m}. \tag{2.4}$$

**Hypothesis:** If $\mathbf{j} = \mathbf{0}$ and $\mathbf{m} = \mathbf{0}$, then the only solution of (2.1) in $\mathbf{X}_1(\mu) \cap \mathbf{X}_2(\varepsilon)$ is $\mathbf{h} = \mathbf{0}$, $\mathbf{e} = \mathbf{0}$. In other words, we assume that $\omega$ *is not a Maxwell eigenvalue.*

To define the global differential operators associated with (2.1), we first define the following spaces and inner products:

$$L^2_\beta(\Omega) := \{u : \Omega \to \mathbb{R} \,:\, \beta^{1/2}\,u \in L^2(\Omega)\}, \qquad (u, v)_\beta := (\beta\,u, v)_\Omega, \qquad \beta \in \{\mu, \varepsilon\}.$$

Next, following the notation of [23], we consider the following four operators:

$$\begin{aligned} \mathbf{curl}_1 : \mathbf{L}^2_\mu(\Omega) \to \mathbf{H}^{-1}(\Omega) \qquad & \langle \mathbf{curl}_1 \mathbf{h}, \boldsymbol{\phi} \rangle := (\mathbf{h}, \mu^{-1} \nabla \times \boldsymbol{\phi})_\mu = (\mathbf{h}, \nabla \times \boldsymbol{\phi})_\Omega,\ \boldsymbol{\phi} \in \mathbf{H}^1_0(\Omega), \\ \mathbf{curl}_2 : \mathbf{L}^2_\varepsilon(\Omega) \to \widetilde{\mathbf{H}}^{-1}(\Omega) \qquad & \langle \mathbf{curl}_2 \mathbf{e}, \boldsymbol{\psi} \rangle := (\mathbf{e}, \varepsilon^{-1} \nabla \times \boldsymbol{\psi})_\varepsilon = (\mathbf{e}, \nabla \times \boldsymbol{\psi})_\Omega,\ \boldsymbol{\psi} \in \mathbf{H}^1(\Omega), \\ \mathrm{div}_{1,\mu} : \mathbf{L}^2_\mu(\Omega) \to \widetilde{H}^{-1}(\Omega) \qquad & \langle \mathrm{div}_{1,\mu} \mathbf{h}. \psi \rangle := -(\mathbf{h}, \nabla \psi)_\mu = -(\mu\,\mathbf{h}, \nabla \psi)_\Omega,\ \psi \in H^1(\Omega), \\ \mathrm{div}_{2,\varepsilon} : \mathbf{L}^2_\varepsilon(\Omega) \to H^{-1}(\Omega) \qquad & \langle \mathrm{div}_{2,\varepsilon} \mathbf{e}, \phi \rangle := -(\mathbf{e}, \nabla \phi)_\varepsilon = -(\varepsilon\,\mathbf{e}, \nabla \phi)_\Omega,\ \phi \in H^1_0(\Omega). \end{aligned}$$

Note that $\mathbf{curl}_1$ is the distributional curl acting on elements of $\mathbf{L}^2_\mu(\Omega) \equiv \mathbf{L}^2(\Omega)$ and that $\mathrm{div}_{2,\varepsilon} = \nabla \cdot (\varepsilon\,\cdot)$, with the divergence operator taken in the sense of distributions acting on $\mathbf{L}^2(\Omega)$

vector fields. The two remaining operators cannot be understood as distributional differentiation operators. We also need two multiplication operators

$$\mu : \mathbf{L}^2_\mu(\Omega) \to \mathbf{H}^{-1}(\Omega) \qquad \mu\,\mathbf{e} := (\mathbf{e},\cdot)_\mu = (\mu\,\mathbf{e},\cdot)_\Omega : \mathbf{H}^1_0(\Omega) \to \mathbb{R},$$
$$\varepsilon : \mathbf{L}^2_\varepsilon(\Omega) \to \widetilde{\mathbf{H}}^{-1}(\Omega) \qquad \varepsilon\,\mathbf{e} := (\mathbf{e},\cdot)_\varepsilon = (\varepsilon\,\mathbf{e},\cdot)_\Omega : \mathbf{H}^1(\Omega) \to \mathbb{R}.$$

These are multiplication operators followed by *compact* inclusions in the corresponding right side of the Gelfand triples in (2.2).

Next, we define the global operators associated with (2.1). We let

$$\mathbb{M} := \mathbf{L}^2_\mu(\Omega) \times \mathbf{L}^2_\varepsilon(\Omega) \equiv \mathbb{M}',$$

endowed with its weighted product norm and inner product and

$$\mathbb{V} := \mathbf{H}^1_0(\Omega) \times \mathbf{H}^1(\Omega) \times H^1(\Omega) \times H^1_0(\Omega),$$

endowed with the norm

$$\|(\boldsymbol{\phi},\boldsymbol{\psi},\psi,\phi)\|^2_{\mathbb{V}} := \|\nabla\boldsymbol{\phi}\|^2_\Omega + \|\boldsymbol{\psi}\|^2_\Omega + \|\nabla\boldsymbol{\psi}\|^2_\Omega + \|\psi\|^2_\Omega + \|\nabla\psi\|^2_\Omega + \|\nabla\phi\|^2_\Omega.$$

The corresponding inner product on $\mathbb{V}\times\mathbb{V}$ is denoted by $a_0(\cdot,\cdot)$. If we define $\mathbf{v} := (\boldsymbol{\phi},\boldsymbol{\psi},\psi,\phi)$ and $\mathbf{p} := (\mathbf{h},\mathbf{e})$, then the bilinear form $b(\cdot,\cdot)$ on $\mathbb{V}\times\mathbb{M}$ is defined by

$$b(\mathbf{v},\mathbf{p}) := b((\boldsymbol{\phi},\boldsymbol{\psi},\psi,\phi),(\mathbf{h},\mathbf{e})) := (\nabla\times\boldsymbol{\phi} - \omega\mu\boldsymbol{\psi} - \mu\nabla\psi,\mathbf{h}) + (-\omega\epsilon\boldsymbol{\phi} + \nabla\times\boldsymbol{\psi} - \epsilon\nabla\phi,\mathbf{e}).$$

With the form $b(\cdot,\cdot)$, we associate the operators $B : \mathbb{V}\to\mathbb{M}$ and $B^* : \mathbb{M}\to\mathbb{V}'$ given by the matrix operators

$$B := \begin{bmatrix} \mu^{-1}\nabla\times & -\omega & -\nabla & 0 \\ -\omega & \varepsilon^{-1}\nabla\times & 0 & -\nabla \end{bmatrix}, \qquad B^* := \begin{bmatrix} \mathbf{curl}_1 & -\omega\varepsilon \\ -\omega\mu & \mathbf{curl}_2 \\ \mathrm{div}_{1,\mu} & 0 \\ 0 & \mathrm{div}_{2,\varepsilon} \end{bmatrix}. \tag{2.5}$$

Following Section 2 of [23], the weak form of equations (2.1), incorporating the information of (2.4), is

$$B^* \begin{bmatrix} \mathbf{h} \\ \mathbf{e} \end{bmatrix} = \mathbf{f} := \begin{bmatrix} \mathbf{j} \\ \mathbf{m} \\ -\omega^{-1}\nabla\cdot\mathbf{m} \\ -\omega^{-1}\nabla\cdot\mathbf{j} \end{bmatrix}. \tag{2.6}$$

All boundary conditions in (2.1) are hidden in the dualization process for the definition of $B^*$ (see [23] for details). The corresponding variational formulation of (2.6) is:
Find $\mathbf{p} = (\mathbf{h},\mathbf{e}) \in \mathbb{M}$ such that

$$b(\mathbf{v},\mathbf{p}) = \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v} = (\boldsymbol{\phi},\boldsymbol{\psi},\psi,\phi) \in \mathbb{V}. \tag{2.7}$$

The relevant information that we need for solving (2.1) via the weak formulation (2.6) or (2.7) is concentrated in the following theorem proved in [23].

