

# SUBSPACE INTERPOLATION WITH APPLICATIONS TO ELLIPTIC REGULARITY

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ABSTRACT. In this paper, we prove new embedding results by means of subspace interpolation theory and apply them to establishing regularity estimates for the biharmonic Dirichlet problem, and for the Stokes and the Navier-Stokes systems on polygonal domains. The main result of the paper gives a stability estimate for the biharmonic problem at the threshold index of smoothness. The classical regularity estimates for the biharmonic problem are deduced as a simple corollary of the main result. The subspace interpolation tools and techniques presented in this paper can be applied to establishing sharp regularity estimates for other elliptic boundary value problems on polygonal domains.

## 1. INTRODUCTION

Regularity estimates or shift estimates for the solutions of elliptic Boundary Value Problems (BVPs) in terms of Sobolev-Besov norms, are of a significant interest in discretizing BVPs by finite differences methods, spectral methods and finite element methods. Shift estimates for the Laplace operator with Dirichlet boundary conditions on non-smooth domains are widely studied in the literature (see, e.g., [3, 8, 19, 22, 26, 5, 6, 7]). Shift estimates for the biharmonic problem on non-smooth domains, are presented in [12, 14, 4]. In this paper we extend the results of [4] and [14] by studying the regularity at the critical case. The shift results are proved by using *the real method of interpolation* of Lions and Peetre [9, 23, 24] and the *subspace interpolation theory* introduced by Kellogg in [19]. The approach based on subspace interpolation theory proves to be essential for the threshold regularity case. We describe a typical subspace interpolation problem as follows. Let  $X$  and  $Y$  be Sobolev spaces of integer order with  $X$  embedded in  $Y$  ( $X \subset Y$ ), and let  $X_K$  be a subspace of  $X$  of finite codimension. Then, we need to characterize the interpolation spaces between  $Y$  and  $X_K$ . In particular, we are looking for embedding relations between the intermediate subspaces  $[Y, X_K]_{s, q_0}$  and the standard interpolation spaces  $[Y, X]_{s, q_1}$ . Here,  $s \in (0, 1)$  can be viewed as the smoothness index, and  $q_0, q_1$  can be viewed as tuning parameters.

In proving shift estimates for the biharmonic problem, we follow Kellogg's approach in dealing with subspace interpolation. First, we study subspace interpolation embeddings on Sobolev spaces defined on all of  $\mathbb{R}^2$ . Then, we apply the results to the polygonal domain case by finding "extension" and "restriction" operators connecting the Sobolev spaces defined on bounded domains and the Sobolev spaces

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defined on  $\mathbb{R}^2$ . In contrast with Kellogg interpolation results presented in [19], we consider subspaces of codimension higher than one, and allow the tuning parameter  $q$  to be not necessarily 2. The shift estimates for Stokes and Navier-Stokes systems are based on the main regularity estimate for the biharmonic problem. Next, we describe the main regularity result of the paper.

Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$ . Let  $\partial\Omega$  be the polygonal curve  $P_1P_2\cdots P_mP_1$ . Assume that  $\partial\Omega$  does not self intersect. For each point  $P_j$ , we denote by  $\omega_j$  the measure of the angle with the vertex at  $P_j$  and sides aligned with  $\partial\Omega$  (measured from inside  $\Omega$ ). Let  $\omega := \max\{\omega_j : j = 1, 2, \dots, m\}$ .

We consider the biharmonic problem: Given  $f \in L^2(\Omega)$ , find  $u$  such that

$$(1.1) \quad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that the solution of (1.1) satisfies

$$(1.2) \quad \|u\|_{H^{2+2s}(\Omega)} \leq c \|f\|_{H^{-2+2s}(\Omega)}, \quad \text{for all } f \in H^{-2+2s}(\Omega), \quad 0 \leq s < s_0,$$

where  $c$  is a positive constant, and  $s_0 = s_0(\omega) \in (0, 1)$  depends on the maximum angle of  $\partial\Omega$  only, (see e.g., [4, 14, 25]). We will prove in this paper that for some positive constant  $c$ , we have

$$(1.3) \quad \|u\|_{B_\infty^{2+2s_0}(\Omega)} \leq c \|f\|_{B_1^{-2+2s_0}(\Omega)}, \quad \text{for all } f \in B_1^{-2+2s_0}(\Omega),$$

where

$$B_\infty^{2+2s_0}(\Omega) := [H^2(\Omega), H^4(\Omega)]_{s_0, \infty} \quad \text{and} \quad B_1^{-2+2s_0}(\Omega) := [H^{-2}(\Omega), L^2(\Omega)]_{s_0, 1}.$$

Note that

$$H^{2+2s}(\Omega) = [H^2(\Omega), H^4(\Omega)]_{s, 2} \quad \text{and} \quad H^{-2+2s}(\Omega) = [H^{-2}(\Omega), L^2(\Omega)]_{s, 2}.$$

In other words, we prove that the estimate (1.2) holds for the threshold regularity index  $s = s_0$ , provided the second index of interpolation  $q$  is taken  $q = 1$  for measuring the data  $f$ , and  $q$  is taken  $q = \infty$  for measuring the solution  $u$ . The precise definitions of the interpolation spaces are given in the next section. We note that (1.2) can be easily proved as a consequence of (1.3) and the reiteration theorem. In addition, the proof of (1.3) is considerably simpler than the proof of (1.2) as given in [4], (see Section 4).

Section 2 and Section 3 contain interesting theoretical embedding results which can be used in deriving regularity estimates for other elliptic BVPs that are not contained in this paper. For example, one can recover the regularity estimates at the critical case for the Laplace operator with Dirichlet boundary conditions which are presented in [8] by means of multilevel representation of norms. In addition, one can use the approach of this paper and prove estimates at the threshold regularity index for the Laplace operator with mixed Dirichlet-Neuman boundary conditions.

The remaining part of the paper is organized as follows. In Section 2 we present subspace interpolation results in an abstract setting. In Section 3, we prove embedding results for interpolation spaces defined on the whole  $\mathbb{R}^2$ . The main shift estimate (1.3) is proved in Section 4. Regularity estimates for the Stokes and Navier-Stokes systems are presented in Section 5. In the Appendix, we present the technical proofs that were omitted in the prior sections.

