

LEAST SQUARES PRECONDITIONING FOR MIXED METHODS WITH NONCONFORMING TRIAL SPACES

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ABSTRACT. We consider a preconditioning technique for mixed methods with a conforming test space and a nonconforming trial space. Our method is based on the classical saddle point discretization theory for mixed methods and the theory of preconditioning symmetric positive definite operators. Efficient iterative processes for solving the discrete mixed formulations are proposed and choices for discrete compatible spaces are provided. For discretization, a basis is needed only for the test spaces and assembly of a global saddle point system is avoided. We provide approximation properties for the discretization and iteration errors and also provide a sharp estimate for the convergence rate of the proposed algorithm in terms of the condition number of the elliptic preconditioner and the discrete inf – sup and sup – sup constants of the pair of discrete spaces. We focus on applications to elliptic PDEs with discontinuous coefficients. Numerical results for two and three dimensional domains are included to support the proposed method.

1. INTRODUCTION

The Saddle Point Least Squares (SPLS) method and its versions can be found in [8, 11]. A recent technique, by the same authors, on preconditioning *conforming* SPLS discretization is presented in [9]. In this paper, we build on the work of [8, 9, 11] to obtain a general preconditioning technique for mixed methods with a possible nonconforming trial space. While applications of the method can be found in modeling many phenomena that reduce to well posed mixed variational formulations, we restrict in this paper to applications for solving elliptic interface problems that are reformulated as primal mixed formulations. Elliptic interface problems model many practical problems in material science and composite materials (that are built from essentially different components, see [3, 13, 23, 25]). In addition, elliptic interface formulations appear also in fluid dynamics, modeling several layers of fluids with different viscosities or diffusion through heterogeneous porous media [15, 22].

We now present the main ideas and concepts for the (possible) nonconforming discretization and preconditioning of the general mixed problem:

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Given $F \in V^*$, find $p \in Q$ such that

$$(1.1) \quad b(v, p) = \langle F, v \rangle \quad \text{for all } v \in V,$$

where V and Q are Hilbert spaces and $b(\cdot, \cdot)$ is a continuous bilinear form on $V \times Q$ satisfying an inf – sup condition. If $a(\cdot, \cdot)$ is the inner product on V , then we have that (see [10, 21]) p is the unique solution of (1.1) if and only if $(w = 0, p)$ is the unique solution to: Find $(w, p) \in V \times Q$ such that

$$(1.2) \quad \begin{aligned} a(w, v) + b(v, p) &= \langle F, v \rangle & \text{for all } v \in V, \\ b(w, q) &= 0 & \text{for all } q \in Q, \end{aligned}$$

where F satisfies a *compatibility condition* (see (2.3) and Section 2). Thus, (1.2) is a saddle point reformulation of (1.1).

For finite dimensional approximation, we consider a discrete conforming test spaces $V_h \subset V$ and discrete trial space \mathcal{M}_h that in general, might not be a subspace of Q . This is because, for our proposed SPLS method, the trial space is built from the action of the continuous differential operator B , associated with problem (1.1), on the test space V_h followed by a smoothing projection that might have the range outside of the continuous trial space Q . We will assume in this paper that the trial space \mathcal{M}_h is a subspace of a space \tilde{Q} that contains Q as a proper closed subspace. Thus, $\mathcal{M}_h \subset \tilde{Q}$, where $Q \subset \tilde{Q}$, and \tilde{Q} is, in most cases, an L^2 type space. We also assume that Q and \tilde{Q} have the same inner product and the form $b(\cdot, \cdot)$ has a continuous extension to $V \times \tilde{Q}$.

Further, we assume a discrete inf – sup condition for the pair (V_h, \mathcal{M}_h) and consider the discrete problem of finding $(w_h, p_h) \in V_h \times \mathcal{M}_h$ such that

$$(1.3) \quad \begin{aligned} a(w_h, v_h) + b(v_h, p_h) &= \langle F, v_h \rangle & \text{for all } v_h \in V_h, \\ b(w_h, q_h) &= 0 & \text{for all } q_h \in \mathcal{M}_h, \end{aligned}$$

which, under standard assumptions, approximates the solution $(w = 0, p)$ of (1.2). As presented in [8], the discrete variational formulation (1.3) is a *saddle point least squares discretization* of (1.1), see Section 2.

Due to the nature of the discrete trial spaces \mathcal{M}_h that we propose (see Section 2.1), finding bases for the discrete trial space \mathcal{M}_h and assembling a block stiffness matrix for (1.3) can be difficult. To bypass this inconvenience, one can apply an Uzawa type algorithm that requires the exact inversion of the operator A_h associated with the inner product $a(\cdot, \cdot)$ on V_h from the first equation of (1.3). In this case, a basis is required only for the test space V_h . The action of A_h^{-1} could be costly, and due to a possible large condition number $\kappa(A_h)$ (when $h \rightarrow 0$), the solver associated with A_h^{-1} might not be accurate. To avoid the exact inversion of A_h , as presented in [9], we consider a form $\tilde{a}(\cdot, \cdot)$ on V_h , which leads to an equivalent norm on V_h , and introduce a preconditioned discrete saddle point problem: Find $(\tilde{w}_h, \tilde{p}_h) \in V_h \times \mathcal{M}_h$ such that

$$(1.4) \quad \begin{aligned} \tilde{a}(\tilde{w}_h, v_h) + b(v_h, \tilde{p}_h) &= \langle F, v_h \rangle & \text{for all } v_h \in V_h, \\ b(\tilde{w}_h, q_h) &= 0 & \text{for all } q_h \in \mathcal{M}_h, \end{aligned}$$

where the action of the operator \tilde{A}_h^{-1} associated with the inner product $\tilde{a}(\cdot, \cdot)$ on V_h is assumed to be fast and easy to implement.

We will show in this paper that the results of SPLS discretization with nonconforming trial spaces [8] hold under the preconditioning reformulation. More concretely, we will prove that under standard assumptions, the proposed SPLS choices for the discrete pairs (V_h, \mathcal{M}_h) lead to a good approximation for the continuous problem, i.e., the component solution \tilde{p}_h of (1.4) approximates the solution p of (1.1). Furthermore, we will analyze the convergence of an iterative solver for (1.4) and estimate its convergence rate.

In contrast with the SPLS work in [9, 11, 12], where both the test and trial spaces were chosen to be conforming finite element spaces, this paper considers trial spaces which are nonconforming finite element spaces. This allows efficient treatment of PDEs with discontinuous coefficients.

The paper is organized as follows. In Section 2, we review notation and the main considerations for the general nonconforming (nc) SPLS method as presented in [8]. In Section 3, we describe the general preconditioning theory, the corresponding discrete approximation theory, and estimate convergence rates for the proposed iterative solver. In Section 4, we apply the proposed nc SPLS theory to approximating the solution of a model elliptic interface problem. Numerical results for the n-c SPLS discretization with preconditioning are presented in Section 5.

2. THE GENERAL NC SPLS APPROACH

Following [8], we let V and \tilde{Q} be infinite dimensional Hilbert spaces and assume the inner products $a(\cdot, \cdot)$ and $(\cdot, \cdot)_{\tilde{Q}}$ induce the norms $|\cdot|_V = \|\cdot\| = a(\cdot, \cdot)^{1/2}$ and $\|\cdot\|_{\tilde{Q}} = \|(\cdot, \cdot)_{\tilde{Q}}\|^{1/2}$. We denote the dual of V by V^* and the duality pairing on $V^* \times V$ by $\langle \cdot, \cdot \rangle$. With the inner product $a(\cdot, \cdot)$, we associate the operator $\mathcal{A} : V \rightarrow V^*$ defined by

$$\langle \mathcal{A}u, v \rangle = a(u, v) \quad \text{for all } u, v \in V.$$

Next, we let Q be a closed subspace of \tilde{Q} equipped with the induced inner product (from \tilde{Q}).