**Theorem 2.1.** *Assume that $\omega$ is not a Maxwell eigenvalue. The operator $B^* : \mathbb{M}\to\mathbb{V}'$ is injective and has closed range. If $\mathbf{j}\in\mathbf{H}(\mathrm{div},\Omega)$ and $\mathbf{m}\in\mathbf{H}_0(\mathrm{div},\Omega)$, then $(\mathbf{j},\mathbf{m},-\omega^{-1}\nabla\cdot\mathbf{m},-\omega^{-1}\nabla\cdot\mathbf{j})^\top$ is in the range of $B^*$ and the unique solution of (2.6) or (2.7) is a strong solution of (2.1).*

The theorem allows us to approximate the solution of (2.1) by discretizing the mixed weak formulation (2.7).

## 3. The general SPLS approach

In this section, we will review the SPLS method as presented in [12, 13]. The general theory presented in this section will be used to define the SPLS discretization of (2.7).

The general problem that can be discretized using the SPLS method is:
Given $f \in \mathbb{V}^*$, find $p \in \mathbb{M}$ such that

$$b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathbb{V}, \quad \text{or} \quad B^* p = \mathbf{f}. \tag{3.1}$$

Here, $\mathbb{V}$ and $\mathbb{M}$ are infinite dimensional Hilbert spaces and $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbb{V} \times \mathbb{M}$ that satisfies a standard $\inf - \sup$ condition. We assume that on $\mathbb{V}$ and $\mathbb{M}$ the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ induce the norms $|\cdot|_{\mathbb{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_{\mathbb{M}} = \|\cdot\| = (\cdot,\cdot)^{1/2}$, respectively. The dual pairing on $\mathbb{V}^* \times \mathbb{V}$ is denoted by $\langle \cdot, \cdot \rangle$. In this section, $\mathbb{V}^*$ and $\mathbb{M}^*$ denote the duals of $\mathbb{V}$ and $\mathbb{M}$, respectively. Since our application needs $\mathbb{M}$ to be an $L^2$ type space, in what follows we identify $\mathbb{M}^*$ with $\mathbb{M}$ ($\mathbb{M}^* \equiv \mathbb{M}$). With the inner product $a_0(\cdot,\cdot)$, we associate the operator $A : \mathbb{V} \to \mathbb{V}^*$ defined by

$$\langle A\mathbf{u}, \mathbf{v} \rangle = a_0(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{V}.$$

We assume that $b(\cdot,\cdot)$ satisfies

$$\sup_{p \in \mathbb{M}} \sup_{\mathbf{v} \in \mathbb{V}} \frac{b(\mathbf{v}, p)}{\|p\| \, |\mathbf{v}|} = M < \infty, \tag{3.2}$$

and that the following $\inf - \sup$ condition holds,

$$\inf_{p \in \mathbb{M}} \sup_{\mathbf{v} \in \mathbb{V}} \frac{b(\mathbf{v}, p)}{\|p\| \, |\mathbf{v}|} = m > 0. \tag{3.3}$$

With the form $b(\cdot,\cdot)$, we associate the linear operators $B : \mathbb{V} \to \mathbb{M}^* \equiv \mathbb{M}$ and $B^* : \mathbb{M} \to \mathbb{V}^*$ defined by

$$(B\mathbf{v}, q) = b(\mathbf{v}, q) = \langle B^* q, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathbb{V}, \ q \in \mathbb{M}.$$

Under the assumption (3.3), the operator $B : \mathbb{V} \to \mathbb{M}$ is onto, see e.g. [5]. We let $\mathbb{V}_0$ be the kernel of $B$. It is well known that if a bounded form $b : \mathbb{V} \times \mathbb{M} \to \mathbb{R}$ satisfies (3.3), then the data $\mathbf{f} \in \mathbb{V}^*$ satisfies the *compatibility condition*

$$\langle \mathbf{f}, \mathbf{v} \rangle = 0 \quad \text{for all } \mathbf{v} \in \mathbb{V}_0, \tag{3.4}$$

if and only if the problem (3.1) has a unique solution, see e.g. [4, 5].

With the mixed problem (3.1), we associate the SPLS formulation:
Find $(\mathbf{u}, p) \in (\mathbb{V}, \mathbb{M})$ such that

$$\begin{array}{lcll} a_0(\mathbf{u}, \mathbf{v}) & + \quad b(\mathbf{v}, p) & = \langle \mathbf{f}, \mathbf{v} \rangle & \text{for all } \mathbf{v} \in \mathbb{V}, \\ b(\mathbf{u}, q) & & = 0 & \text{for all } q \in \mathbb{M}. \end{array} \tag{3.5}$$

The following statement summarizes the connection between the two variational formulations.

**Proposition 3.1.** *In the presence of the continuous* $\inf - \sup$ *condition* (3.3) *and the compatibility condition* (3.4)*, we have that* $p$ *is the unique solution of* (3.1) *if and only if* $(\mathbf{u} = 0, p)$ *is the unique solution of* (3.5)*.*

### 3.1. SPLS discretization.

The SPLS discretization of (3.1) is defined as a standard saddle point discretization of (3.5). We let $\mathbb{V}_h \subset \mathbb{V}$ and $\mathbb{M}_h \subset \mathbb{M}$ be finite dimensional approximation spaces and consider the restrictions of the forms $a_0(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to the discrete spaces $\mathbb{V}_h$ and $\mathbb{M}_h$. Assume that for the pair $(\mathbb{V}_h, \mathbb{M}_h)$, the following discrete $\inf - \sup$ condition holds:

$$\inf_{p_h \in \mathbb{M}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|p_h\| \, |\mathbf{v}_h|} = m_h > 0. \tag{3.6}$$

We define $\mathbb{V}_{h,0}$ to be the kernel of the discrete operator $B_h$, i.e.,

$$\mathbb{V}_{h,0} := \{\mathbf{v}_h \in \mathbb{V}_h |\ b(\mathbf{v}_h, q_h) = 0 \quad \text{for all } q_h \in \mathbb{M}_h\}.$$

Let $\mathbb{V}_{h,0}^{\perp}$ denote the orthogonal complement of $\mathbb{V}_{h,0}$ with respect to the inner product $a_0(\cdot,\cdot)$ on $\mathbb{V}_h$. If $\mathbb{V}_{h,0} \subset \mathbb{V}_0$, then the compatibility condition (3.4) implies a discrete compatibility condition. Consequently, under the discrete stability assumption (3.6), the problem of finding $p_h \in \mathbb{M}_h$ such that

$$b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \text{for all } \mathbf{v}_h \in \mathbb{V}_h, \tag{3.7}$$

has a unique solution. For a general choice of discrete spaces where the compatibility condition (3.4) does not hold on $\mathbb{V}_{h,0}$, the standard saddle point discrete variational formulation,
Find $(\mathbf{u}_h, p_h) \in \mathbb{V}_h \times \mathbb{M}_h$ such that

$$\begin{array}{lcll} a_0(\mathbf{u}_h, \mathbf{v}_h) & + & b(\mathbf{v}_h, p_h) & = \langle \mathbf{f}, \mathbf{v}_h \rangle \qquad \text{for all } \mathbf{v}_h \in \mathbb{V}_h, \\ b(\mathbf{u}_h, q_h) & & & = 0 \qquad \text{for all } q_h \in \mathbb{M}_h, \end{array} \tag{3.8}$$

has a unique solution. The variational formulation (3.8) is called the *saddle point least squares discretization* of (3.1). It is easy to check that the $p_h$ part of the solution of (3.8) is in fact the solution of the normal equation associated with (3.7). The following error estimate for $\|p - p_h\|$ was proved in [12].