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## 2. INTERPOLATION RESULTS

In this section, we start with some basic definitions and results concerning interpolation between subspaces of Hilbert spaces using the real method of interpolation of Lions and Peetre (see [23]). In the second subsection, we present *subspace interpolation* theory.

**2.1. Interpolation between Banach and Hilbert spaces.** Let  $X, Y$  be Banach spaces satisfying the following conditions:

$$(2.1) \quad \begin{cases} X \text{ is a dense subset of } Y \text{ and} \\ \|u\|_Y \leq c\|u\|_X \quad \text{for all } u \in X, \end{cases}$$

for some positive constant  $c$ . For  $s \in (0, 1)$ ,  $1 \leq q \leq \infty$ , the intermediate space  $[Y, X]_{s,q}$  is defined using the  $K$  function. For  $u \in Y$  and  $t > 0$ , we define

$$K(t, u, Y, X) = K(t, u) := \inf_{u_0 \in X} (\|u - u_0\|_Y^2 + t^2\|u_0\|_X^2)^{1/2}.$$

For  $s \in (0, 1)$  and  $1 \leq q \leq \infty$ , the space  $[Y, X]_{s,q}$  is defined by

$$[Y, X]_{s,q} := \{u \in Y : \|u\|_{[Y,X]_{s,q}} < \infty\},$$

where

$$\|u\|_{[Y,X]_{s,q}} = \left( \int_0^\infty (t^{-s} K(t, u))^q \frac{dt}{t} \right)^{1/q} \quad \text{for } q < \infty,$$

and

$$\|u\|_{[Y,X]_{s,\infty}} = \sup_{t>0} t^{-s} K(t, u).$$

By definition, we take

$$[Y, X]_{0,q} := Y \quad \text{and} \quad [Y, X]_{1,q} := X.$$

The main properties of the interpolation between Banach spaces can be found in [9, 10, 13]. Next, we will focus on interpolation between Hilbert spaces.

Let us assume that, in addition,  $X, Y$  are separable Hilbert spaces with inner products  $(\cdot, \cdot)_X$  and  $(\cdot, \cdot)_Y$ , and the norms on the two spaces are the norms induced by the corresponding inner products ( $\|u\|_X^2 = (u, u)_X$  and  $\|u\|_Y^2 = (u, u)_Y$ ). Let  $D(S)$  denote the subset of  $X$  consisting of all elements  $u$  such that the anti-linear form

$$(2.2) \quad v \rightarrow (u, v)_X, \quad v \in X,$$

is continuous in the topology induced by  $Y$ . For any  $u$  in  $D(S)$  the anti-linear form (2.2) can be extended to a continuous anti-linear form on  $Y$ . Then, by the Riesz representation theorem (see e.g., [1, 29]), there exists an element  $Su$  in  $Y$  such that

$$(2.3) \quad (u, v)_X = (Su, v)_Y \quad \text{for all } v \in X.$$

In this way,  $S$  is a well defined operator in  $Y$  with domain  $D(S)$ . The next result illustrates the properties of  $S$ , see, e.g. [23, 28].

**Proposition 2.1.** *The domain  $D(S)$  of the operator  $S$  is dense in  $X$  and consequently  $D(S)$  is dense in  $Y$ . The operator  $S : D(S) \subset Y \rightarrow Y$  is a bijective, self-adjoint and positive definite operator. The inverse operator  $S^{-1} : Y \rightarrow D(S) \subset Y$  is a bounded symmetric positive definite operator and*

$$(2.4) \quad (S^{-1}z, u)_X = (z, u)_Y \quad \text{for all } z \in Y, u \in X.$$

If  $X, Y$  are Hilbert spaces, we have a better representations of the intermediate space  $[Y, X]_{s,q}$ . The next lemma provides the relation between  $K(t, u)$  and the connecting operator  $S$ , and consequently, gives the representation of  $[Y, X]_{s,q}$  in terms of  $S$ . A similar result can be found in [23].

**Lemma 2.2.** *For all  $u \in Y$  and  $t > 0$ ,*

$$(2.5) \quad K(t, u)^2 = ((I + (t^2S)^{-1})^{-1}u, u)_Y,$$

$$(2.6) \quad \|u\|_{[Y, X]_{s,q}} = \left( \int_0^\infty t^{sq} ((I + t^2S^{-1})^{-1}u, u)_Y^{q/2} \frac{dt}{t} \right)^{1/q}, \quad 1 \leq q < \infty,$$

and

$$(2.7) \quad \|u\|_{[Y, X]_{s,\infty}} = \sup_{t>0} t^s ((I + t^2S^{-1})^{-1}u, u)_Y^{1/2}.$$

*Proof.* Using the density of  $D(S)$  in  $X$ , we have

$$K(t, u)^2 = \inf_{u_0 \in D(S)} (\|u_0\|_X^2 + t^2\|u - u_0\|_Y^2)$$

Let  $v = Su_0$ . Then

$$(2.8) \quad K(t, u)^2 = \inf_{v \in Y} (\|u - S^{-1}v\|_Y^2 + t^2\|(S^{-1}v, v)_Y).$$

By solving the minimization problem (2.8), we obtain that the element  $v$ , which gives the optimum, satisfies the following equations:

$$(t^2I + S^{-1})v = u,$$

and

$$\|u - S^{-1}v\|_Y^2 + t^2\|(S^{-1}v, v)_Y = ((I + (t^2S)^{-1})^{-1}u, u)_Y.$$

To obtain (2.6) and (2.7) we replace  $K$  in the definition of  $[Y, X]_{s,q}$  with the expression given by (2.5) and apply the change of variable  $1/t = \tau$ .  $\square$

**Lemma 2.3.** *Let  $X_0$  be a closed subspace of  $X$ , and let  $Y_0$  be a closed subspace of  $Y$ . Let  $X_0$  and  $Y_0$  be equipped with the topologies and the geometries induced by  $X$  and  $Y$ , respectively, and assume that the pair  $(X_0, Y_0)$  satisfies (2.1). Then, for  $s \in [0, 1], 1 \leq q \leq \infty$ ,*

$$[Y_0, X_0]_{s,q} \subset [Y, X]_{s,q} \cap Y_0.$$

*Proof.* For any  $u \in Y_0$ , we have

$$K(t, u, Y, X) \leq K(t, u, Y_0, X_0).$$

Thus,

$$(2.9) \quad \|u\|_{[Y, X]_{s,q}} \leq \|u\|_{[Y_0, X_0]_{s,q}} \quad \text{for all } u \in [Y_0, X_0]_{s,q},$$

which proves the lemma.  $\square$

The validity of the other inclusion will be addressed in the next subsection for cases related to the regularity for the biharmonic problem.