We assume that $b(\cdot, \cdot)$ is a continuous bilinear form on $V \times \tilde{Q}$ satisfying

$$(2.1) \quad \sup_{p \in \tilde{Q}} \sup_{v \in V} \frac{b(v, p)}{|v| \|p\|} = M < \infty,$$

and the following inf – sup condition on $V \times Q$,

$$(2.2) \quad \inf_{p \in Q} \sup_{v \in V} \frac{b(v, p)}{|v| \|p\|} = m > 0.$$

With the form b , we associate the linear operators $B : V \rightarrow \tilde{Q}$ and $B^* : \tilde{Q} \rightarrow V^*$ defined by

$$(Bv, q)_{\tilde{Q}} = b(v, q) = \langle B^*q, v \rangle \quad \text{for all } v \in V, q \in \tilde{Q}.$$

Note that the operator B is defined using the inner product on \tilde{Q} , while B^* is defined using the duality pairing on $V^* \times V$. Hence, B is the Hilbert dual of $\mathcal{A}^{-1}B^*$. Lastly, we define V_0 to be the kernel of B , i.e.,

$$V_0 := \text{Ker}(B) = \{v \in V \mid Bv = 0\}.$$

We consider problems of the form: Given $F \in V^*$, find $p \in Q$ such that (1.1) holds. We note here that for the existence and uniqueness of the solution of the continuous problem (1.1), we use the trial space Q . However, for discretization purposes, we need to consider the form $b(\cdot, \cdot)$ on $V \times \tilde{Q}$, where \tilde{Q} is an extension of Q . The existence and uniqueness of (1.1) was first studied by Aziz and Babuška in [2]. If a bounded form $b : V \times \tilde{Q} \rightarrow \mathbb{R}$ satisfies (2.2) and the data $F \in V^*$ satisfies the *compatibility condition*

$$(2.3) \quad \langle F, v \rangle = 0 \quad \text{for all } v \in V_0,$$

then the mixed problem (1.1) has a unique solution, see e.g., [2, 4]. With the mixed problem (1.1), we associate the SPLS formulation: Find $(w, p) \in (V, Q)$ such that

$$(2.4) \quad \begin{aligned} a(w, v) + b(v, p) &= \langle F, v \rangle & \text{for all } v \in V, \\ b(w, q) &= 0 & \text{for all } q \in Q. \end{aligned}$$

The following statement can be found in [10, 21] and is essential in our approach.

Proposition 2.1. *In the presence of the continuous inf – sup condition (2.2) and the compatibility condition (2.3), we have that p is the unique solution of (1.1) if and only if $(w = 0, p)$ is the unique solution of (2.4).*

2.1. nc SPLS discretization. The nonconforming *SPLS discretization* of (1.1) is defined as a (trial) nonconforming saddle point discretization of (2.4). We consider finite dimensional approximation spaces $V_h \subset V$ and $\mathcal{M}_h \subset \tilde{Q}$ (larger than Q in general) and restrict the forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ to the discrete spaces $V_h \times V_h$ and $V_h \times \mathcal{M}_h$. Assume that the following discrete inf – sup condition holds for the pair (V_h, \mathcal{M}_h) :

$$(2.5) \quad \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \|p_h\|} > 0.$$

We define $V_{h,0}$ to be the kernel of the discrete operator B_h , i.e.,

$$V_{h,0} := \{v_h \in V_h \mid b(v_h, q_h) = 0 \quad \text{for all } q_h \in \mathcal{M}_h\}.$$

If $V_{h,0} \subset V_0$, then the compatibility condition (2.3) implies a discrete compatibility condition. Consequently, under the discrete stability assumption (2.5), the problem of finding $p_h \in \mathcal{M}_h$ such that

$$(2.6) \quad b(v_h, p_h) = \langle f_h, v_h \rangle \quad \text{for all } v_h \in V_h,$$

has a unique solution. In the above equation,

$$\langle f_h, v_h \rangle := \langle F, v_h \rangle \quad \text{for all } v_h \in V_h.$$

In general, the compatibility condition (2.3) might not hold on $V_{h,0}$. Hence, the discrete problem (2.6) may not be well-posed. In any case, under the assumption (2.5), the standard discrete saddle point problem of finding $(w_h, p_h) \in V_h \times \mathcal{M}_h$ such that (1.3) holds has a unique solution. We call the variational formulation (1.3) the *nonconforming saddle point least squares discretization* of (1.1).

2.2. The discrete spaces. Let V_h be a *finite element subspace* of V . As presented in [8], using a simplified notation, we provide two types of general trial spaces M_h that can be considered for the SPLS discretization. The first choice for M_h , the *no projection trial space*, can be viewed as a conforming trial space, already investigated in [9, 11, 12]. We review the *no projection trial space* here, because it helps analyzing the second choice of M_h , the *projection trial space*.

2.2.1. No projection trial space. We first consider the case when \mathcal{M}_h is given by

$$\mathcal{M}_h := BV_h \subset \tilde{Q}.$$

In this case, we take $\tilde{Q} = Q$ and have that $V_{h,0} \subset V_0$. As presented in [8], a discrete inf – sup condition holds:

$$(2.7) \quad m_{h,0} := \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \|p_h\|} > 0.$$

Thus, we have that both variational formulations (2.6) and (1.3) have a unique solution $p_h \in \mathcal{M}_h$. Furthermore, using Proposition 2.1 for the discrete pair (V_h, \mathcal{M}_h) , we have that $(w_h = 0, p_h)$ is the solution of (1.3).

In addition, if p is the solution of (1.1) and p_h is the solution of (2.6), or $(0, p_h)$ is the solution of (1.3), then from (1.1) and (2.6) we obtain

$$0 = b(v_h, p - p_h) = (Bv_h, p - p_h)_{\tilde{Q}} \quad \text{for all } v_h \in V_h.$$

Thus, p_h is the orthogonal projection of p onto \mathcal{M}_h which gives us

$$\|p - p_h\| = \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|.$$

This result is optimal, and in contrast with the standard approximation estimates for saddle point problems, it does not depend on $m_{h,0}$.

2.2.2. Projection type trial space. Let $\tilde{\mathcal{M}}_h \subset \tilde{Q}$ be a finite dimensional subspace equipped with the inner product $(\cdot, \cdot)_h$. The corresponding induced norm on $\tilde{\mathcal{M}}_h$ will be denoted by $\|\cdot\|_h$. Define the representation operator $R_h : \tilde{Q} \rightarrow \tilde{\mathcal{M}}_h$ by

$$(2.8) \quad (R_h p, q_h)_h := (p, q_h)_{\tilde{Q}} \quad \text{for all } q_h \in \tilde{\mathcal{M}}_h.$$

Here, $R_h p$ is the Riesz representation of $p \rightarrow (p, q_h)_{\tilde{Q}}$ as a functional on $(\tilde{\mathcal{M}}_h, (\cdot, \cdot)_h)$. In the case when $(\cdot, \cdot)_h$ coincides with the inner product on \tilde{Q} , we have that R_h is the orthogonal projection onto $\tilde{\mathcal{M}}_h$.

Since the space $\tilde{\mathcal{M}}_h$ is finite dimensional, there exist constants k_1, k_2 such that

$$(2.9) \quad k_1 \|q_h\| \leq \|q_h\|_h \leq k_2 \|q_h\| \quad \text{for all } q_h \in \tilde{\mathcal{M}}_h.$$

We further assume that the equivalence is uniform with respect to h , i.e., the constants k_1, k_2 are independent of h . Using the operator R_h , we define \mathcal{M}_h as

$$\mathcal{M}_h := R_h B V_h \subset \tilde{\mathcal{M}}_h \subset \tilde{Q}.$$

The following proposition gives a sufficient condition on R_h to ensure the discrete inf – sup condition is satisfied and relates the stability of the families of spaces $\{(V_h, B V_h)\}$ and $\{(V_h, R_h B V_h)\}$. The result was proved in [8].

Proposition 2.2. *Assume that*

$$(2.10) \quad \|R_h q_h\|_h \geq \tilde{c} \|q_h\| \quad \text{for all } q_h \in B V_h,$$

with a constant \tilde{c} independent of h . Then $V_{h,0} \subset V_0$. Furthermore, if $m_{h,0}$ defined in (2.7) satisfies $m_{h,0} > c_0 > 0$ for some constant c_0 independent of h , then

$$(2.11) \quad \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \|p_h\|_h} \geq \tilde{c} m_{h,0} \geq \tilde{c} c_0 > 0.$$

As a consequence of Proposition 2.2, we have that under the assumption (2.10), both variational formulations (2.6) and (1.3) have unique solution $p_h \in \mathcal{M}_h$. Furthermore, using Proposition 2.1 for the discrete pair (V_h, \mathcal{M}_h) , we have that $(w_h = 0, p_h)$ is the solution of (1.3).

Regarding the *approximability property* of the projection type trial space, the following proposition was proved in [8].

Proposition 2.3. *If p is the solution of (1.1), p_h is the solution of (2.6) (or the nc SPLS solution of (1.3)), and R_h satisfies (2.10), then*

$$\|p - p_h\| \leq C \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|,$$

where $C = 1 + \frac{1}{\tilde{c} k_1}$ and \tilde{c}, k_1 are defined in (2.10) and (2.9), respectively.