**Theorem 3.2.** *Let $b : \mathbb{V} \times \mathbb{M} \to \mathbb{R}$ satisfy* (3.2) *and* (3.3) *and assume that $\mathbf{f} \in \mathbb{V}^*$ is given and satisfies* (3.4)*. Assume that $p$ is the solution of* (3.1) *and $\mathbb{V}_h \subset \mathbb{V}$, $\mathbb{M}_h \subset \mathbb{M}$ are chosen such that the discrete* inf − sup *condition* (3.6) *holds. If $(\mathbf{u}_h, p_h)$ is the solution of* (3.8)*, then the following error estimate holds:*

$$\frac{1}{M}|u_h| \le \|p - p_h\| \le \frac{M}{m_h} \inf_{q_h \in \mathbb{M}_h} \|p - q_h\|. \tag{3.9}$$

The considerations made regarding the SPLS discretization in this section make sense if the form $a_0(\cdot,\cdot)$, as an inner product on $\mathbb{V}_h$, is replaced by another inner product which gives rise to an equivalent norm on $\mathbb{V}_h$ independent of $h$. The error estimate (3.9) remains valid with different estimating constants that factor in the norm equivalence constants. If $A_h$ is the discrete operator associated with the form $a_0(\cdot,\cdot)$ on $\mathbb{V}_h$ and an Uzawa type algorithm is involved to solve (3.8), then the action of the discrete operator $A_h^{-1}$ can be replaced by the action of any equivalent preconditioner.

3.2. **A note on the connection with the Bramble-Pasciak least squares method.** Using our notation, the Bramble-Pasciak approach [22, 23] in discretizing (3.1) reduces to solving for $p_h \in \mathbb{M}_h$ such that

$$b(A_h^{-1}B_h^* q_h, p_h) = \langle \mathbf{f}, A_h^{-1}B_h^* q_h \rangle \quad \text{for all } \ q_h \in \mathbb{M}_h, \tag{3.10}$$

where $A_h$ and $B_h$ are the corresponding discrete versions of $A$ and $B$. For the action of $A_h^{-1}$, a discrete space $\mathbb{V}_h$ is required and bases for both $\mathbb{V}_h$ and $\mathbb{M}_h$ are needed. In addition, the action of the stiffness matrix associated with the left hand side of (3.10) is needed at every step of the global iterative solver. The stability of the family $\{(\mathbb{V}_h, \mathbb{M}_h)\}$ is also necessary.

Our SPLS approach is different from the way the trial space $\mathbb{M}_h$ is chosen as well as the way the iterative process is chosen. The *SPLS discretization* solves the coupled saddle point system (3.8) for $p_h$ by an Uzawa type iterative process and bases and matrix assembly is required only for the test space $\mathbb{V}_h$. We do have to compute the action of $A_h^{-1}$ on $\mathbb{V}_h$ or the action of an approximate inverse at each step of the global iterative solver.

3.3. **Uzawa type iterative solvers.** The *saddle point least squares discretization* system (3.8) might be difficult to assemble and solve. The SPLS discretization approach solves the system using an Uzawa type iteration that benefits from the fact that inversion processes are needed only on the test space $\mathbb{V}_h$ and explicit bases for the trial space $\mathbb{M}_h$ are not required. The proposed idea in [12, 13] is to use an iterative process such as the Uzawa (U), Uzawa Gradient (UG), or the Uzawa Conjugate Gradient (UCG).

Following [6, 12], for the Uzawa algorithm we have to choose a fixed number $\alpha = \alpha_0$ in the interval $\left(0, \frac{2}{M}\right)$. For the UG algorithm, the parameter $\alpha$ is chosen by the algorithm to impose orthogonality of consecutive residuals associated with the second equation in (3.8). The first step for Uzawa is identical with the first step of UG. We combine the two algorithms as follows:

**Algorithm 3.3.** *(U-UG) Algorithms*

**Step 1: Set** $\mathbf{u}_0 = 0 \in \mathbb{V}_h$, $p_0 \in \mathbb{M}_h$, **compute** $\mathbf{u}_1 \in \mathbb{V}_h$, $q_1 \in \mathbb{M}_h$ *by*

$$\begin{aligned} a_0(\mathbf{u}_1, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle - b(\mathbf{v}, p_0) && \text{for all } \mathbf{v} \in \mathbb{V}_h, \\ (q_1, q) &= b(\mathbf{u}_1, q) && \text{for all } q \in \mathbb{M}_h. \end{aligned}$$

**Step 2: For** $j = 1, 2, \ldots,$ **compute** $\mathbf{h}_j, \alpha_j, p_j, \mathbf{u}_{j+1}, q_{j+1}$ *by*

$$\begin{aligned} &(\mathbf{U}-\mathbf{UG1}) && a_0(\mathbf{h}_j, \mathbf{v}) = -b(\mathbf{v}, q_j) && \text{for all } \mathbf{v} \in \mathbb{V}_h \\ &(\mathbf{U}\alpha) && \alpha_j = \alpha_0 && \text{for the Uzawa algorithm or} \\ &(\mathbf{UG}\alpha) && \alpha_j = -\frac{(q_j, q_j)}{b(\mathbf{h}_j, q_j)} && \text{for the UG algorithm} \\ &(\mathbf{U}-\mathbf{UG2}) && p_j = p_{j-1} + \alpha_j\, q_j \\ &(\mathbf{U}-\mathbf{UG3}) && \mathbf{u}_{j+1} = \mathbf{u}_j + \alpha_j\, \mathbf{h}_j \\ &(\mathbf{U}-\mathbf{UG4}) && (q_{j+1}, q) = b(\mathbf{u}_{j+1}, q) && \text{for all } q \in \mathbb{M}_h. \end{aligned}$$

To obtain the UCG algorithm, the UG algorithm is modified as in [20, 45] by the following steps. First, we define $d_1 := q_1$ in **Step 1**. Then, we modify **Step 2** by replacing $b(\cdot, q_j)$ with $b(\cdot, d_j)$, where $\{d_j\}$ is a sequence of conjugate directions. The resulting algorithm is as follows:

**Algorithm 3.4.** *(UCG) Algorithm*

**Step 1: Set** $\mathbf{u}_0 = 0 \in \mathbb{V}_h$, $p_0 \in \mathbb{M}_h$. **Compute** $\mathbf{u}_1 \in \mathbb{V}_h$, $q_1, d_1 \in \mathbb{M}_h$ *by*

$$\begin{aligned} a_0(\mathbf{u}_1, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle - b(\mathbf{v}, p_0) && \text{for all } \mathbf{v} \in \mathbb{V}_h, \\ (q_1, q) &= b(\mathbf{u}_1, q) && \text{for all } q \in \mathbb{M}_h, \quad d_1 := q_1. \end{aligned}$$

**Step 2: For** $j = 1, 2, \ldots,$ **compute** $\mathbf{h}_j, \alpha_j, p_j, \mathbf{u}_{j+1}, q_{j+1}, \beta_j, d_{j+1}$ *by*

$$\begin{aligned} &(\mathbf{UCG1}) && a_0(\mathbf{h}_j, \mathbf{v}) = -\, b(\mathbf{v}, d_j) && \text{for all } \mathbf{v} \in \mathbb{V}_h \\ &(\mathbf{UCG}\alpha) && \alpha_j = -\,\frac{(q_j, q_j)}{b(\mathbf{h}_j, q_j)} \\ &(\mathbf{UCG2}) && p_j = p_{j-1} + \alpha_j\, d_j \\ &(\mathbf{UCG3}) && \mathbf{u}_{j+1} = \mathbf{u}_j + \alpha_j\, \mathbf{h}_j \\ &(\mathbf{UCG4}) && (q_{j+1}, q) = b(\mathbf{u}_{j+1}, q) && \text{for all } q \in \mathbb{M}_h \\ &(\mathbf{UCG}\beta) && \beta_j = \frac{(q_{j+1}, q_{j+1})}{(q_j, q_j)} \\ &(\mathbf{UCG6}) && d_{j+1} = q_{j+1} + \beta_j d_j. \end{aligned}$$

We choose spaces $\mathbb{M}_h$ such that bases for computing $q_j$ are not needed and at each iteration step *the only one essential inversion* is the one involving the form $a_0(\cdot,\cdot)$. Each one of the described algorithms can be applied to approximate the solution $(\mathbf{u}_h, p_h)$ of (3.8). The following *sharp error estimation* result was proved in [6].