**2.2. Subspace interpolation on Hilbert spaces.** Let  $\mathcal{K} = \text{span}\{\varphi_1, \dots, \varphi_n\}$  be a  $n$ -dimensional subspace of  $X$ , and let  $X_{\mathcal{K}}$  be the orthogonal complement of  $\mathcal{K}$  in  $X$  in the  $(\cdot, \cdot)_X$  inner product. We are interested in determining the interpolation spaces of  $Y$  and  $X_{\mathcal{K}}$ , where on  $X_{\mathcal{K}}$  we consider again the  $(\cdot, \cdot)_X$  inner product. For certain spaces  $X, \mathcal{K}, Y$ ,  $n = 1$ , and  $q = 2$ , this problem was studied in [19]. To apply the interpolation results from the previous section we need to check that the density part of the condition (2.1) is satisfied for the pair  $(X_{\mathcal{K}}, Y)$ . For  $\varphi \in \mathcal{K}$ , we define the linear functional  $\Lambda_{\varphi} : X \rightarrow C$  by

$$\Lambda_{\varphi} u := (u, \varphi)_X, \quad u \in X.$$

The following lemma can be found in [4].

**Lemma 2.4.** *The space  $X_{\mathcal{K}}$  is dense in  $Y$  if and only if the following condition is satisfied:*

$$(2.10) \quad \begin{cases} \Lambda_{\varphi} \text{ is not bounded in the topology of } Y \\ \text{for all } \varphi \in \mathcal{K}, \varphi \neq 0. \end{cases}$$

For the remaining part of this section, we assume that the condition (2.10) holds. By the above lemma, the condition (2.1) is satisfied. It follows from the previous section that the operator  $S_{\mathcal{K}} : D(S_{\mathcal{K}}) \subset Y \rightarrow Y$  defined by

$$(2.11) \quad (u, v)_X = (S_{\mathcal{K}} u, v)_Y \quad \text{for all } v \in X_{\mathcal{K}},$$

has the same properties as  $S$  has. Consequently, the norm on the intermediate space  $[Y, X_{\mathcal{K}}]_{s,q}$  is given by

$$(2.12) \quad \|u\|_{[Y, X_{\mathcal{K}}]_{s,q}} = \left( \int_0^{\infty} t^{sq} \left( (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u \right)_Y^{q/2} \frac{dt}{t} \right)^{1/q}, \quad 1 \leq q < \infty,$$

and

$$(2.13) \quad \|u\|_{[Y, X_{\mathcal{K}}]_{s,\infty}} = \sup_{t>0} t^s \left( (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u \right)_Y^{1/2}.$$

In this section, our aim is to study the embedding relation between  $[Y, X_{\mathcal{K}}]_{s,q_1}$  and  $[Y, X]_{s,q_2}$ .

**Lemma 2.5.** *Assume that for the fixed parameters  $s_0 \in (0, 1)$ ,  $q_0, q_1 \in [1, \infty]$ , we have the following continuous embedding*

$$(2.14) \quad [Y, X]_{s_0, q_1} \subset [Y, X_{\mathcal{K}}]_{s_0, q_0}.$$

Then,

$$[Y, X_{\mathcal{K}}]_{s,q} = [Y, X]_{s,q}, \quad 0 < s < s_0, \quad q \in [1, \infty].$$

*Proof.* From Lemma 2.3, we have that

$$[Y, X_{\mathcal{K}}]_{s,q} \subset [Y, X]_{s,q}, \quad 0 \leq s < s_0.$$

On the other hand, from (2.14), the inclusion operator is continuous from  $[Y, X]_{s_0, q_1}$  to  $[Y, X_{\mathcal{K}}]_{s_0, q_0}$ , and clearly, it is continuous from  $[Y, X]_{0,2} = Y$  to  $[Y, X_{\mathcal{K}}]_{0,2} = Y$ . By interpolation, and by the reiteration theorem (see Appendix 6.2), we get

$$[Y, X]_{s,q} \subset [Y, X_{\mathcal{K}}]_{s,q}, \quad 0 < s < s_0,$$

which completes the proof.  $\square$

We note here that, for a fixed  $s_0 \in (0, 1)$ , the weakest assumption of type (2.14) is for  $q_0 = \infty$  and  $q_1 = 1$ . Next, we study the validity of (2.14) for  $q_0 = \infty$  and  $q_1 = 1$ . First, we note that the operators  $S_{\mathcal{K}}$  and  $S$  are related by the following identity:

$$(2.15) \quad S_{\mathcal{K}}^{-1} = (I - Q_{\mathcal{K}})S^{-1},$$

where  $Q_{\mathcal{K}} : X \rightarrow \mathcal{K}$  is the orthogonal projection onto  $\mathcal{K}$ . The proof of (2.15) follows easily from the definitions of the operators  $S, Q_{\mathcal{K}}$  and  $S_{\mathcal{K}}$ . The identity (2.15) leads to a formula relating the norms on  $[Y, X_{\mathcal{K}}]_{s,q}$  and  $[Y, X]_{s,q}$ . To get the formula, we introduce first the following notation. Let

$$(2.16) \quad (u, v)_{X,t} := ((I + t^2 S^{-1})^{-1} u, v)_X \quad \text{for all } u, v \in X,$$

and

$$(2.17) \quad (u, v)_{Y,t} := ((I + t^2 S^{-1})^{-1} u, v)_Y \quad \text{for all } u, v \in Y.$$