Remark 2.4. *The no projection trial space described in Section 2.2.1 can be viewed as the special case of the projection type trial space when $R_h = I$ and the inner product $(\cdot, \cdot)_h$ on \mathcal{M}_h to be the original inner product on $\tilde{Q} = Q$. Thus, in what follows we will consider \mathcal{M}_h to be equipped with the inner product $(\cdot, \cdot)_h$ for both the no projection and projection type trial spaces.*

2.3. An Uzawa CG iterative solver. Note that a global linear system may be difficult to assemble when solving (1.3) on $(V_h, \mathcal{M}_h = R_h B V_h)$, especially if the operator R_h involves a nonlocal projection. In this case, bases for the trial spaces \mathcal{M}_h might be difficult to find. One can solve (1.3) and avoid building a basis for \mathcal{M}_h by using an Uzawa type algorithm. To simplify the presentation, we will focus on the Uzawa Conjugate Gradient (UCG) algorithm. Other Uzawa type algorithms are discussed in [8]. For

the convergence analysis of the (UCG) Algorithm, we first define A_h to be the discrete version of the operator \mathcal{A} , i.e., A_h satisfies

$$\langle A_h u_h, v_h \rangle = a(u_h, v_h) \quad \text{for all } u_h, v_h \in V_h.$$

The discrete operators $B_h : V_h \rightarrow \mathcal{M}_h$ and $B_h^* : \mathcal{M}_h \rightarrow V_h^*$ are defined by

$$(B_h v_h, q_h)_h = b(v_h, q_h) = \langle B_h^* q_h, v_h \rangle \quad \text{for all } v_h \in V_h, q_h \in \mathcal{M}_h.$$

Note that the operator B_h is defined using the inner product on \mathcal{M}_h and not with the duality on $\mathcal{M}_h^* \times \mathcal{M}_h$. Also, B_h is the Hilbert transpose of $A_h^{-1} B_h^*$, i.e., $B_h = (A_h^{-1} B_h^*)^T$. Thus, we can define the discrete Schur complement $S_h : \mathcal{M}_h \rightarrow \mathcal{M}_h$ as $S_h = B_h A_h^{-1} B_h^*$. It is well known that the spectrum of S_h satisfies $\sigma(S_h) \subset [m_h^2, M_h^2]$, where

$$(2.12) \quad m_h := \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \|p_h\|_h} > 0,$$

and

$$(2.13) \quad M_h := \sup_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \|p_h\|_h} \leq M < \infty.$$

In addition, we have that m_h^2, M_h^2 are (the extreme) eigenvalues of S_h . It is easy to check that the p_h part of the solution of (1.3) is the solution of the *normal equation* associated with (2.6).

Algorithm 2.5. (UCG) Algorithm

Step 1: Choose any $p_0 \in \mathcal{M}_h$. **Compute** $w_1 \in V_h, q_1, d_1 \in \mathcal{M}_h$ by

$$\begin{aligned} a(w_1, v_h) &= \langle f_h, v_h \rangle - b(v, p_0) && \text{for all } v_h \in V_h, \\ (q_1, q)_h &= b(w_1, q) && \text{for all } q \in \mathcal{M}_h, \quad d_1 := q_1. \end{aligned}$$

Step 2: For $j = 1, 2, \dots$, **compute** $h_j, \alpha_j, p_j, w_{j+1}, q_{j+1}, \beta_j, d_{j+1}$ by

$$\text{(UCG1)} \quad a(h_j, v_h) = -b(v_h, d_j) \quad \text{for all } v_h \in V_h$$

$$\text{(UCG}\alpha) \quad \alpha_j = -\frac{(q_j, q_j)_h}{b(h_j, q_j)}$$

$$\text{(UCG2)} \quad p_j = p_{j-1} + \alpha_j d_j$$

$$\text{(UCG3)} \quad w_{j+1} = w_j + \alpha_j h_j$$

$$\text{(UCG4)} \quad (q_{j+1}, q)_h = b(w_{j+1}, q) \quad \text{for all } q \in \mathcal{M}_h$$

$$\text{(UCG}\beta) \quad \beta_j = \frac{(q_{j+1}, q_{j+1})_h}{(q_j, q_j)_h}$$

$$\text{(UCG6)} \quad d_{j+1} = q_{j+1} + \beta_j d_j.$$

Note that, if the action of B_h does not require an inversion matrix, the only inversion needed at each step of the algorithm involves the form $a(\cdot, \cdot)$ in **Step 1** or **(UCG1)**. In operator form, these steps become

$$(2.14) \quad w_1 = A_h^{-1}(f_h - B_h^* p_0), \text{ or } \quad h_j = -A_h^{-1}(B_h^* d_j), \text{ respectively.}$$

In practical implementations of Algorithm 2.5, we will replace the action of A_h^{-1} with the action of a suitable preconditioner.

Regarding the convergence of the UCG algorithm, it is well known that if (w_h, p_h) is the discrete solution of (1.3) and (w_{j+1}, p_j) is the j^{th} iteration for the UCG algorithm, then $(w_{j+1}, p_j) \rightarrow (w_h, p_h)$. In the next section, we investigate the stability and approximability results of Section 2.2 for the case when $a(\cdot, \cdot)$ is replaced by a uniformly equivalent form $\tilde{a}(\cdot, \cdot)$.

3. A PRECONDITIONING TECHNIQUE

A preconditioning theory for SPLS discretization with conforming trial spaces was introduced in [9]. In this section, we develop a similar general preconditioning framework to approximate the solution of (1.1) based on the preconditioned saddle point discretization (1.4). From the solver point of view, this corresponds to modifying the UCG algorithm such that the action of the elliptic operator A_h^{-1} on V_h^* is replaced by the action of a preconditioning operator P_h . From the formulation and analysis point of view, we replace the original form $a(\cdot, \cdot)$ in (1.3) with a uniformly equivalent form $\tilde{a}(\cdot, \cdot)$ (on V_h) that leads to an implementably fast operator $\tilde{A}_h^{-1} = P_h$.

For the rest of this section, we assume that $V_h \subset V$ and $\mathcal{M}_h \subset \tilde{Q}$ are finite dimensional approximation spaces satisfying (2.12) and (2.13). We note that the inner product $(\cdot, \cdot)_h$ on \mathcal{M}_h is chosen to satisfy (2.9).

3.1. The preconditioned saddle point problem. First, we introduce a general preconditioner operator $P_h : V_h^* \rightarrow V_h$ that is equivalent to A_h^{-1} . We assume that $P_h A_h : V_h \rightarrow V_h$ is symmetric with respect to the $a(\cdot, \cdot)$ inner product and that

$$(3.1) \quad m_1^2 |v_h|^2 \leq a(P_h A_h v_h, v_h) \leq m_2^2 |v_h|^2,$$

where the positive constants m_1^2, m_2^2 are the smallest and largest eigenvalues of $P_h A_h$, respectively. The assumption (3.1) gives us that the condition number of $P_h A_h$ satisfies

$$(3.2) \quad \kappa(P_h A_h) = \frac{m_2^2}{m_1^2}.$$

With the preconditioner $P_h : V_h^* \rightarrow V_h$, we define the form $\tilde{a} : V_h \times V_h \rightarrow \mathbb{R}$ by

$$(3.3) \quad \tilde{a}(u_h, v_h) := a((P_h A_h)^{-1} u_h, v_h) \quad \text{for all } u_h, v_h \in V_h.$$

We note that the form $\tilde{a}(\cdot, \cdot)$ is symmetric and that (3.1), (3.3) imply

$$(3.4) \quad \frac{1}{m_2^2} |v_h|^2 \leq \tilde{a}(v_h, v_h) \leq \frac{1}{m_1^2} |v_h|^2 \quad \text{for all } v_h \in V_h.$$

Thus, $\tilde{a}(\cdot, \cdot)$ is another inner product on V_h that induces an equivalent norm on V_h . Let $|v_h|_P := \tilde{a}(v_h, v_h)^{1/2}$ be the norm induced by the inner

product $\tilde{a}(\cdot, \cdot)$ and define the operator $\tilde{A}_h : V_h \rightarrow V_h^*$ by

$$\langle \tilde{A}_h u_h, v_h \rangle := \tilde{a}(u_h, v_h) \quad \text{for all } u_h, v_h \in V_h.$$