*If $(\mathbf{u}_h, p_h)$ is the discrete solution of* (3.8)*, and $(\mathbf{u}_{j+1}, p_j)$ is the $j^{th}$ iteration for U, UG, or UCG, then $(\mathbf{u}_{j+1}, p_j) \to (\mathbf{u}_h, p_h)$ and*

$$\begin{aligned} \frac{1}{M^2}\,\|q_{j+1}\| \le \|p_j - p_h\| \quad &\le \frac{1}{m_h^2}\,\|q_{j+1}\|, \\ \frac{m_h}{M^2}\,\|q_{j+1}\| \le |\mathbf{u}_{j+1} - \mathbf{u}_h| &\le \frac{M}{m_h^2}\,\|q_{j+1}\|. \end{aligned} \tag{3.11}$$

The estimates (3.11) *entitle* $\|q_{j+1}\|$ as a *computable, robust, efficient, and uniform*-modulo $m_h$ *estimator for the iteration error for all three algorithms.*

If the discretization error order is available, say $O(\|p - p_h\|)$, and an estimate for $m_h$ is also available, the iteration error can match the discretization error by imposing the stopping criterion

$$\|q_{j+1}\| \le c_0\, m_h^2\; O(\|p - p_h\|), \tag{3.12}$$

where $c_0$ is a constant independent of $h$.

3.4. **The SPLS method.** In [12, 13], the following five steps that define the *saddle point least squares discretization* method were introduced:

Step 1) Write the general problem (3.1) as a *saddle point least squares* formulation (3.5) using the natural inner product $a_0(\cdot,\cdot)$ on $\mathbb{V}\times\mathbb{V}$.

Step 2) Choose a standard *conforming approximation space* $\mathbb{V}_h$ for the *variational space* $\mathbb{V}$.

Step 3) Construct a discrete *trial space* $\mathbb{M}_h \subset \mathbb{M}$ using the operator $B$ associated with the form $b(\cdot,\cdot)$. For example, take $\mathbb{M}_h := B\mathbb{V}_h$ or $\mathbb{M}_h := \tilde{\mathcal{Q}}_h B\mathbb{V}_h$, where $\tilde{\mathcal{Q}}_h$ is an *orthogonal projection* from $\mathbb{M}$ to a subspace $\tilde{\mathbb{M}}_h \subset \mathbb{M}$. The pair $(\mathbb{V}_h, \mathbb{M}_h)$ will automatically satisfy a discrete $\inf-\sup$ condition.

Step 4) Write the discrete version of the SPLS formulation given in (3.8) and replace $a_0(\cdot,\cdot)$ by an equivalent form $a_{prec}(\cdot,\cdot)$ on $\mathbb{V}_h\times\mathbb{V}_h$ if available.

Step 5) Solve the new discrete SPLS problem using an Uzawa type process that requires only the action of $A_h^{-1}$ (or preconditioner) and the action of $B$ or $\tilde{\mathcal{Q}}_h B$ on functions in $\mathbb{V}_h$.

3.5. **Special discrete spaces.** In this subsection, we discuss in more detail the two choices of discrete trial spaces introduced in Step 3 of the SPLS method that we consider in this paper when discretizing the Maxwell equations. Let $\mathbb{V}_h$ be a *finite element subspace* of $\mathbb{V}$. Assume that the action of $B$ on functions in $\mathbb{V}_h$ is easy to obtain.

3.5.1. *Choice 1: No projection trial space.* The first choice for $\mathbb{M}_h \subset \mathbb{M}$ is

$$\mathbb{M}_h := B\mathbb{V}_h. \tag{3.13}$$

In this case, we have that $\mathbb{V}_{h,0} \subset \mathbb{V}_0$ and a discrete $\inf-\sup$ condition holds. More precisely, in [13] it was proved that

$$\inf_{p_h\in\mathbb{M}_h}\sup_{\mathbf{v}_h\in\mathbb{V}_h}\frac{b(\mathbf{v}_h,p_h)}{\|p_h\|\;|\mathbf{v}_h|} \ge \inf_{\mathbf{w}_h\in\mathbb{V}_{h,0}^{\perp}}\frac{\|B\mathbf{w}_h\|}{|\mathbf{w}_h|} := m_{h,0} > 0. \tag{3.14}$$

We note that for this choice of $\mathbb{M}_h$ the residual $q_{j+1}$ from **Step 1**, **(U-UG4)**, and **(UCG4)** can be computed by a simple action of the $B$ operator, i.e.,

$$q_{j+1} = B\mathbf{u}_{j+1}.$$

*Approximability for Choice 1.* It was proved in [13] that if $p$ is the solution of (3.1) and $p_h$ is the solution of (3.7) (or the SPLS solution of (3.8)) we have that $p_h$ is the orthogonal projection of $p$ onto $\mathbb{M}_h$. Consequently,

$$\|p - p_h\| = \inf_{q_h \in \mathbb{M}_h} \|p - q_h\|. \tag{3.15}$$

The estimate (3.15) is optimal and independent of the inf − sup constant $m_h$ which makes it a better estimation than (3.9) where the right hand side contains the factor $1/m_h$. Using that $B$ is onto $\mathbb{M}$, we can represent $p = B\mathbf{w}$ for some $\mathbf{w} \in \mathbb{V}$ and write $q_h = B\mathbf{v}_h$ for some $\mathbf{v}_h \in \mathbb{V}_h$. Thus, we have

$$\|p - p_h\| = \inf_{\mathbf{v}_h \in \mathbb{V}_h} \|B\mathbf{w} - B\mathbf{v}_h\| \leq M \inf_{\mathbf{v}_h \in \mathbb{V}_h} |\mathbf{w} - \mathbf{v}_h|. \tag{3.16}$$

Even though the trial space $\mathbb{M}_h$ is not chosen to be a standard approximation space, the estimate (3.16) reveals that the approximability of the solution $p \in \mathbb{M}$ with discrete functions in $\mathbb{M}_h$ is at least as good as the approximability of the best smooth representation $\mathbf{w} \in \mathbb{V}$ of $p$ by discrete functions in the standard approximation test space $\mathbb{V}_h$.

3.5.2. *Choice 2: Projection trial space.* Let $\tilde{\mathbb{M}}_h$ be a finite dimensional subspace of $\mathbb{M}$ that has good approximability properties. Typical examples of spaces $\tilde{\mathbb{M}}_h$ are the spaces of piecewise polynomials. We equip $\tilde{\mathbb{M}}_h$ with the restriction of the inner product on $\mathbb{M}$. If $\tilde{\mathcal{Q}}_h : \mathbb{M} \to \tilde{\mathbb{M}}_h$ is the orthogonal projection onto $\tilde{\mathbb{M}}_h$, we define the space $\mathbb{M}_h$ by

$$\mathbb{M}_h := \tilde{\mathcal{Q}}_h B \mathbb{V}_h. \tag{3.17}$$

A discrete inf − sup condition always holds for $(\mathbb{V}_h, \mathbb{M}_h)$, see Section 5 of [12]. In [13], the stability of this type of discretization was analyzed and the following result was established: If the projection operator $\tilde{\mathcal{Q}}$ satisfies

$$\|\tilde{\mathcal{Q}}_h q_h\| \geq \tilde{c}\, \|q_h\| \quad \text{for all } q_h \in B\mathbb{V}_h, \tag{3.18}$$

with a constant $\tilde{c}$ independent of $h$, then

$$m_h := \inf_{p_h \in \mathbb{M}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|p_h\|\ |\mathbf{v}_h|} \geq \tilde{c} \inf_{\mathbf{w}_h \in \mathbb{V}_{h,0}^{\perp}} \frac{\|B\mathbf{w}_h\|}{|\mathbf{w}_h|} = \tilde{c}\, m_{h,0}.$$

The estimate proves that under assumption (3.18) the stability of the family $\{(\mathbb{V}_h, B\mathbb{V}_h)\}$ is a sufficient condition for stability of the family $\{(\mathbb{V}_h, \tilde{\mathcal{Q}}_h B\mathbb{V}_h)\}$.