Denote by  $M_t$  the Gram matrix associated with the set of vectors  $\{\varphi_1, \dots, \varphi_n\}$  in the  $(\cdot, \cdot)_{X,t}$  inner product, i.e.,

$$(M_t)_{ij} := (\varphi_j, \varphi_i)_{X,t}, \quad i, j \in \{1, \dots, n\}.$$

**Theorem 2.6.** *For any  $u \in Y$ , we have*

$$(2.18) \quad \|u\|_{[Y, X_{\mathcal{K}}]_{s,2}}^2 = \|u\|_{[Y, X]_{s,2}}^2 + \int_0^\infty t^{2+2s} \langle M_t^{-1} y_t, y_t \rangle \frac{dt}{t},$$

$$(2.19) \quad \|u\|_{[Y, X_{\mathcal{K}}]_{s,\infty}}^2 = \sup_{t>0} (t^{2s} (u, u)_{Y,t} + t^{2+2s} \langle M_t^{-1} y_t, y_t \rangle),$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathbb{C}^n$ , and  $y_t$  is the  $n$ -dimensional vector in  $\mathbb{C}^n$  whose components are

$$(y_t)_i := (u, \varphi_i)_{Y,t}, \quad i = 1, \dots, n.$$

The proof is given in Appendix 6.1. For  $n = 1$ , let  $\mathcal{K} = \text{span}\{\varphi\}$  and denote  $X_{\mathcal{K}}$  by  $X_\varphi$ . Then, the formulas (2.18) and (2.19) become

$$(2.20) \quad \|u\|_{[Y, X_\varphi]_{s,2}}^2 = \|u\|_{[Y, X]_{s,2}}^2 + \int_0^\infty t^{2+2s} \frac{|(u, \varphi)_{Y,t}|^2}{(\varphi, \varphi)_{X,t}} \frac{dt}{t},$$

and

$$(2.21) \quad \|u\|_{[Y, X_\varphi]_{s,\infty}}^2 = \sup_{t>0} \left( t^{2s} (u, u)_{Y,t} + t^{2+2s} \frac{|(u, \varphi)_{Y,t}|^2}{(\varphi, \varphi)_{X,t}} \right),$$

respectively. The next theorem gives sufficient conditions for (2.14) to be satisfied. Before we state the result, we fix  $s_i \in (0, 1), i = 0, 1, \dots, n$ , and define  $s_0 = \min\{s_1, s_2, \dots, s_n\}$ . We introduce the following two conditions:

**(A.1)**  $[Y, X]_{s_0,1} \subset [Y, X_{\varphi_i}]_{s_0,\infty}$ , for all  $i = 1, \dots, n$ .

**(A.2)** There exist  $\delta > 0$  and  $\gamma > 0$  such that

$$\sum_{i=1}^n |\alpha_i|^2 (\varphi_i, \varphi_i)_{X,t} \leq \gamma \langle M_t \alpha, \alpha \rangle \quad \text{for all } \alpha = (\alpha_1, \dots, \alpha_n)^{\mathbf{t}} \in \mathbb{C}^n, \quad t \in (\delta, \infty).$$

**Theorem 2.7.** *Assume that the conditions (A.1) and (A.2) hold. Then*

$$(2.22) \quad [Y, X]_{s_0,1} \subset [Y, X_{\mathcal{K}}]_{s_0,\infty},$$

and

$$(2.23) \quad [Y, X_{\mathcal{K}}]_{s,q} = [Y, X]_{s,q}, \quad 0 < s < s_0, \quad q \in [1, \infty].$$

*Proof.* Due to (A.1), for a fixed  $u$  in  $[Y, X]_{s_0,1}$ , we have that  $\|u\|_{[Y, X_{\varphi_i}]_{s_i, \infty}} < \infty$ . Using the notation of the previous theorem, we get

$$\|u\|_{[Y, X_{\mathcal{K}}]_{s_0, \infty}}^2 = \sup_{t>0} (t^{2s_0} (w_{\mathcal{K}}, u)_Y).$$

On the other hand,

$$\begin{aligned} (w_{\mathcal{K}}, u)_Y &= ((I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u)_Y = (u, u)_Y - t^2 (S_{\mathcal{K}}^{-1} (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u)_Y \\ &\leq (u, u)_Y \leq c \|u\|_{[Y, X]_{s_0,1}}^2 \end{aligned}$$

Thus, from (6.2) and (2.7), we obtain

$$\begin{aligned} \|u\|_{[Y, X_{\mathcal{K}}]_{s_0, \infty}}^2 &\leq \sup_{0 < t \leq \delta} (t^{2s_0} (w_{\mathcal{K}}, u)_Y) + \sup_{t > \delta} (t^{2s_0} (w_{\mathcal{K}}, u)_Y) \\ &\leq c(\delta) \|u\|_{[Y, X]_{s_0,1}}^2 + \sup_{t > \delta} (t^{2s_0} (u, u)_{Y,t}) + \sup_{t > \delta} (t^{2+2s_0} \langle M_t^{-1} y_t, y_t \rangle) \\ &\leq c \|u\|_{[Y, X]_{s_0,1}}^2 + \sup_{t > \delta} (t^{2+2s_0} \langle M_t^{-1} y_t, y_t \rangle). \end{aligned}$$

Next, (A.2) is equivalent to

$$\langle M_t^{-1} \alpha, \alpha \rangle \leq \gamma \sum_{i=1}^n |\alpha_i|^2 (\varphi_i, \varphi_i)_{X,t}^{-1} \quad \text{for all } \alpha = (\alpha_1, \dots, \alpha_n)^t \in \mathbb{C}^n, \quad t \in (\delta, \infty).$$

In particular, for  $\alpha_i = (y_t)_i = (u, \varphi_i)_{Y,t}$ ,  $i = 1, \dots, n$ , we obtain

$$\langle M_t^{-1} y_t, y_t \rangle \leq \gamma \sum_{i=1}^n \frac{|(u, \varphi_i)_{Y,t}|^2}{(\varphi_i, \varphi_i)_{X,t}} \quad \text{for all } t \in (\delta, \infty), u \in [Y, X]_{s_0,1}.$$

Consequently, using (2.21) and (A.1), we have

$$\begin{aligned} \|u\|_{[Y, X_{\mathcal{K}}]_{s_0, \infty}}^2 &\leq c \|u\|_{[Y, X]_{s_0,1}}^2 + \gamma \sup_{t > \delta} \left( t^{2+2s_0} \sum_{i=1}^n \frac{|(u, \varphi_i)_{Y,t}|^2}{(\varphi_i, \varphi_i)_{X,t}} \right) \\ &\leq c \|u\|_{[Y, X]_{s_0,1}}^2 + \gamma \sum_{i=1}^n \|u\|_{[Y, X_{\varphi_i}]_{s_0, \infty}}^2 \leq c(n, \delta, \gamma) \|u\|_{[Y, X]_{s_0,1}}^2. \end{aligned}$$