Note that for any $u_h, v_h \in V_h$ we have that

$$\begin{aligned} \langle \tilde{A}_h u_h, v_h \rangle &= \tilde{a}(u_h, v_h) = a((P_h A_h)^{-1} u_h, v_h) \\ &= \langle A_h (P_h A_h)^{-1} u_h, v_h \rangle, \end{aligned}$$

which implies $\tilde{A}_h = A_h (P_h A_h)^{-1} = P_h^{-1}$. Hence, we can view $\tilde{a}(\cdot, \cdot)$ as a preconditioned version of the form $a(\cdot, \cdot)$. The preconditioned discrete saddle point problem consists of finding $(\tilde{w}_h, \tilde{p}_h) \in V_h \times \mathcal{M}_h$ such that (1.4) holds. To simplify the notation, we will drop the $\tilde{\cdot}$ notation from $(\tilde{w}_h, \tilde{p}_h)$. Thus, for the remainder of this paper, the *preconditioned saddle point least squares* formulation is: Find $(w_h, p_h) \in V_h \times \mathcal{M}_h$ such that

$$(3.5) \quad \begin{aligned} \tilde{a}(w_h, v_h) + b(v_h, p_h) &= \langle F, v_h \rangle = \langle f_h, v_h \rangle & \text{for all } v_h \in V_h, \\ b(w_h, q_h) &= 0 & \text{for all } q_h \in \mathcal{M}_h. \end{aligned}$$

Using that $V_h \subset V$ and $\mathcal{M}_h \subset \tilde{Q}$ satisfy (2.12) and (2.13), we obtain

$$(3.6) \quad \tilde{m}_h := \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h|_P \|p_h\|_h} \geq m_1 m_h > 0,$$

and

$$(3.7) \quad \tilde{M}_h := \sup_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h|_P \|p_h\|_h} \leq m_2 M_h \leq m_2 M.$$

Hence, the *preconditioned saddle point least squares* formulation (3.5) has a unique solution. The Schur complement associated with problem (3.5) is

$$\tilde{S}_h = B_h \tilde{A}_h^{-1} B_h^* = B_h P_h B_h^*.$$

Solving for p_h from (3.5), we obtain

$$(3.8) \quad \tilde{S}_h p_h = B_h (P_h B_h^*) p_h = B_h P_h f_h.$$

We call the component p_h of the solution (w_h, p_h) of (3.5) the *preconditioned saddle point least squares* approximation of the solution p of the original mixed problem (1.1). Next we discuss about how well p_h approximates p . Due to the fact that $V_{h,0}$ is independent of the norm on V_h and the projection R_h satisfies the coercivity condition (2.10), we have that if p_h is the solution of (2.6) (or the nc preconditioned SPLS solution of (3.5)) and p is the solution of (1.1), then the Proposition 2.3 remains valid. More precisely, we have

$$\|p - p_h\| \leq C \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|,$$

where $C = 1 + \frac{1}{\tilde{c} k_1}$ and \tilde{c} , k_1 are defined in (2.10) and (2.9), respectively.

3.2. An iterative solver for the preconditioned variational formulation. We use a modified version of Algorithm 2.5 to solve (3.5) by replacing the form $a(\cdot, \cdot)$ by $\tilde{a}(\cdot, \cdot)$ in **Step 1** and **(UCG1)**. With this modification, we obtain the following (Uzawa) Preconditioned Conjugate Gradient (PCG) algorithm for mixed methods.

Algorithm 3.1. (PCG) Algorithm for Mixed Methods

Step 1: Choose any $p_0 \in \mathcal{M}_h$. **Compute** $w_1 \in V_h$, $q_1, d_1 \in \mathcal{M}_h$ by

$$\begin{aligned} w_1 &= P_h(f_h - B_h^* p_0) \\ q_1 &= B_h w_1, \quad d_1 := q_1. \end{aligned}$$

Step 2: For $j = 1, 2, \dots$, **compute** $h_j, \alpha_j, p_j, w_{j+1}, q_{j+1}, \beta_j, d_{j+1}$ by

$$\begin{aligned} \text{(PCG1)} \quad & h_j = -P_h(B_h^* d_j) \\ \text{(PCG}\alpha) \quad & \alpha_j = -\frac{(q_j, q_j)_h}{b(h_j, q_j)} \\ \text{(PCG2)} \quad & p_j = p_{j-1} + \alpha_j d_j \\ \text{(PCG3)} \quad & w_{j+1} = w_j + \alpha_j h_j \\ \text{(PCG4)} \quad & q_{j+1} = B_h w_{j+1}, \\ \text{(PCG}\beta) \quad & \beta_j = \frac{(q_{j+1}, q_{j+1})_h}{(q_j, q_j)_h} \\ \text{(PCG6)} \quad & d_{j+1} = q_{j+1} + \beta_j d_j. \end{aligned}$$

Note that only the actions of P_h , B_h , and B_h^* are needed in the above algorithm. For any preconditioner P_h and trial space \mathcal{M}_h that is not defined via a global projection, these actions do not involve inversion processes.

Remark 3.2. Algorithm 3.1 recovers in particular the steps of the conjugate gradient algorithm for solving problem (3.8). Hence, the rate of convergence for $\|p_j - p_h\|_{\tilde{S}_h}$ or $\|p_j - p_h\|_h$ depends on the condition number of \tilde{S}_h , which is $\kappa(\tilde{S}_h) = \frac{\tilde{M}_h^2}{\tilde{m}_h^2}$.

The following Theorem discusses the convergence of Algorithm 3.1.

Theorem 3.3. If (w_h, p_h) is the discrete solution of (3.5) and (w_{j+1}, p_j) is the j^{th} iteration for Algorithm 3.1, then

$$(3.9) \quad \|p_h - p_j\|_{\tilde{S}_h} \leq 2 \left(\frac{\tilde{M}_h - \tilde{m}_h}{\tilde{M}_h + \tilde{m}_h} \right)^j \|p_h - p_0\|_{\tilde{S}_h},$$

and the following estimates hold:

$$(3.10) \quad \begin{aligned} \frac{1}{M^2} \frac{1}{m_2^2} \|q_j\|_h &\leq \|p_j - p_h\|_h \leq \frac{1}{m_h^2} \frac{1}{m_1^2} \|q_j\|_h, \\ \frac{m_h}{M^2} \frac{m_1^2}{m_2^2} \|q_j\|_h &\leq |w_{j+1} - w_h| \leq \frac{M}{m_h^2} \frac{m_2^2}{m_1^2} \|q_j\|_h. \end{aligned}$$

Consequently, $(w_{j+1}, p_j) \rightarrow (w_h, p_h)$.

Proof. The convergence estimate (3.9) is a direct consequence of Remark 3.2 and the general convergence result of the Conjugate Gradient algorithm [16, 24]. Hence, $p_j \rightarrow p_h$. By induction over j , we have that

$$\tilde{a}(w_j, v_h) + b(v_h, p_{j-1}) = \langle f_h, v_h \rangle \quad \text{for all } v_h \in V_h.$$

Combining this with the first equation of (3.5) gives us

$$(3.11) \quad \tilde{a}(w_j - w_h, v_h) = b(v_h, p_h - p_{j-1}) \quad \text{for all } v_h \in V_h.$$

Note that $\sigma(\tilde{S}_h) \subset [\tilde{m}_h^2, \tilde{M}_h^2]$. Hence,

$$(3.12) \quad \tilde{m}_h \|q\|_h = (\tilde{S}_h q, q)_h^{1/2} \leq \tilde{M}_h \|q\|_h \quad \text{for all } q \in \mathcal{M}_h.$$

By substituting $v_h = \tilde{A}_h^{-1} B_h^*(p_h - p_{j-1})$ into (3.11),

$$|w_j - w_h|_P^2 = (\tilde{S}_h(p_h - p_{j-1}), p_h - p_{j-1})_h = \|p_h - p_{j-1}\|_{\tilde{S}_h}^2.$$

The above equality, (3.4), and (3.12) gives us that

$$(3.13) \quad m_1 \tilde{m}_h \|p_h - p_{j-1}\|_h \leq |w_j - w_h| \leq m_2 \tilde{M}_h \|p_h - p_{j-1}\|_h.$$

From **(PCG4)**, the second equation of (3.5), and (3.11) we have that

$$q_j = B_h w_j = B_h(w_j - w_h) = \tilde{S}_h(p_h - p_{j-1}).$$

Thus,

$$(3.14) \quad \tilde{m}_h^2 \|p_h - p_{j-1}\|_h \leq \|\tilde{S}_h(p_h - p_{j-1})\|_h = \|q_j\|_h \leq \tilde{M}_h^2 \|p_h - p_{j-1}\|_h.$$

The inequalities (3.10) follow from (3.13), (3.14), and the fact that $\tilde{m}_h \geq m_h m_1$ and $\tilde{M}_h \leq M m_2$. From (3.10), we conclude that $w_j \rightarrow w_h$. \square

As a direct consequence of (3.6), (3.7), (3.9), and the formula $\kappa(\tilde{S}_h) = \frac{\tilde{M}_h^2}{\tilde{m}_h^2}$, we obtain the following.