We note that for this choice of $\mathbb{M}_h$, the residual $q_{j+1}$ from **Step 1**, **(U-UG4)**, and **(UCG4)** can be computed using the action of the operator $B$ followed by the action of the projection $\tilde{\mathcal{Q}}_h$, i.e.,

$$q_{j+1} = \tilde{\mathcal{Q}}_h(B\mathbf{u}_{j+1}).$$

*Approximability for Choice 2.* If $p = B\mathbf{w}$ is a representation for $p$ for some $\mathbf{w} \in \mathbb{V}$, it is easy to check as done in [13] that

$$\|p - p_h\| \leq \frac{M}{m_h} \|B\mathbf{w} - \tilde{\mathcal{Q}}_h B\mathbf{w}\| + \frac{M^2}{m_h} \inf_{\mathbf{v}_h \in \mathbb{V}_h} |\mathbf{w} - \mathbf{v}_h|. \tag{3.19}$$

Consequently, if SPLS discretization is used with the pair $(\mathbb{V}_h, \tilde{\mathcal{Q}}_h B\mathbb{V}_h)$ to approximate the solution $p$ of (3.1), it would be enough to ask for some regularity of a solution $\mathbf{w}$ of $B\mathbf{w} = p$, for approximation properties of the projection $\tilde{\mathcal{Q}}_h : \mathbb{M} \to \tilde{\mathbb{M}}_h$, and for approximation properties of the test space $\mathbb{V}_h$.

In our numerical experiments presented in Section 5, we observed higher order approximability than the estimate (3.19) suggests. While (3.9) could lead to a sharper estimate, due to the non-standard choice of the space $\mathbb{M}_h$ an order of approximation of $\|p - p_h\|$ obtained via the right part of (3.9) is more difficult to investigate.

## 4. SPLS discretization for Maxwell Equations

In this section, we apply the abstract discretization theory of Section 3 to the operator $B$ associated with the Maxwell equations. We recall that $B : \mathbb{V} \to \mathbb{M}$ is given by the matrix operator

$$B := \begin{bmatrix} \mu^{-1}\nabla\times & -\omega & -\nabla & 0 \\ -\omega & \varepsilon^{-1}\nabla\times & 0 & -\nabla \end{bmatrix}.$$

Let $\mathcal{T}_h$ be a shape regular tetrahedralization of $\Omega$. Consider the lowest order finite element spaces

$$H_h := \{u_h \in \mathcal{C}(\overline{\Omega}) \,:\, u_h|_K \in \mathbb{P}_1 \quad \forall K \in \mathcal{T}_h\}, \qquad H_h^0 := H_h \cap H_0^1(\Omega),$$

and the corresponding product spaces

$$\mathbb{V}_h := \mathbf{H}_h^0 \times \mathbf{H}_h \times H_h \times H_h^0. \tag{4.1}$$

In this section, we will assume that the coefficients $\mu, \varepsilon$ are continuous functions.

First, note that

$$B(\mathbb{V}_h) \subset \mathcal{P}_0(\mathcal{T}_h)^6 + H_h^6 \subset \mathcal{P}_1(\mathcal{T}_h)^6, \tag{4.2}$$

where $\mathcal{P}_k(\mathcal{T}_h)$ is the space of discontinuous piecewise $\mathcal{P}_k$ functions (polynomial of degree $k$) on the tetrahedral partition $\mathcal{T}_h$.

Next, we are going to give three different choices for the discrete trial space $\mathbb{M}_h \subset \mathbb{M}$.

Case 1) *No projection space:* We simply take $\mathbb{M}_h = B(\mathbb{V}_h)$. The inner product on $\mathbb{M}_h$ that is needed in the UCG algorithm is taken as the inner product on $\mathbb{M}$. We recall that the inner product on $\mathbb{M}$ contains the weight $\mu$ for the first three components and the weight $\varepsilon$ for the last three. A discrete $\inf-\sup$ condition for the pair $(\mathbb{V}_h, \mathbb{M}_h)$ holds as described in the abstract case, see (3.14). The approximability in this case is given by (3.15). In the next subsection, we will address the stability of the pair $(\mathbb{V}_h, \mathbb{M}_h)$.

Case 2) *Orthogonal projection space:* We start with defining the space $\tilde{\mathbb{M}}_h := H_h^6$ with the inner product on $\mathbb{M}$. Consider now the orthogonal projection

$$\tilde{\mathcal{Q}}_h : \mathbb{M} \longrightarrow H_h^6,$$

and define the *orthogonal projection* trial space by

$$\mathbb{M}_h := \tilde{\mathcal{Q}}_h(B(\mathbb{V}_h)) \subset H_h^6.$$

A discrete $\inf-\sup$ condition for the pair $(\mathbb{V}_h, \mathbb{M}_h)$ holds as described in the abstract case and an estimate for approximability in this case is given by (3.19). The (numerical) stability of the pair $(\mathbb{V}_h, \mathbb{M}_h)$ will be analyzed via the estimate (3.18) in the next subsection.

Case 3) *Lump projection space:* We start with defining the same space $\tilde{\mathbb{M}}_h := H_h^6$, but then change the definition of the projection by lumping the mass matrix. In other words, we define another inner product on $\mathbb{M}$. To be more precise, given the nodal Lagrange basis $\{\varphi_1, \ldots, \varphi_N\}$ of $H_h$ and the function $\beta \in \{\varepsilon, \mu\}$, the $\beta-$weighted lumped mass matrix is the diagonal matrix with elements

$$d_i := \int_\Omega \beta\varphi_i = \sum_{j=1}^{N} \int_\Omega \beta\, \varphi_i\, \varphi_j. \tag{4.3}$$

The associated inner product is defined by

$$H_h \times H_h \ni \big(\sum_j u_j\varphi_j, \sum_i v_i\varphi_i\big) \longrightarrow (u_h, v_h)_{\beta,\text{lump}} := \sum_i u_i\, d_i\, v_i.$$

The inner product $(\cdot,\cdot)_{\mathbb{M},\text{lump}}$ in $H_h^6$ is defined by lumping the mass matrices of each of the six components. The lump orthogonal projection with respect to the inner product $(\cdot,\cdot)_{\mathbb{M},\text{lump}}$ is $\tilde{\mathcal{Q}}_h^{\text{lump}} : \mathbb{M} \longrightarrow H_h^6$. We define the *lump projection* trial space as

$$\mathbb{M}_h := \tilde{\mathcal{Q}}_h^{\text{lump}}(B(\mathbb{V}_h)) \subset H_h^6.$$

The discrete stability for the pair $(\mathbb{V}_h, \mathbb{M}_h)$ and an estimate for approximability follow as in Case 2.