This concludes (2.22). The identity (2.23) follows from (2.22) and Lemma 2.5.  $\square$

### 3. INTERPOLATION BETWEEN SUBSPACES OF $H^\beta(\mathbb{R}^N)$ AND $H^\alpha(\mathbb{R}^N)$ .

In this section, we apply our abstract interpolation result to the particular case  $Y = H^\alpha(\mathbb{R}^N)$ ,  $X = H^\beta(\mathbb{R}^N)$ . The codimension one case with  $q = 2$  was analyzed for the first time by Kellogg in [19]. A finite codimension case with  $q = 2$  is considered in [4]. Using Theorem 2.7, we prove a new subspace interpolation embedding and recover the main interpolation results of [19] and [4]. The proofs are based on techniques used in [19] and [4], but are considerably simplified. For the proof of the regularity for the biharmonic problem we will need only the last theorem of this

section. The reader not interested in the proof of the embedding tools might skip this section.

Let  $\alpha \in \mathbb{R}$  and let  $H^\alpha(\mathbb{R}^N)$  be defined by means of the Fourier transform. For a smooth function  $u$  with compact support in  $\mathbb{R}^N$ , the Fourier transform  $\hat{u}$  is defined by

$$\hat{u}(\xi) = (2\pi)^{-N/2} \int u(x) e^{-ix \cdot \xi} dx,$$

where the integral is taken over the whole  $\mathbb{R}^N$ . For  $u$  and  $v$  smooth functions the  $\alpha$ -inner product is defined by

$$\langle u, v \rangle_\alpha = \int (1 + |\xi|^2)^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.$$

The space  $H^\alpha(\mathbb{R}^N)$  is the closure of smooth functions with compact support in the norm induced by the  $\alpha$ -inner product. For  $\alpha, \beta$  real numbers,  $\alpha < \beta$ , and  $s \in [0, 1]$  it is easy to check, using (2.6), that

$$[H^\alpha(\mathbb{R}^N), H^\beta(\mathbb{R}^N)]_{s,2} = H^{(1-s)\alpha + s\beta}(\mathbb{R}^N).$$

For  $\varphi \in H^\beta(\mathbb{R}^N)$ , we are interested in determining the embedding relation between  $[H^\alpha(\mathbb{R}^N), H_\varphi^\beta(\mathbb{R}^N)]_{s,q_0}$  and  $[H^\alpha(\mathbb{R}^N), H^\beta(\mathbb{R}^N)]_{s,q_1}$ . We recall that  $H_\varphi^\beta$  is the orthogonal complement of  $\text{span}\{\varphi\}$  in  $H^\beta$ . For simplicity we shall start with the case  $\alpha = 0$ . We note that  $H^0(\mathbb{R}^N) = L^2(\mathbb{R}^N)$ . The operator  $S$  associated with the pair  $(X, Y) = (H^\beta(\mathbb{R}^N), H^0(\mathbb{R}^N))$  is given by

$$\widehat{S}u = \mu^{2\beta} \hat{u}, \quad u \in D(S) = H^{2\beta}(\mathbb{R}^N),$$

where  $\mu(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$ ,  $\xi \in \mathbb{R}^N$ . For the remaining part of this chapter,  $H^\beta$  denotes the space  $H^\beta(\mathbb{R}^N)$  and  $\hat{H}^\beta$  is the space  $\{\hat{u} \mid u \in H^\beta\}$ . For  $\hat{u}, \hat{v} \in \hat{H}^\beta$ , we define the inner product and the norm by

$$(\hat{u}, \hat{v})_\beta = \int \mu^{2\beta} \hat{u} \overline{\hat{v}} d\zeta, \quad \|\hat{u}\|_\beta = (\hat{u}, \hat{u})_\beta^{1/2}.$$

To simplify the notation, we denote the the inner products  $(\cdot, \cdot)_0$  and  $\langle \cdot, \cdot \rangle_0$  by  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$ , respectively. The norm  $\|\cdot\|_0$  on  $H^0$  or  $\hat{H}^0$  is simply  $\|\cdot\|$ . Thus, for  $X = \hat{H}^\beta$  and  $Y = \hat{H}^0$ , we have

$$(\hat{u}, \phi)_{Y,t} = \left( \frac{\mu^{2\beta} \hat{u}}{\mu^{2\beta} + t^2}, \phi \right) \quad \text{and} \quad (\hat{u}, \phi)_{X,t} = \left( \frac{\mu^{4\beta} \hat{u}}{\mu^{2\beta} + t^2}, \phi \right).$$

Let  $\phi \in \hat{H}^\beta$  be such that for some constants  $\epsilon > 0$  and  $c > 0$ ,

$$(3.1) \quad \begin{cases} |\phi(\xi) - b(\omega) \rho^{-\frac{N}{2} - 2\beta + \alpha_0}| < c \rho^{-\frac{N}{2} - 2\beta + \alpha_0 - \epsilon} & \text{for all } \rho > 1, \\ 0 < \alpha_0 < \beta, \end{cases}$$

where  $\rho \geq 0$  and  $\omega \in S^{N-1}$  (the unit sphere of  $\mathbb{R}^N$ ) are the spherical coordinates of  $\xi \in \mathbb{R}^N$ , and where  $b(\omega)$  is a bounded measurable function on  $S^{N-1}$ , which is non zero on a set of positive measure.