Proposition 3.4. *The condition number of the Schur complement $\tilde{S}_h = B_h P_h B_h^*$ satisfies*

$$(3.15) \quad \kappa(\tilde{S}_h) \leq \frac{M_h^2 m_2^2}{m_h^2 m_1^2} = \kappa(S_h) \cdot \kappa(P_h A_h).$$

Consequently, the convergence rate ρ_h for $\|p_j - p_h\|_{\tilde{S}_h}$ satisfies

$$\rho_h \leq \frac{\frac{M_h m_2}{m_h m_1} - 1}{\frac{M_h m_2}{m_h m_1} + 1}.$$

4. NC SPLS FOR SECOND ORDER ELLIPTIC INTERFACE PROBLEMS

To illustrate our SPLS discretization with preconditioning technique we apply the proposed strategy to the second order elliptic interface problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain with $\{\Omega_j\}_{j=1}^N$ a partition of Ω and \mathbf{n}_j be the outward unit normal vector to $\partial\Omega_j$. Define $\Gamma_{km} := \partial\Omega_k \cap \partial\Omega_m$ to be the interface between Ω_k and Ω_m for $1 \leq k < m \leq N$. Given $f \in L^2(\Omega)$, we consider the problem of finding $u \in H_0^1(\Omega)$ such that

$$(4.1) \quad -\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega,$$

with the continuity of the co-normal derivative condition

$$\llbracket A\nabla u \cdot \mathbf{n} \rrbracket_{\Gamma_{km}} = (A_k \nabla u_k \cdot \mathbf{n}_k + A_m \nabla u_m \cdot \mathbf{n}_m)|_{\Gamma_{km}} = 0 \quad \text{for all } k < m.$$

We assume the matrix A is symmetric and satisfies

$$(4.2) \quad a_{\min} |\xi|_e^2 \leq \langle A(x)\xi, \xi \rangle_e \leq a_{\max} |\xi|_e^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d,$$

for positive constants $a_{\min} \leq a_{\max}$ where $\langle \cdot, \cdot \rangle_e$ and $|\cdot|_e$ denote the standard Euclidean inner product and norm for vectors in \mathbb{R}^d , respectively. In addition, the entries could be discontinuous, with possibly large jumps, across the subdomain boundaries. For the remainder of this section, (\cdot, \cdot) and $\|\cdot\|$ will denote the standard L^2 inner product and norm for both scalar and vector functions.

The primal mixed variational formulation of (4.1) we consider is: Find $p = A\nabla u$, with $u \in H_0^1(\Omega)$, such that

$$(4.3) \quad (p, \nabla v) = (A\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega).$$

To fit (4.3) into the abstract formulation (1.1), we let $V := H_0^1(\Omega)$, $\tilde{Q} := L^2(\Omega)^d$, $Q := A\nabla V$, and define $b : V \times \tilde{Q} \rightarrow \mathbb{R}$ by

$$b(v, q) := (q, \nabla v) \quad \text{for all } v \in V, q \in \tilde{Q}.$$

Also, define

$$\langle F, v \rangle := (f, v) \quad \text{for all } v \in V.$$

On V , we consider the weighted inner product

$$a(u, v) := (A\nabla u, \nabla v) \quad \text{for all } u, v \in V.$$

On \tilde{Q} , we define a similar weighted inner product

$$(p, q)_{\tilde{Q}} := (p, A^{-1}q) \quad \text{for all } p, q \in \tilde{Q}.$$

With these inner products on V and \tilde{Q} , we have that the operator $B : V \rightarrow \tilde{Q}$ is given by

$$Bv = A\nabla v \quad \text{for all } v \in V.$$

Hence,

$$V_0 = \operatorname{Ker}(B) = \{v \in V | Bv = 0\} = \{v \in H_0^1(\Omega) | A\nabla v = 0\} = \{0\},$$

The continuity constant satisfies

$$\begin{aligned}
M &= \sup_{q \in \tilde{Q}} \sup_{v \in V} \frac{b(v, q)}{|v|_V \|q\|_{\tilde{Q}}} = \sup_{q \in \tilde{Q}} \sup_{v \in V} \frac{(q, \nabla v)}{|v|_V \|q\|_{\tilde{Q}}} \\
&= \sup_{q \in \tilde{Q}} \sup_{v \in V} \frac{(q, A\nabla v)_{\tilde{Q}}}{|v|_V \|q\|_{\tilde{Q}}} \\
(4.4) \quad &\leq \sup_{v \in V} \frac{\|A\nabla v\|_{\tilde{Q}}}{(A\nabla v, \nabla v)^{1/2}} = 1.
\end{aligned}$$

Also, the inf – sup constant satisfies

$$\begin{aligned}
m &= \inf_{q=A\nabla u \in Q} \sup_{v \in V} \frac{b(v, q)}{|v|_V \|q\|_{\tilde{Q}}} \\
(4.5) \quad &= \inf_{u \in V} \sup_{v \in V} \frac{(A\nabla u, \nabla v)}{(A\nabla u, \nabla u)^{1/2} (A\nabla v, \nabla v)^{1/2}} \geq 1.
\end{aligned}$$

Consequently, the variational problem (4.3) is well-posed and suitable for nc SPLS formulation and discretization with preconditioning.

4.1. nc SPLS discretization for second order elliptic interface problems. We take $V_h \subset V = H_0^1(\Omega)$ to be the space of continuous piecewise polynomials of degree k with respect to the *interface-fitted* triangular mesh \mathcal{T}_h . We note that while the *no projection trial space* case is similar with the work presented in [12], the *projection trial space* is analyzed using the non-conforming trial space setting and leads to new stability and approximability estimates for the discontinuous coefficients (or interface) case.

4.1.1. *No projection trial space.* Following Section 2.2.1, we define the trial space as

$$\mathcal{M}_h := BV_h = A\nabla V_h.$$

By similar arguments used to show (4.5), we obtain

$$(4.6) \quad m_h := \inf_{q_h=A\nabla u_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{|v_h|_V \|q_h\|_{\tilde{Q}}} \geq 1.$$

Thus, we do have stability in this case. The discrete mixed variational formulation is: Find $p_h = A\nabla u_h$, with $u_h \in V_h$, such that

$$(4.7) \quad (p_h, \nabla v_h) = (A\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in V_h.$$

The SPLS discretization (1.3) to be solved is: Find $(w_h, p_h = A\nabla u_h)$ such that

$$(4.8) \quad \begin{aligned} (A\nabla w_h, \nabla v_h) + (p_h, \nabla v_h) &= (f, v_h) & \text{for all } v_h \in V_h, \\ A\nabla w_h &= 0. \end{aligned}$$

4.1.2. *Projection type trial space.* We define $\tilde{\mathcal{M}}_h \subset \tilde{Q} = L^2(\Omega)^d$ to be

$$\tilde{\mathcal{M}}_h := \bigoplus_{i=1}^N AM_{h,0}|_{\Omega_i},$$

where N is the number of subdomains and where each component of $M_{h,0}|_{\Omega_i}$ consists of continuous piecewise polynomials of degree k with respect to the mesh $\mathcal{T}_{h,i} := \mathcal{T}_h|_{\Omega_i}$ with no restrictions on the boundary. Two different options for the projection type trial space are outlined in Sections 4.1.3 and 4.1.5. They will be referred to as type I and type II, respectively.