4.1. **Stability and numerical stability of the proposed discretizations.** In order to estimate the $\inf-\sup$ constant $m_h$ for a pair $(\mathbb{V}_h, \mathbb{M}_h)$, we use that $m_h$ is the square root of the Schur complement $S_h$ associated with the system (3.8), see e.g. [5]. The action of $S_h$ can be computed by slightly modifying the Uzawa Algorithm 3.3. By using a standard power method for the Schur complement, estimates for the eigenvalues of $S_h$ can be easily obtained. We considered $\mathcal{T}_h$ for the unit cube obtained by uniform refinement. We split each cube into eight cubes of half side and then split each small cube into six tetrahedra by a standard procedure. To get an idea about the behavior of $m_h = m_h(\omega)$, we applied this technique for (3.8) for the Maxwell discretization in the *no projection case* for four levels of uniform refinement and for $\varepsilon = \mu = 1$.

TABLE 1. Approximations of $m_h(\omega)$ for the *no projection case*.

| level | $\omega = 1$ | $\omega = 2$ | $\omega = 4$ | $\omega = 16$ | $\omega = 64$ | $\omega = 256$ |
|---|---|---|---|---|---|---|
| | $m_h$ | $m_h$ | $m_h$ | $m_h$ | $m_h$ | $m_h$ |
| 1 | 0.1003 | 0.1976 | 0.1145 | 1.6253 | 6.4516 | 25.7893 |
| 2 | 0.0697 | 0.1172 | 1.1185 | 0.5075 | 2.858 | 10.9398 |
| 3 | 0.0465 | 0.0766 | 1.0072 | 1.1426 | 2.5893 | 14. 7614 |
| 4 | 0.0271 | 0.9991 | 1.0150 | 1.2425 | 3.6897 | 17.2878 |

We observed that the stability of $m_h$ depends on $\omega$, but even in the worse case of $\omega = 1$ we still have $m_h \geq O(h)$. Here, $h = 1/2^{level}$. For small values of $\omega > 0$, we noticed stability with respect to $h$ and $\omega$. The investigation of stability in the *no projection case* is equivalent with building a stable right inverse for the operator $B$ restricted to $\mathbb{V}_h$. This problem is complicated when, for example $B$ is a simpler first order differential operator such as the divergence operator [3, 43, 44]. Even though we believe that under special conditions for the mesh $\mathcal{T}_h$ stability can be proved to hold, in this paper we do not plan to investigate such conditions.

For the other two discretizations, by proving the estimate (3.18) we have that the stability is at least as good as the stability for the *no projection case* according to Section 3.5.2. The next theorem proves (3.18) for the projection trial spaces of Case 2 and Case 3.

**Theorem 4.1.** *If $\tilde{\mathcal{Q}}_h : \mathbb{M} \longrightarrow H_h^6$ is the orthogonal projection, then*

$$\|\tilde{\mathcal{Q}}_h \mathbf{q}_h\| \geq \tilde{c}\,\|\mathbf{q}_h\| \quad \text{for all } \mathbf{q}_h \in B(\mathbb{V}_h), \tag{4.4}$$

*with a constant $\tilde{c}$ independent of $h$. A similar result holds for the lump projection $\tilde{\mathcal{Q}}_h^{\text{lump}}$.*

*Proof.* Since $B(\mathbb{V}_h) \subset \mathcal{P}_0(\mathcal{T}_h)^6 + H_h^6 \subset \mathcal{P}_1(\mathcal{T}_h)^6$, it would be enough to prove that the estimate (4.4) holds component-wise on $\mathcal{P}_1(\mathcal{T}_h)$, i.e.,

$$\|Q_{h,\beta}\, q_h\|_\beta \geq \tilde{c}\,\|q_h\|_\beta \quad \text{for all } q_h \in \mathcal{P}_1(\mathcal{T}_h), \tag{4.5}$$

where $Q_{h,\beta} : L^2_\beta(\Omega) \to H_h$ is the orthogonal projection with respect to the $L^2_\beta$-inner product. Without loss of generality, we assume that

$$0 < \beta_0 < \beta < \beta_1 \text{ on } \Omega.$$

The constant $c$ that appears in this proof is generic and might be different at different occurrences, but is always independent of $h$. Next, we let $\{\varphi_1, \ldots, \varphi_N\}$ be the nodal Lagrange basis

of $H_h$, $M_{h,\beta}$ be the mass matrix with entries $(\varphi_i,\varphi_j)_\beta$, and $D_{h,\beta}$ be the corresponding diagonal matrix with diagonal entries $d_i$ defined in (4.3). In this section, by $\langle\cdot,\cdot\rangle$ we will denote the inner product on $\mathbb{R}^N$. Since the mesh $\mathcal{T}_h$ is uniform, we can assume that $(1,\varphi_i)\approx h^3$. It is easy to check that

$$\langle M_{h,\beta}\,\gamma,\gamma\rangle \le c\,\frac{\beta_1}{\beta_0}\,\langle D_{h,\beta}\,\gamma,\gamma\rangle \quad \text{for all } \gamma\in\mathbb{R}^N, \tag{4.6}$$

with a constant $c$ independent of $h$. Indeed, for any $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_N)\in\mathbb{R}^N$ we let $p_h := \sum_j \gamma_j\,\varphi_j$. Then we have

$$\begin{aligned}\langle M_{h,\beta}\gamma,\gamma\rangle &= \sum_{i,j=1}^{N}\gamma_i\gamma_j(\varphi_i,\varphi_j)_\beta = \|p_h\|_\beta^2 \le \beta_1\,\|p_h\|^2\\ &\le c\beta_1\,h^3\sum_i\gamma_i^2 \le c\frac{\beta_1}{\beta_0}\sum_i(1,\varphi_i)_\beta\,\gamma_i^2 = c\,\frac{\beta_1}{\beta_0}\,\langle D_{h,\beta}\,\gamma,\gamma\rangle,\end{aligned}$$

which proves (4.6). As a consequence of (4.6), we have

$$\langle M_{h,\beta}^{-1}\,\gamma,\gamma\rangle \ge c\,\frac{\beta_0}{\beta_1}\,\langle D_{h,\beta}^{-1}\,\gamma,\gamma\rangle \quad \text{for all } \gamma\in\mathbb{R}^N, \tag{4.7}$$

with a constant $c$ independent of $h$. Now, we are ready to prove (4.5). Let $q_h\in\mathcal{P}_1(\mathcal{T}_h)$ and let $\tilde{q_h}$ denote the dual vector of $q_h$, i.e.,

$$\tilde{q_h} = \left((q_h,\varphi_1)_\beta,\cdots,(q_h,\varphi_N)_\beta\right)^T.$$

We define $\alpha=(\alpha_1,\cdots,\alpha_N)^T\in\mathbb{R}^N$ such that

$$Q_{h,\beta}\,q_h = \sum_i \alpha_i\,\varphi_i.$$

Hence, $\alpha = M_{h,\beta}^{-1}\,\tilde{q_h}$. Then, using (4.7) we obtain

$$\begin{aligned}\|Q_{h,\beta}\,q_h\|_\beta^2 = \langle M_{h,\beta}\alpha,\alpha\rangle &= \langle M_{h,\beta}^{-1}\,\tilde{q_h},\tilde{q_h}\rangle \ge c\,\frac{\beta_0}{\beta_1}\,\langle D_{h,\beta}^{-1}\,\tilde{q_h},\tilde{q_h}\rangle\\ &= c\,\frac{\beta_0}{\beta_1}\sum_i\frac{(q_h,\varphi_i)_\beta^2}{(1,\varphi_i)_\beta} \ge c\,\frac{\beta_0^3}{\beta_1^2}\sum_i\frac{(q_h,\varphi_i)^2}{(1,\varphi_i)}.\end{aligned}$$

From the uniformity of the mesh, the spectral properties of the local mass matrix, and the fact that $q_h$ is linear on all tetrahedra of $\mathcal{T}_h$, it is easy to check that

$$\sum_i\frac{(q_h,\varphi_i)^2}{(1,\varphi_i)} \ge c\|q_h\|^2 \ge c\frac{1}{\beta_1}\|q_h\|_\beta^2.$$

Combining the last two estimates lead to (4.5) with $\tilde{c}=c\left(\frac{\beta_0}{\beta_1}\right)^{3/2}$.