**Remark 3.1.** *By using Lemma 2.4, under the assumption (3.1) about  $\phi$ , we have that*

$$(3.2) \quad \hat{H}_\phi^\beta \text{ is dense in } \hat{H}^{\alpha_1} \text{ if and only if } \alpha_1 \leq \alpha_0.$$

*Thus, in particular, we have that  $\hat{H}_\phi^\beta$  is dense in  $\hat{H}^0$  and the pair  $(\hat{H}_\phi^\beta, \hat{H}^0)$  satisfies (2.1).*



**Theorem 3.2.** *Let  $\varphi \in H^\beta$  be such that its Fourier transform  $\phi$  satisfies (3.1), and let  $\theta_0 = \alpha_0/\beta$ . Then*

$$(3.3) \quad [H^0, H^\beta]_{\theta_0,1} \subset [H^0, H_\varphi^\beta]_{\theta_0,\infty},$$

and

$$(3.4) \quad [H^0, H_\varphi^\beta]_{\theta,q} = [H^0, H^\beta]_{\theta,q}, \quad 0 < \theta < \theta_0, \quad q \in [1, \infty].$$

*Proof.* By the definition of the inner product on  $H^\beta$ , we have that the embedding (3.3) is equivalent to

$$(3.5) \quad [\hat{H}^0, \hat{H}^\beta]_{\theta_0,1} \subset [\hat{H}^0, \hat{H}_\phi^\beta]_{\theta_0,\infty}.$$

The identity (3.4) follows from (3.3) and Lemma 2.5. Thus, we have to prove only (3.5). From (2.6),

$$(3.6) \quad \|\hat{u}\|_{[\hat{H}^0, \hat{H}_\phi^\beta]_{\theta_0,1}} = \int_0^\infty t^{\theta_0} \left( \int \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta} + t^2} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \frac{dt}{t},$$

and due to (2.21) we get

$$(3.7) \quad \|\hat{u}\|_{[\hat{H}^0, \hat{H}_\phi^\beta]_{\theta_0,\infty}}^2 = \sup_{t>0} \left( t^{2\theta_0} (\hat{u}, \hat{u})_{Y,t} + t^{2+2\theta_0} \frac{|(\hat{u}, \phi)_{Y,t}|^2}{(\phi, \phi)_{X,t}} \right),$$

where  $X = \hat{H}^\beta$  and  $Y = H^0$ . Using (3.1), it is easy to see that, for a large enough  $\delta \geq 1$ , one can find positive constants  $c_1, c_2$ , such that

$$(3.8) \quad c_1 t^{\theta_0-1} \leq ((\phi, \phi)_{X,t})^{1/2} = \left( \frac{\mu^{4\beta} \phi}{\mu^{2\beta} + t^2}, \phi \right)^{1/2} \leq c_2 t^{\theta_0-1}, \quad t \geq \delta,$$

$$|\phi(\xi)| < c_2 |\rho|^{-\frac{N}{2} - 2\beta + \alpha_0} \quad \text{for } |\xi| > 1.$$

Following the proof of Theorem 2.7, we see that, in order to prove (3.5), it is enough to verify the following inequality

$$(3.9) \quad M := \sup_{t>\delta} t^{1+\theta_0} \frac{|(\hat{u}, \phi)_{Y,t}|}{((\phi, \phi)_{X,t})^{1/2}} \leq c \|\hat{u}\|_{[\hat{H}^0, \hat{H}_\phi^\beta]_{\theta_0,1}} \quad \text{for all } \hat{u} \in [\hat{H}^0, \hat{H}_\phi^\beta]_{\theta_0,1},$$

for some positive constants  $c = c(\delta)$ . From (3.8), we get

$$(3.10) \quad \begin{aligned} M &\leq c \sup_{t>\delta} t^2 |(\hat{u}, \phi)_{Y,t}| \leq c \sup_{t>\delta} t^2 \int \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta} + t^2} |\hat{u}(\xi)| |\phi(\xi)| d\xi \\ &\leq \int \mu(\xi)^{2\beta} |\hat{u}(\xi)| |\phi(\xi)| d\xi = \int_{|\xi|<1} \mu(\xi)^{2\beta} |\hat{u}||\phi| d\xi + \int_{|\xi|>1} \mu(\xi)^{2\beta} |\hat{u}||\phi| d\xi. \end{aligned}$$

The last two integrals can be estimated as follows:

$$(3.11) \quad \int_{|\xi|<1} \mu(\xi)^{2\beta} |\hat{u}||\phi| d\xi \leq c \|\hat{u}\| \|\phi\| \leq c(\phi) \|\hat{u}\|_{[\hat{H}^0, \hat{H}_\phi^\beta]_{\theta_0,1}},$$

and, using again (3.8), we have

$$\begin{aligned}
(3.12) \quad \int_{|\xi|>1} \mu(\xi)^{2\beta} |\hat{u}||\phi| \, d\xi &= \int_{|\xi|>1} \frac{2}{\pi} \int_0^\infty \frac{\mu(\xi)^{3\beta}}{\mu(\xi)^{2\beta} + t^2} |\hat{u}(\xi)||\phi(\xi)| \, dt \, d\xi \\
&= \frac{2}{\pi} \int_0^\infty \int_{|\xi|>1} \frac{\mu^{3\beta}}{\mu^{2\beta} + t^2} |\hat{u}||\phi| \, d\xi \, dt \\
&= \frac{2}{\pi} \int_0^\infty \int_{|\xi|>1} \frac{\mu^\beta |\hat{u}|}{(\mu^{2\beta} + t^2)^{1/2}} \frac{\mu^{2\beta} |\phi|}{(\mu^{2\beta} + t^2)^{1/2}} \, d\xi \, dt \\
&\leq \frac{2}{\pi} \int_0^\infty \left( \int_{|\xi|>1} \frac{\mu^{2\beta} |\hat{u}|^2}{\mu^{2\beta} + t^2} \, d\xi \right)^{1/2} \left( \int_{|\xi|>1} \frac{\mu^{4\beta} |\phi|^2}{\mu^{2\beta} + t^2} \, d\xi \right)^{1/2} \, dt \\
&\leq \frac{2}{\pi} \int_0^\infty \left( \int_{|\xi|>1} \frac{\mu^{2\beta} |\hat{u}|^2}{\mu^{2\beta} + t^2} \, d\xi \right)^{1/2} ((\phi, \phi)_{X,t})^{1/2} \, dt \\
&\leq c \int_0^\infty t^{\theta_0} \left( \int_{|\xi|>1} \frac{\mu^{2\beta} |\hat{u}|^2}{\mu^{2\beta} + t^2} \, d\xi \right)^{1/2} \frac{dt}{t} \leq c \|\hat{u}\|_{[\hat{H}^0, \hat{H}^\beta]_{\theta_0, 1}}.
\end{aligned}$$

Combining (3.10)-(3.12), we conclude the validity of (3.9) and the proof of the theorem.  $\square$