4.1.3. *Projection trial space type I.* We equip $\tilde{\mathcal{M}}_h$ with the inner product

$$(A\tilde{q}_h, A\tilde{p}_h)_h = \sum_{i=1}^N (A\tilde{q}_h, A\tilde{p}_h)_{\tilde{Q},\Omega_i} \quad \text{for all } A\tilde{q}_h, A\tilde{p}_h \in \tilde{\mathcal{M}}_h.$$

Here, $(\cdot, \cdot)_{\tilde{Q},\Omega_i}$ is the inner product on \tilde{Q} restricted to the subdomain Ω_i . Using the definition of R_h given in (2.8), we have that $R_h p$ is the orthogonal projection of p onto $\tilde{\mathcal{M}}_h$ with respect to the $(\cdot, \cdot)_{\tilde{Q}}$ inner product. In turn, this implies $R_h p|_{\Omega_j}$ is the orthogonal projection onto $\tilde{\mathcal{M}}_h|_{\Omega_j} = AM_{h,0}|_{\Omega_j}$ with respect to the $(\cdot, \cdot)_{\tilde{Q}}$ inner product. We then define the trial space as

$$\mathcal{M}_h := R_h A \nabla V_h.$$

The discrete mixed variational formulation in this case is: Find $p_h = R_h A \nabla u_h$, with $u_h \in V_h$, such that

$$(4.9) \quad (p_h, \nabla v_h) = (R_h A \nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in V_h.$$

The nc SPLS discretization (1.3) to be solved is: Find $(w_h, p_h = R_h A \nabla u_h)$ such that

$$(4.10) \quad \begin{aligned} (A \nabla w_h, \nabla v_h) + (p_h, \nabla v_h) &= (f, v_h) & \text{for all } v_h \in V_h, \\ R_h A \nabla w_h &= 0. \end{aligned}$$

4.1.4. *Piecewise linear test space.* In this section, we discuss the stability for the family of spaces $\{(V_h, \mathcal{M}_h)\}$, where \mathcal{M}_h is as outlined in Section 4.1.3. We improve upon the result presented in [8] for the case that the matrix A is diagonal and has constant coefficients by proving that we have stability independent of the matrix A . For simplicity, we assume $\Omega \subset \mathbb{R}^2$ is a polygonal domain separated into two subdomains by a smooth interface $\Gamma \subset \Omega$. The results can easily be extended to N subdomains as well as polyhedral domains in \mathbb{R}^3 . We also assume that the triangular mesh \mathcal{T}_h is locally quasi-uniform. Let $\{z_{1,i}, \dots, z_{N_i,i}\}$ be the set of all nodes of $\mathcal{T}_{h,i}$ and assume all triangles adjacent to $z_{j,i}$ are of regular shape and their area is of order $h_{j,i}^2$. In this notation, the mesh size of $\mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$ is $h := \max\{h_{1,1}, h_{2,1}, \dots, h_{N_1,1}, h_{1,2}, h_{2,2}, \dots, h_{N_2,2}\}$.

We take V_h to be the space consisting of piecewise linear polynomials with respect to \mathcal{T}_h vanishing on the boundary of Ω . Also, we take $k = 1$. Hence, each component of $M_{h,0}|_{\Omega_i}$ consists of continuous linear piecewise

polynomials with respect to the mesh $\mathcal{T}_{h,i}$. Let $\{\Phi_1^i, \dots, \Phi_{2N_i}^i\}$ be a nodal basis for $M_{h,0|\Omega_i}$ and assume that $\Phi_j^i = (\phi_j^i, 0)^T$ and $\Phi_{N_i+j}^i = (0, \phi_j^i)^T$ for $j = 1, \dots, N_i$. Here, $\{\phi_1^i, \dots, \phi_{N_i}^i\}$ is a nodal basis for the space of continuous piecewise linear polynomials with respect to $\mathcal{T}_{h,i}$. With this notation, we note that $\{A\Phi_j^1\}_{j=1}^{N_1} \cup \{A\Phi_j^2\}_{j=1}^{N_2}$ is a basis for \mathcal{M}_h . Lastly, we define M_{A_i} to be the Gram matrix of the set $\{A\Phi_j^i\}_{j=1}^{N_i}$ with respect to the $(\cdot, \cdot)_{\tilde{Q}}$ inner product and $H_i := \text{diag}(h_{1,i}^2, h_{2,i}^2, \dots, h_{N_i,i}^2)$. Let

$$D_i = \left[\begin{array}{c|c} a_{11}H_i & \\ \hline & a_{22}H_i \end{array} \right],$$

where a_{11}, a_{22} are the entries of the matrix A .

Lemma 4.1. *Under the assumptions of Section 4.1.4, we have that for $i = 1, 2$*

$$(4.11) \quad \langle M_{A_i} \gamma, \gamma \rangle_e \leq c \langle D_i \gamma, \gamma \rangle_e \quad \text{for all } \gamma \in \mathbb{R}^{2N_i}.$$

Consequently,

$$(4.12) \quad \langle M_{A_i}^{-1} \gamma, \gamma \rangle_e \geq c \langle D_i^{-1} \gamma, \gamma \rangle_e \quad \text{for all } \gamma \in \mathbb{R}^{2N_i},$$

where c is independent of h , a_{11} , and a_{22} .

Proof. We will prove the result when $i = 1$. The case when $i = 2$ is similar.

Let $\gamma \in \mathbb{R}^{2N_1}$ and define $q_h := \sum_{j=1}^{2N_1} \gamma_j \Phi_j^1$. Note that

$$(4.13) \quad \langle M_{A_1} \gamma, \gamma \rangle_e = (Aq_h, q_h) = \|Aq_h\|_{\tilde{Q}}^2 = \sum_{\tau \in \mathcal{T}_h} \|Aq_h\|_{\tau, \tilde{Q}}^2.$$

If $\tau = [z_{1\tau}, z_{2\tau}, z_{3\tau}]$, then $q_h|_{\tau} = \begin{pmatrix} \sum_{j=1}^3 \gamma_{j\tau} \phi_{j\tau}^1 \\ \sum_{j=1}^3 \gamma_{(j+N_1)\tau} \phi_{j\tau}^1 \end{pmatrix}$. Hence,

$$(4.14) \quad \|Aq_h\|_{\tau, \tilde{Q}}^2 \leq c |\tau| \left(a_{11} \sum_{j=1}^3 \gamma_{j\tau}^2 + a_{22} \sum_{j=1}^3 \gamma_{(j+N_1)\tau}^2 \right).$$

Using (4.13), (4.14), and the fact that each coefficient γ_k can repeat at most three times, we obtain

$$\langle M_{A_1} \gamma, \gamma \rangle_e \leq c \left(a_{11} \sum_{j=1}^{N_1} h_{j,1}^2 \gamma_j^2 + a_{22} \sum_{j=1}^{N_1} h_{j,1}^2 \gamma_{j+N_1}^2 \right) = c \langle D_1 \gamma, \gamma \rangle_e.$$

The estimate (4.12) follows from (4.11). \square

We now show that (2.10) is satisfied for the operator R_h defined Section 4.1.3.

Lemma 4.2. *Under the assumptions of Section 4.1.4, there exists a constant c , independent of h , a_{11} , and a_{22} , such that*

$$(4.15) \quad \|R_h A \nabla v_h\|_h \geq c \|A \nabla v_h\|_{\tilde{Q}} \quad \text{for all } v_h \in V_h,$$

for the type I projection trial space.

Proof. First, note that $\{A\Phi_1^1, \dots, A\Phi_{2N_1}^1\}$ and $\{A\Phi_1^2, \dots, A\Phi_{2N_2}^2\}$ are nodal bases for $\tilde{\mathcal{M}}_h|_{\Omega_1}$ and $\tilde{\mathcal{M}}_h|_{\Omega_2}$, respectively. Define $v_h^i := v_h|_{\Omega_i}$ for $v_h \in V_h$. For a fixed $A \nabla v_h$ with $v_h \in V_h$ we define the dual vectors $G_h^1 \in \mathbb{R}^{2N_1}$, $G_h^2 \in \mathbb{R}^{2N_2}$ by

$$\begin{aligned} (G_h^1)_i &:= (A \nabla v_h^1, A \Phi_i^1)_{\tilde{Q}} = (A \nabla v_h^1, \Phi_i^1) \quad i = 1, \dots, 2N_1, \\ (G_h^2)_i &:= (A \nabla v_h^2, A \Phi_i^2)_{\tilde{Q}} = (A \nabla v_h^2, \Phi_i^2) \quad i = 1, \dots, 2N_2, \end{aligned}$$

and let

$$R_h A \nabla v_h = \begin{cases} \sum_{i=1}^{2N_1} \alpha_i A \Phi_i^1 & \text{in } \Omega_1, \\ \sum_{i=1}^{2N_2} \beta_i A \Phi_i^2 & \text{in } \Omega_2. \end{cases}$$