The proof for the the lump projection $\tilde{\mathcal{Q}}_h^{\text{lump}}$ is similar. We just note that the corresponding *lump mass matrix* coincides with $D_{h,\beta}$. □

## 5. Numerical results for the SPLS discretization

In this section, we report some numerical results for various ways to approximate the solution of (2.1). In all cases, we considered $\mathcal{T}_h$ to be a tetrahedralization of $\Omega$ and the test space $\mathbb{V}_h$ was taken to be the finite element space of piecewise linear functions as defined in (4.1). We implemented the UCG algorithm for all three types of discretization and tested our solvers for various values of the frequency $\omega$ using five levels of refinement from the original coarse

mesh. Since our sequences of meshes are nested, we used a cascadic-multilevel approach, see [6, 7, 8, 11, 14, 19, 21], for obtaining a multilevel solver. More precisely, we started the UCG algorithm on the coarse level using a zero vector as the initial guess for $p$. At the next level, we used the extension of the computed solution $p_{comp}$ from the previous level to the current level as the initial guess. Our stopping criterion was based on (3.12) with upper best estimates for $O(\|p - p_h\|)$. For the non-convex domain example, we used a zero vector as the initial guess for $p$ on each level.

5.1. **Numerical results on the unit cube.** First, we discretized (2.1) on the unit cube with coefficients $\mu = \varepsilon = 1$. The data was chosen such that the exact solution is

$$\mathbf{h} = (x(1-x), y(1-y), z(1-z))^T,$$
$$\mathbf{e} = (y(1-y)z(1-z), x(1-x)z(1-z), y(1-y)x(1-x))^T.$$

We performed numerical tests for various values of $\omega \in \mathbb{R}$. All three types of CG discretizations that we implemented performed well for the chosen values of $\omega$. It was noticed that the approximation for the *orthogonal projection* trial space is better than the *no projection* trial space, see Tables 2, 3, and 5. The approximation on *lump projection* trial spaces seems to be similar with the approximation on *orthogonal projection* spaces for values $\omega \leq 100$. We also noticed that the solver based on *lump projection* trial spaces performs well (better than the other two) for large values of $\omega$. In Table 4, we show its performance for $\omega \in \{10, 100, 1000\}$. All implemented algorithms showed robustness with respect to values of $\omega$ close to zero. In Table 5, we present the results for $\omega = 1/1000$. For negative values of $\omega$, the behavior of all three types of discretization with the UCG algorithm is very similar with the behavior corresponding to the case $-\omega > 0$.

TABLE 2. Numerical results for CG-SPLS discretization for $\omega = 1$

| CG-P1-B(P1) | $\|\mathbf{h} - \mathbf{h}_h\|$ | Conv. Rate | $\|\mathbf{e} - \mathbf{e}_h\|$ | Conv. Rate | # of iter |
|---|---|---|---|---|---|
| k=1 | 0.1192 | | 0.0269 | | 1 |
| k=2 | 0.0511 | 1.22 | 0.0113 | 1.25 | 2 |
| k=3 | 0.0248 | 1.04 | 0.0055 | 1.03 | 2 |
| k=4 | 0.0125 | 0.99 | 0.0028 | 1.03 | 2 |
| k=5 | 0.0062 | 1.00 | 0.0014 | 1.03 | 3 |
| CG-P1-Q B(P1) | | | | | |
| k=1 | 0.1169 | | 0.0245 | | 1 |
| k=2 | 0.0325 | 1.84 | 0.0090 | 1.45 | 2 |
| k=3 | 0.0089 | 1.87 | 0.0025 | 1.83 | 2 |
| k=4 | 0.0029 | 1.64 | 0.00008 | 1.70 | 2 |
| k=5 | 0.0007 | 1.96 | 0.0002 | 1.87 | 4 |

5.2. **Numerical results on a 3D $L$-shaped domain.** We tested our method on a three dimensional $L$-shaped domain when the component $\mathbf{e}$ of the exact solution for (2.1) is not smooth. More precisely, we defined

$$\Psi = (1-x^2)(1-y^2)(z-z^2)\ r^{2/3}\ \sin(2\theta/3),$$

TABLE 3. Numerical results for CG-SPLS discretization for $\omega = 16$

| CG-P1-B(P1) | $\|\mathbf{h} - \mathbf{h}_h\|$ | Conv. Rate | $\|\mathbf{e} - \mathbf{e}_h\|$ | Conv. Rate | # of iter |
|---|---|---|---|---|---|
| k=1 | 0.1255 | | 0.0352 | | 5 |
| k=2 | 0.0695 | 0.85 | 0.0345 | 0.03 | 8 |
| k=3 | 0.0426 | 0.71 | 0.0229 | 0.60 | 12 |
| k=4 | 0.0248 | 0.78 | 0.0138 | 0.73 | 21 |
| k=5 | 0.0141 | 0.81 | 0.0079 | 0.81 | 34 |
| CG-P1-Q B(P1) | | | | | |
| k=1 | 0.0084 | | 0.0226 | | 3 |
| k=2 | 0.0068 | 0.31 | 0.0154 | 1.45 | 6 |
| k=3 | 0.0047 | 0.56 | 0.0060 | 1.83 | 8 |
| k=4 | 0.0018 | 1.40 | 0.0013 | 1.70 | 10 |
| k=5 | 0.0006 | 1.55 | 0.0004 | 1.68 | 14 |

TABLE 4. Numerical results for CG-SPLS discretization with lumping projection

| P1-Qlump B(P1), $\omega = 10$ | $\|\mathbf{h} - \mathbf{h}_h\|$ | Conv. Rate | $\|\mathbf{e} - \mathbf{e}_h\|$ | Conv. Rate | # of iter |
|---|---|---|---|---|---|
| k=1 | 0.0819 | | 0.0283 | | 3 |
| k=2 | 0.0442 | 0.89 | 0.0214 | 0.03 | 3 |
| k=3 | 0.0149 | 1.57 | 0.0076 | 0.60 | 8 |
| k=4 | 0.0040 | 1.91 | 0.0015 | 0.73 | 11 |
| k=5 | 0.0010 | 2.05 | 0.0003 | 2.06 | 12 |
| P1-Qlump B(P1), $\omega = 100$ | | | | | |
| k=1 | 0.0820 | | 0.0295 | | 8 |
| k=2 | 0.0463 | 0.83 | 0.0245 | 0.27 | 10 |
| k=3 | 0.0191 | 1.27 | 0.0072 | 1.77 | 35 |
| k=4 | 0.0072 | 1.41 | 0.0024 | 1.57 | 56 |
| k=5 | 0.0026 | 1.49 | 0.0012 | 1.00 | 96 |
| P1-Qlump B(P1), $\omega = 1000$ | | | | | |
| k=1 | 0.0820 | | 0.0295 | | 12 |
| k=2 | 0.0463 | 0.82 | 0.0246 | 0.26 | 18 |
| k=3 | 0.0192 | 1.27 | 0.0074 | 1.74 | 64 |
| k=4 | 0.0073 | 1.40 | 0.0025 | 1.54 | 114 |
| k=5 | 0.0027 | 1.45 | 0.0015 | 0.77 | 272 |

where $(r, \theta)$ are the polar coordinates in the $xy$-plane. For $\mu = 1$ and $\epsilon = 1$, we computed the data $\mathbf{f}$ such that the exact solution is

$$\mathbf{e} = \nabla(\Psi), \quad \text{and} \quad \mathbf{h} = (x(1-x)(1+x), y(1-y)(1+y), z(1-z))^T.$$