Next, we prepare for the generalization of the previous result. Let  $\phi_1, \phi_2, \dots, \phi_n \in \hat{H}^\beta(\mathbb{R}^N)$  be such that for some constants  $\epsilon > 0$  and  $c > 0$ , we have

$$(3.13) \quad \begin{cases} |\phi_i(\xi) - \tilde{\phi}_i(\xi)| < c\rho^{-\frac{N}{2}-2\beta+\alpha_i-\epsilon} \text{ for } |\xi| > 1 \\ 0 < \alpha_i < \beta, \quad i = 1, \dots, n, \end{cases}$$

where

$$\tilde{\phi}_i(\xi) = b_i(\omega)\rho^{-\frac{N}{2}-2\beta+\alpha_i}, \quad \xi = (\rho, \omega),$$

and  $b_i(\cdot)$  is a bounded measurable function on  $S^{N-1}$ , which is non zero on a set of positive measure. Using Lemma 2.4 and the Riesz representation theorem, it is easy to check that, under the assumption (3.1), the space  $H_{\mathcal{K}}^\beta$  is dense in  $H^0$ .

**Theorem 3.3.** *Let  $\varphi_1, \varphi_2, \dots, \varphi_n$  be functions in  $H^\beta$  such that the corresponding Fourier transforms  $\phi_1, \phi_2, \dots, \phi_n$  are well defined and satisfy (3.13). Assume that the functions  $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$  defined in (3.13) are linearly independent. Let  $\mathcal{K} = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ ,  $\alpha_0 := \min\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ , and let  $\theta_0 = \alpha_0/\beta$ . Then,*

$$(3.14) \quad [H^0, H^\beta]_{\theta_0, 1} \subset [H^0, H_{\mathcal{K}}^\beta]_{\theta_0, \infty},$$

and

$$(3.15) \quad [H^0, H_{\mathcal{K}}^\beta]_{\theta, q} = [H^0, H^\beta]_{\theta, q}, \quad 0 < \theta < \theta_0, \quad q \in [1, \infty].$$

*Proof.* We apply the Theorem 2.7 for  $X = H^\beta$ ,  $Y = H^0$ ,  $\mathcal{K} = \text{span}\{\varphi_1, \dots, \varphi_n\}$  and  $s_0 = \theta_0$ . By using the hypothesis (3.13) and Theorem 3.2, we get

$$[H^0, H^\beta]_{\theta_0, 1} \subset [H^0, H_{\varphi_i}^\beta]_{\theta_0, \infty}, \text{ for } i = 1, 2, \dots, n.$$

Thus, **(A1)** is satisfied. The proof of **(A2)** follows from (3.13) by using elementary linear algebra and calculus tools, and it is presented in [4]. The identity (3.15), is a direct consequence of Lemma 2.5. The proof is completed.  $\square$

The interpolation problem between  $H^\alpha$  and a subspace of  $H^\beta$  of finite codimension with  $\alpha < \beta$  arbitrary real numbers, can be approached in the light of the previous theorem. Let  $\varphi_1, \varphi_2, \dots, \varphi_n \in H^\beta$  and let  $\alpha < \beta$  be such that the corresponding Fourier transform,  $\phi_1, \phi_2, \dots, \phi_n$  are well defined and satisfy for some positive constants  $c$  and  $\epsilon$ ,

$$(3.16) \quad \begin{cases} |\phi_i(\xi) - \tilde{\phi}_i(\xi)| < c\rho^{-\frac{N}{2}-2\beta+\gamma_i-\epsilon} \text{ for } |\xi| > 1 \\ \alpha < \gamma_i < \beta, \quad i = 1, \dots, n, \end{cases}$$

where

$$\tilde{\phi}_i(\xi) = b_i(\omega)\rho^{-\frac{N}{2}-2\beta+\gamma_i}, \quad \xi = (\rho, \omega),$$

and  $b_i(\cdot)$  is a bounded measurable function on  $S^{N-1}$ , which is non zero on a set of positive measure.

**Theorem 3.4.** *Let  $\varphi_1, \varphi_2, \dots, \varphi_n \in H^\beta$  be such that the corresponding Fourier transforms  $\phi_1, \phi_2, \dots, \phi_n$  are well defined and satisfy (3.16). Assume that the functions  $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$  are linearly independent. Let  $\mathcal{L} = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ ,  $\gamma_0 := \min\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ , and let  $\theta_0 = (\gamma_0 - \alpha)/(\beta - \alpha)$ . Then,*

$$(3.17) \quad [H^\alpha, H^\beta]_{\theta_0, 1} \subset [H^\alpha, H_{\mathcal{L}}^\beta]_{\theta_0, \infty},$$

and

$$(3.18) \quad [H^\alpha, H_{\mathcal{L}}^\beta]_{\theta, q} = [H^\alpha, H^\beta]_{\theta, q}, \quad 0 < \theta < \theta_0, \quad q \in [1, \infty].$$

*Proof.* The proof is a direct consequence of Theorem 3.3 and the fact that  $T : H^\alpha \rightarrow H^0$  defined by  $\hat{T}u = \mu^\alpha \hat{u}$ ,  $u \in H^\alpha$ , is an isometry from  $H^\alpha$  to  $H^0$  and from  $H^\beta$  to  $H^{\beta-\alpha}$ .  $\square$

#### 4. SHIFT ESTIMATES FOR THE BIHARMONIC OPERATOR ON POLYGONAL DOMAINS

Let  $\Omega$  be a polygonal domain in  $R^2$  with boundary  $\partial\Omega$ . Let  $\partial\Omega$  be the polygonal curve  $P_1P_2 \cdots P_mP_1$ . At each point  $P_j$ , we denote by  $\omega_j$  the measure of the angle at  $P_j$  measured from inside  $\Omega$ . Let  $\omega := \max\{\omega_j : j = 1, 2, \dots, m\}$ . We consider the biharmonic problem: Given  $f \in L^2(\Omega)$ , find  $u$  such that (1.1). Let  $V = H_0^2(\Omega)$  and

$$a(u, v) := \sum_{1 \leq i, j \leq 2} \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx, \quad u, v, \in V.$$