Thus, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2N_1})^T$ and $\beta = (\beta_1, \beta_2, \dots, \beta_{2N_2})^T$ are solutions to

$$M_{A_1} \alpha = G_h^1, \quad \text{and} \quad M_{A_2} \beta = G_h^2,$$

respectively. Using (4.12), we obtain

$$\begin{aligned} \|R_h A \nabla v_h\|_h^2 &= \sum_{i,j=1}^{2N_1} \alpha_i \alpha_j (A \Phi_i^1, \Phi_j^1) + \sum_{i,j=1}^{2N_2} \beta_i \beta_j (A \Phi_i^2, \Phi_j^2) \\ &= \langle M_{A_1}^{-1} G_h^1, G_h^1 \rangle_e + \langle M_{A_2}^{-1} G_h^2, G_h^2 \rangle_e \\ &\geq c_1 \langle D_1^{-1} G_h^1, G_h^1 \rangle_e + c_2 \langle D_2^{-1} G_h^2, G_h^2 \rangle_e. \end{aligned}$$

We recall by definition of H_1, H_2 that we have $h_{i,1} = h_{i+N_1,1}$ for $i = 1, \dots, N_1$ and $h_{i,2} = h_{i+N_2,2}$ for $i = 1, \dots, N_2$. Thus,

$$\begin{aligned} \langle D_1^{-1} G_h^1, G_h^1 \rangle_e &= \sum_{i=1}^{N_1} h_{i,1}^{-2} \left[a_{11} \left(\frac{\partial v_h^1}{\partial x}, \phi_i^1 \right)^2 + a_{22} \left(\frac{\partial v_h^1}{\partial y}, \phi_i^1 \right)^2 \right] \\ &= \sum_{i=1}^{N_1} \sum_{\tau \subset \text{supp}(\phi_i^1)} h_{i,1}^{-2} (1, \phi_i^1)_\tau \left[a_{11} \left| \frac{\partial v_h^1}{\partial x} \right|_\tau^2 + a_{22} \left| \frac{\partial v_h^1}{\partial y} \right|_\tau^2 \right] \\ &\geq c_1 \|A \nabla v_h^1\|_{\Omega_1, \tilde{Q}}^2. \end{aligned}$$

Similarly, we can show

$$\langle D_2^{-1} G_h^2, G_h^2 \rangle_e \geq c_2 \|A \nabla v_h^2\|_{\Omega_2, \tilde{Q}}^2.$$

Thus,

$$\|R_h A \nabla v_h\|_h^2 \geq c \left(\|A \nabla v_h^1\|_{\Omega_1, \tilde{Q}}^2 + \|A \nabla v_h^2\|_{\Omega_2, \tilde{Q}}^2 \right) = c \|A \nabla v_h\|_{\tilde{Q}}^2.$$

□

As a consequence of Lemma 4.2, equation (4.6), and Proposition 2.2, we have the following result.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\{T_h\}$ be a family of locally quasi-uniform meshes for Ω . For each h , let V_h be the space of continuous linear functions with respect to the mesh $\{T_h\}$ that vanish on $\partial\Omega$ and \mathcal{M}_h be the corresponding projection type I trial space defined in Section 4.1.3. Then the family of spaces $\{(V_h, \mathcal{M}_h)\}$ is stable.*

4.1.5. *Projection trial space type II.* For simplicity, we present the second type of projection trial space for the case when $N = 1$ (no interface). We consider the following inner product on $\tilde{\mathcal{M}}_h$:

$$(A\Phi_i, A\Phi_j)_h := \delta_{ij}(1, A\Phi_i).$$

Note that

$$\left(\sum_i \frac{(p, A\Phi_i)_{\tilde{Q}}}{(1, A\Phi_i)} A\Phi_i, A\Phi_j \right)_h = (p, A\Phi_j)_{\tilde{Q}},$$

which implies $R_h : \tilde{Q} \rightarrow \tilde{\mathcal{M}}_h$ is given by

$$R_h p = \sum_i \frac{(p, A\Phi_i)_{\tilde{Q}}}{(1, A\Phi_i)} A\Phi_i = \sum_i \frac{(p, \Phi_i)}{(1, A\Phi_i)} A\Phi_i,$$

by (2.8). For the application to the elliptic interface problem, we simply apply R_h locally on each subdomain with respect to the $(\cdot, \cdot)_h$ inner product just as in Section 4.1.3. We then define the trial space as

$$\mathcal{M}_h := R_h A \nabla V_h.$$

The problem to be solved using the type II projection trial space is identical to (4.10). Furthermore, a similar estimate holds as in (4.15) with a constant independent of h , a_{11} , and a_{22} . Consequently, a version of Theorem 4.3 holds for this type of projection trial space.

5. NUMERICAL RESULTS

We implemented the nc SPLS discretization method with preconditioning on second order elliptic PDE of the form (4.1). For all examples presented, Ω is taken to be a bounded polygonal or polyhedral domain and the test space $V_h \subset H_0^1(\Omega)$ is chosen to be the space of continuous piecewise linear polynomials with respect to the quasi-uniform, or locally quasi-uniform, meshes \mathcal{T}_h . We use Algorithm 3.1 to approximate the flux $A \nabla u$ using both the no projection type of trial space outlined in Section 4.1.1, as well as both

types of projection trial spaces outlined in Sections 4.1.3 and 4.1.5. Based on the first inequality of (3.10), we used a stopping criterion of

$$(5.1) \quad \|q_j\|_h \leq c_0 h^2,$$

on each level of refinement. In **Step 1** and **(PCG1)** of Algorithm 3.1, we consider the cases when P_h is given by the BPX and Multigrid preconditioners [17, 18, 30, 28]. For a thorough analysis of these preconditioners for elliptic interface problems, we refer to [18, 29, 27].

5.1. Intersecting interface example. For this example, $\Omega = (0, 1) \times (0, 1)$ with interface $\Gamma := \Omega \cap \{(x, y) \mid x = 1/2 \text{ or } y = 1/2\}$ as considered in [14]. The family of interface-fitted, locally quasi-uniform meshes $\{\mathcal{T}_h\}$ was obtained by a standard uniform refinement strategy starting with a uniform coarse mesh. We computed f such that for

$$A(x, y) = a(x, y)I_2, \text{ where } a(x, y) = \begin{cases} 1 & \text{if } (x, y) \in [0, 1/2]^2 \cup [1/2, 1]^2, \\ c & \text{if } (x, y) \in \Omega \setminus ([0, 1/2]^2 \cup [1/2, 1]^2), \end{cases}$$

the exact solution is $u(x, y) = a(x, y)^{-1} \sin(2\pi x) \sin(2\pi y)$. Table 1 shows results for the no projection type trial space and BPX preconditioner. Tables 2 and 3 show results for the type II projection trial space using the BPX and Multigrid preconditioners, respectively. Table 4 shows results for the type I projection trial space with the BPX preconditioner.

$$\text{error} = \|A\nabla u - A\nabla u_h\|$$

$h = 2^{-k}$	$c = 1/10$			$c = 1/100$			$c = 1/1000$		
	error	rate	it	error	rate	it	error	rate	it
1	7.045		1	21.349		1	67.209		1
2	3.933	0.841	3	11.918	0.841	3	37.520	0.841	4
3	2.025	0.957	7	6.137	0.957	8	19.320	0.957	9
4	1.020	0.989	10	3.092	0.989	12	9.733	0.989	13
5	0.511	0.997	13	1.549	0.997	15	4.876	0.997	16
6	0.256	0.999	16	0.775	0.999	18	2.439	0.999	19

Table 1: No projection trial space with BPX preconditioner.

$$\text{error} = \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$$

$h = 2^{-k}$	$c = 1/10$			$c = 1/100$			$c = 1/1000$		
	error	rate	it	error	rate	it	error	rate	it
1	4.344		1	13.162		1	41.437		1
2	1.743	1.317	3	5.282	1.317	3	16.627	1.317	3
3	0.599	1.540	6	1.815	1.541	8	5.710	1.542	9
4	0.208	1.526	14	0.627	1.534	18	1.971	1.534	23
5	0.073	1.515	23	0.218	1.521	32	0.685	1.525	45
6	0.026	1.483	33	0.076	1.513	54	0.239	1.517	82

Table 2: Type II projection trial space, BPX preconditioner.

$$\text{error} = \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$$

$h = 2^{-k}$ k	$c = 1/10$			$c = 1/100$			$c = 1/1000$		
	error	rate	it	error	rate	it	error	rate	it
1	4.344		1	13.162		1	41.437		1
2	1.796	1.274	3	5.281	1.317	4	16.626	1.317	4
3	0.620	1.535	4	1.814	1.542	8	5.716	1.540	9
4	0.216	1.520	7	0.639	1.505	12	1.983	1.527	20
5	0.076	1.509	9	0.226	1.499	15	0.700	1.502	27
6	0.027	1.512	11	0.079	1.513	19	0.247	1.506	32

Table 3: Type II projection trial space, Multigrid preconditioner.

$$\text{error} = \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$$

$h = 2^{-k}$ k	$c = 1/10$			$c = 1/100$			$c = 1/1000$		
	error	rate	it	error	rate	it	error	rate	it
1	5.176		1	15.686		1	49.383		1
2	1.258	2.041	4	3.812	2.041	4	12.001	2.041	4
3	0.339	1.893	10	1.026	1.893	12	3.231	1.893	13
4	0.093	1.868	24	0.281	1.868	26	0.885	1.868	31
5	0.025	1.877	48	0.076	1.880	59	0.240	1.880	66
6	0.007	1.865	80	0.021	1.893	107	0.065	1.895	130

Table 4: Type I projection trial space, BPX preconditioner.