Note that $\mathbf{e} \notin \mathbf{H}^1(\Omega)$. We implemented the UCG algorithm using both uniform and non-uniform refinement strategies for the first two cases of discrete trial spaces: *no projection* and *orthogonal projection*. The family of locally quasi uniform meshes $\{\mathcal{T}_h\}$ that we used for discretization in the case of non-uniform refinement were obtained by a graded refinement strategy using the simple coordinate transformation

$$x_j := x_j \cdot |x_j|^{-1+1/q} \qquad j = 1, 2,$$

as shown in [1, 2]. The numerical results for both uniform and non-uniform refinement are shown in Table 6 and Table 7, respectively. In the case of non-uniform refinement, the parameter $q$ in

TABLE 5. Numerical results for CG-SPLS discretization for $\omega = 1/1000$

| P1-B(P1) | $\|\mathbf{h}-\mathbf{h}_h\|$ | Conv. Rate | $\|\mathbf{e}-\mathbf{e}_h\|$ | Conv. Rate | # of iter |
|---|---|---|---|---|---|
| k=1 | 30.2359 | | 0.0218 | | 3 |
| k=2 | 9.3126 | 1.70 | 0.0107 | 1.03 | 3 |
| k=3 | 2.5041 | 1.89 | 0.0060 | 0.85 | 8 |
| k=4 | 0.6396 | 1.97 | 0.0027 | 1.13 | 11 |
| k=5 | 0.1609 | 1.99 | 0.0014 | 0.99 | 12 |
| P1-Q B(P1) | | | | | |
| k=1 | 36.0886 | | 0.0208 | | 4 |
| k=2 | 10.6594 | 1.76 | 0.0086 | 1.28 | 9 |
| k=3 | 2.8557 | 1.90 | 0.0025 | 1.78 | 17 |
| k=4 | 0.9462 | 1.60 | 0.0005 | 2.24 | 21 |
| k=5 | 0.5093 | 1.49 | 0.0002 | 1.45 | 36 |
| P1-Qlump B(P1) | | | | | |
| k=1 | 36.2702 | | 0.0220 | | 6 |
| k=2 | 10.4984 | 1.79 | 0.0063 | 1.81 | 12 |
| k=3 | 2.6543 | 1.98 | 0.0023 | 1.44 | 20 |
| k=4 | 0.6726 | 1.98 | 0.0006 | 1.87 | 25 |
| k=5 | 0.1824 | 1.88 | 0.0001 | 2.12 | 37 |

the coordinate transformation was chosen to be $q = 0.9$ for the *no projection* discrete trial space and $q = 0.55$ for the case of *orthogonal projection*.

TABLE 6. Numerical results for CG-SPLS discretization on non-convex domain with uniform refinement and $\omega = 1$

| CG-P1-B(P1) | $\|\mathbf{h}-\mathbf{h}_h\|$ | Conv. Rate | $\|\mathbf{e}-\mathbf{e}_h\|$ | Conv. Rate | # of iter |
|---|---|---|---|---|---|
| k=1 | 0.2838 | | 0.2672 | | 2 |
| k=2 | 0.1607 | 0.82 | 0.1595 | 0.74 | 1 |
| k=3 | 0.0834 | 0.95 | 0.0852 | 0.90 | 2 |
| k=4 | 0.0423 | 0.98 | 0.0441 | 0.95 | 1 |
| k=5 | 0.0212 | 1.00 | 0.0229 | 0.95 | 3 |
| CG-P1-Q B(P1) | | | | | |
| k=1 | 0.2465 | | 0.2108 | | 2 |
| k=2 | 0.0765 | 1.69 | 0.0718 | 1.55 | 2 |
| k=3 | 0.0225 | 1.77 | 0.0342 | 1.07 | 2 |
| k=4 | 0.0062 | 1.86 | 0.0155 | 1.14 | 3 |
| k=5 | 0.0021 | 1.59 | 0.0086 | 0.86 | 5 |

The regularity of the $\mathbf{h}$ component of the solution is higher than the regularity of the $\mathbf{e}$ component. This is reflected in the approximation of the solution using the *orthogonal projection* trial space as shown by the numerical results in Table 6 and Table 7. The component $\mathbf{e}$ of the solution is not in $\mathbf{H}^{2/3}(\Omega)$, but the order of $\|\mathbf{e}-\mathbf{e}_h\|$ is higher than $2/3$ due to the use of the graded meshes.

The fact that the operators $B$ and $B^*$ depend on the parameters $\varepsilon, \mu, \omega$ affects the stability of the problem at the continuous and discrete levels. In our first numerical example, the operator

TABLE 7. Numerical results for CG-SPLS discretization on non-convex domain with non-uniform refinement and $\omega = 1$

| CG-P1-B(P1) | $\|\mathbf{h} - \mathbf{h}_h\|$ | Conv. Rate | $\|\mathbf{e} - \mathbf{e}_h\|$ | Conv. Rate | # of iter |
|---|---|---|---|---|---|
| k=1 | 0.2783 | | 0.2699 | | 2 |
| k=2 | 0.1634 | 0.77 | 0.1628 | 0.73 | 3 |
| k=3 | 0.0862 | 0.92 | 0.0872 | 0.90 | 4 |
| k=4 | 0.0439 | 0.97 | 0.0449 | 0.96 | 6 |
| k=5 | 0.0221 | 0.99 | 0.0230 | 0.97 | 8 |
| CG-P1-Q B(P1) | | | | | |
| k=1 | 0.1554 | | 0.1783 | | 4 |
| k=2 | 0.0764 | 1.02 | 0.0587 | 1.60 | 6 |
| k=3 | 0.0266 | 1.52 | 0.0169 | 1.79 | 7 |
| k=4 | 0.0081 | 1.72 | 0.0049 | 1.79 | 9 |
| k=5 | 0.0024 | 1.75 | 0.0015 | 1.72 | 12 |

$B$ depends on $\omega$. Thus, the condition number of the discrete Schur complement depends on $\omega$. Consequently, the number of iterations for the Uzawa algorithms depends on $\omega$. From Table 2-Table 5, we can see that the number of iterations increase as $\omega \to \infty$ and $h \to 0$, but the increase is slow. For a large range of values of $\omega$, we get the order of approximation to be close to or higher than one with just piecewise linear approximation using the SPLS discretizations.

## 6. CONCLUSION

We proposed a new least squares discretization method for the time-harmonic Maxwell equations as a first order system. The essential novelties of the proposed discretization are in the way the discrete trial spaces are chosen and in the way a global iterative process is designed for solving the discrete saddle point variational formulation of the problem.

The approximability of the proposed discretizations depends on how well the solution $\mathbf{p}$ can be represented as $\mathbf{p} = B\mathbf{w}$. The higher the regularity of the representant function $\mathbf{w}$, the better the SPLS approximation $\mathbf{p}_h$ of $\mathbf{p}$ gets. At least for the *no projection* choice of the trial space, the discretization error $\|\mathbf{p} - \mathbf{p}_h\|$ is independent of the inf $-$ sup constant $m_h$ associated with the SPLS discrete system. Due to the lack of bases for the trial spaces, to solve the corresponding SPLS discrete system an Uzawa type algorithm has to be considered and the performance of such an algorithm depends on $m_h$. Using the *projection* choices for the trial space, the approximability is better if compared with other finite element approximation techniques that rely on piecewise linear approximation functions. The algorithms are robust with respect to the frequency parameter $\omega$. Efficient preconditioning techniques, such as multigrid, could be easily considered to speed up the proposed iterative processes. The method is efficient for both convex and non-convex domains and for a various range of frequencies.

We plan to further investigate whether the SPLS method can be applied to solve other mixed variational formulations such as mixed formulations for linear elasticity problems or Maxwell equations with different types of boundary conditions. We believe that the SPLS discretization can be applied to a large class of PDEs that can be reduced to lower order systems.

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