5.1.1. *Comparison with a different choice of inner product.* For the intersecting interface problem, we demonstrate the benefit of choosing the weighted inner product on $V = H_0^1(\Omega)$ (See Section 4) as compared to choosing the inner product $a_0(u_h, v_h) := (\nabla u_h, \nabla v_h)$. Table 5 displays results using the type II projection trial space using this inner product with the Multigrid preconditioner for the same values of the jump in the coefficients.

$$\text{error} = \|A\nabla u - R_h A\nabla u_h\|$$

$h = 2^{-k}$ k	$c = 1/10$			$c = 1/100$			$c = 1/1000$		
	error	rate	it	error	rate	it	error	rate	it
1	4.344		1	13.162		1	41.437		1
2	1.975	1.137	5	6.133	1.102	9	19.804	1.065	13
3	0.638	1.632	9	2.030	1.600	19	6.425	1.624	43
4	0.227	1.487	11	0.739	1.453	26	2.439	1.397	68
5	0.080	1.510	13	0.247	1.581	36	0.791	1.623	101
6	0.028	1.524	17	0.087	1.512	44	0.286	1.479	112

Table 5: Type II projection trial space with Multigrid preconditioner and inner product $(\nabla u_h, \nabla v_h)$.

In comparison with Table 3, we see a significant increase in the number of iterations when this inner product is chosen. This is due to the fact that in this case the factor $\kappa(S_h)$, in estimate (3.15), depends on the size of the jump

in the coefficients. In contrast, using the weighted inner product eliminates the influence of the factor $\kappa(S_h)$ from the condition number.

5.2. 3-D example. For this example, $\Omega \subset \mathbb{R}^3$ is taken to be the unit cube with interface $\Gamma := \Omega \cap \{(x, y, z) \mid x = 1/2\}$. We computed f such that for

$$A(x, y, z) = a(x, y, z)I_3, \text{ where } a(x, y, z) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ c & \text{if } x \geq \frac{1}{2}, \end{cases}$$

the exact solution is

$$u(x, y, z) = \begin{cases} cx(x - \frac{1}{2})y(y - 1)z(z - 1) & \text{if } x < \frac{1}{2}, \\ (x - \frac{1}{2})(x - 1)y(y - 1)z(1 - z) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Table 6 shows results for the no projection type trial space and BPX preconditioner. Tables 7 and 8 show results for the type II projection trial space and the BPX and Multigrid preconditioners, respectively. Table 9 shows results for the type I projection trial space and the BPX preconditioner.

error = $\|A\nabla u - A\nabla u_h\|_{\tilde{Q}}$

$h = 2^{-k}$	$c = 100$			$c = 1000$			$c = 10000$		
	error	rate	it	error	rate	it	error	rate	it
1	0.837		1	8.337		1	83.334		1
2	0.572	0.549	2	5.700	0.549	3	56.972	0.549	4
3	0.320	0.838	6	3.188	0.838	8	31.864	0.838	11
4	0.165	0.953	11	1.647	0.953	15	16.462	0.953	18
5	0.083	0.987	19	0.831	0.988	24	8.302	0.988	29

Table 6: 3D - no projection trial space, BPX preconditioner.

error = $\|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$

$h = 2^{-k}$	$c = 100$			$c = 1000$			$c = 10000$		
	error	rate	it	error	rate	it	error	rate	it
1	0.837		1	8.337		1	83.334		1
2	0.312	1.426	1	2.995	1.477	2	29.774	1.485	5
3	0.120	1.374	4	1.139	1.395	8	11.390	1.386	14
4	0.046	1.397	8	0.414	1.458	20	4.141	1.460	29
5	0.017	1.436	13	0.148	1.485	32	1.463	1.500	57

Table 7: 3D - Type II projection trial space, BPX preconditioner.

$$\text{error} = \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$$

$h = 2^{-k}$	$c = 100$			$c = 1000$			$c = 10000$		
	error	rate	it	error	rate	it	error	rate	it
1	0.837		1	8.337		1	83.334		1
2	0.314	1.413	1	2.995	1.477	2	29.777	1.484	5
3	0.117	1.431	3	1.139	1.394	6	11.391	1.386	10
4	0.044	1.400	4	0.414	1.459	10	4.141	1.460	17
5	0.016	1.451	5	0.148	1.488	14	1.464	1.500	28

Table 8: 3D - Type II projection trial space, Multigrid preconditioner.

$$\text{error} = \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$$

$h = 2^{-k}$	$c = 100$			$c = 1000$			$c = 10000$		
	error	rate	it	error	rate	it	error	rate	it
1	0.837		1	8.337		1	83.334		1
2	0.294	1.511	2	2.923	1.512	5	29.221	1.512	5
3	0.078	1.920	8	0.772	1.921	13	7.717	1.921	18
4	0.021	1.876	16	0.210	1.881	27	2.095	1.881	37
5	0.006	1.871	31	0.057	1.887	46	0.566	1.889	71

Table 9: 3D - Type I projection trial space, BPX preconditioner.

5.3. Remarks about the nc SPLS with preconditioning. We observe for both problems presented that the approximation of the flux is *super-linear* for the case of a projection type trial space, and we obtain a significant decrease in the error when compared to the trial space of no projection type. Also, the order of convergence for the flux is robust with respect to the jump in the coefficients for both choices of projection trial spaces.

The number of iterations depends on the size of the jump as well as the mesh size. According with Remark 3.2 and the estimate (3.15) of Proposition 3.4, the number of iterations of the PCG Algorithm 3.1 depends on the condition number of the Schur complement of the unpreconditioned problem $\kappa(S_h)$ and the condition number of the elliptic preconditioner, $\kappa(P_h A_h)$. From Proposition 2.10, Lemma 4.2, and the estimates (4.4), (4.6), we have

$$\kappa(S_h) \leq \frac{M^2}{m_h^2} \leq c,$$

with c independent of the jump and the mesh size. For both BPX and Multigrid preconditioners we used in our numerical tests, according to [29], we have

$$\kappa(P_h A_h) \leq c \min \left\{ c_d(h), \frac{a_{max}}{a_{min}} \right\},$$

where $c_d(h) = |\log h|^2$ when $d = 2$ and $c_d(h) = h^{-1}$ when $d = 3$ (d refers to the dimension). Combining (3.15) with the above two inequalities we obtain

$$\kappa(\tilde{S}_h) \leq C \min \left\{ c_d(h), \frac{a_{max}}{a_{min}} \right\}.$$

We also note that a slight dependence on h is also due to the imposed stopping criterion (5.1).

6. CONCLUSION

We presented a saddle point least squares method with nonconforming trial spaces for discretization of mixed variational formulations with a focus on applications to second order PDEs with discontinuous coefficients. Discretization analysis and preconditioning to other problems requires non-trivial discrete stability analysis and will be discussed in separate projects. The proposed method is easy to implement using an Uzawa PCG type algorithm and leads to higher order approximation of the flux when compared with standard finite element (non-mixed) techniques based on linear element approximation. The preconditioning of the mixed formulation reduces to elliptic preconditioning associated with inner products on test spaces, usually H^1 type spaces.

We plan to further combine the nc SPLS discretization method with known multilevel and adaptive techniques [1, 5, 6, 7, 10, 19, 20, 26] for designing robust iterative solvers for more general first and second order elliptic PDEs or systems of PDEs that are parameter dependent and could exhibit singular solutions due to non-convex domains or rough coefficients or data.

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