The bilinear form  $a(\cdot, \cdot)$  defines a scalar product on  $V$  and the induced norm is equivalent to the standard norm on  $H_0^2(\Omega)$ . The variational form of (1.1) is : Find  $u \in V$  such that

$$(4.1) \quad a(u, v) = \int_{\Omega} f v dx \quad \text{for all } v \in V.$$

Let  $S_\omega$  denote a sector domain defined by

$$S_\omega = \{(r, \theta), 0 < r < r_\omega, -\omega/2 < \theta < \omega/2\},$$

where, without loss of generality, we assume that  $S_\omega \subset \Omega$  for a sufficiently small  $r_\omega$ , and that the vertex corresponding to the largest angle of  $\Omega$  is the vertex of the sector domain  $S_\omega$ . We associate to (1.1), with  $\Omega = S_\omega$ , the following characteristic equation

$$(4.2) \quad \sin^2(z\omega) = z^2 \sin^2 \omega.$$

In order to simplify the exposition of the proof, we assume that

$$(4.3) \quad \sin \sqrt{\frac{\omega^2}{\sin \omega^2} - 1} \neq \pm \sqrt{1 - \frac{\sin \omega^2}{\omega^2}}$$

and

$$(4.4) \quad \operatorname{Re} z \neq 2 \text{ for any solution } z \text{ of (4.2).}$$

The restriction (4.3) assures that the equation (4.2) has only simple roots. Let  $z_1, z_2, \dots, z_n$  be all the roots of (4.2) such that  $0 < \operatorname{Re}(z_j) < 2$ . It is known that, under the assumptions (4.3) and (4.4), the solution  $u$  of (4.1) can be written as

$$(4.5) \quad u = u_R + \sum_{j=1}^n k_j S_j,$$

where  $u_R \in H^4(S_\omega)$  is the regular part of  $u$ , the  $S_j$ 's are called the singular functions, and the  $k_j$ 's are the coefficients of the singular functions. More precisely, for  $j = 1, 2, \dots, n$ , we have  $S_j(r, \theta) = r^{1+z_j} u_j(\theta)$ , with  $u_j$  smooth function on  $[-\omega/2, \omega/2]$  satisfying  $u_j(-\omega/2) = u_j(\omega/2) = u_j'(-\omega/2) = u_j'(\omega/2) = 0$ , and  $k_j = c_j \int_{S_\omega} f \varphi_j dx$ , with  $c_j$  being nonzero and depending only on  $\omega$ . The function  $\varphi_j$  is called the dual singular function of the singular function  $S_j$  and  $\varphi_j(r, \theta) = \eta(r) r^{1-z_j} u_j(\theta) - w_j$ , where  $w_j \in V$  is defined for a smooth truncation function  $\eta$  to be the solution of (4.1) with  $f = \Delta^2(\eta(r) r^{1-z_j} u_j(\theta))$ . In addition, we have that the regular part  $u_R$  satisfies

$$(4.6) \quad \|u_R\|_{H^4(S_\omega)} \leq c \|f\|, \quad \text{for all } f \in L^2(S_\omega).$$

For the proof of the expansion (4.5), see, e.g., [14, 16, 17, 21, 25]. Next, we define  $\mathcal{K} = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$  and

$$(4.7) \quad s_0 := \frac{1}{2} \min\{\operatorname{Re}(z_j) \mid j = 1, 2, \dots, n\}.$$

The main theorem of the paper is based on the following embedding result:

**Lemma 4.1.** *For  $\Omega = S_\omega$  and  $\mathcal{K}$  defined above, we have*

$$(4.8) \quad [H^{-2}(\Omega), L^2(\Omega)]_{s_0, 1} \subset [H^{-2}(\Omega), L^2(\Omega)]_{\mathcal{K}, s_0, \infty}.$$

To prove it, we first reduce the problem to the case  $\Omega = \mathbb{R}^2$  and then apply the interpolation results of the previous section. A detailed proof is given in Appendix 6.1. Now, we are ready to state the main result of the paper.

**Theorem 4.2.** *Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$ . Assume that all the angles  $\omega_j$  of  $\partial\Omega$  satisfy (4.3) and (4.4). Let  $s_0 = s_0(\omega)$  be the threshold defined by (4.7), where  $\omega$  is the largest inner angle of the polygon  $\partial\Omega$ . If  $u$  is the variational solution of (1.1), then the regularity estimate (1.3) holds. Consequently, the classical estimate (1.2) holds also.*

*Proof.* We consider a covering of  $\Omega$  with  $m + 1$  subdomains. For each angle of  $\partial\Omega$ , we associate a sector domain  $S_{\omega_j} \subset \Omega$  and complete the covering of  $\Omega$  with a domain  $\Omega_0 \subset \Omega$  such that  $\Omega_0$  has smooth boundary and  $\partial\Omega_0$  does not contain any vertex of  $\partial\Omega$ . As done in [3], by using a smooth partition of unity subordinated to the described covering of  $\Omega$ , the problem of deriving a shift estimate on  $\Omega$  can be reduced to deriving similar estimates on each of the subdomains of the covering. Since the amount of regularity of the solution of (1.1) for a sector domain decreases as the angle of the a sector domain increases, (see Figure 1), to prove (1.3), it will be enough to prove the estimate for  $\Omega = S_\omega$  and for  $\Omega = \Omega_0$ . First, let us consider the case  $\Omega = \Omega_0$ . Then, it is known that the solution  $u$  of (4.1) satisfies

$$\|u\|_{H^4(\Omega)} \leq c\|f\|, \quad \text{for all } f \in L^2(\Omega),$$

and

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{H^{-2}(\Omega)}, \quad \text{for all } f \in H^{-2}(\Omega).$$

Interpolating these two inequalities with  $0 < s < 1$  and  $1 \leq q \leq \infty$ , we obtain

$$(4.9) \quad \|u\|_{[H^2(\Omega), H^4(\Omega)]_{s,q}} \leq c\|f\|_{[H^{-2}(\Omega), L^2(\Omega)]_{s,q}}, \quad f \in [H^{-2}(\Omega), L^2(\Omega)]_{s,q}.$$

In particular, taking  $s = s_0$  and  $q = \infty$  we get

$$(4.10) \quad \|u\|_{[H^2(\Omega), H^4(\Omega)]_{s_0,\infty}} \leq c\|f\|_{[H^{-2}(\Omega), L^2(\Omega)]_{s_0,\infty}}, \quad f \in [H^{-2}(\Omega), L^2(\Omega)]_{s_0,\infty}.$$

Using the standard embedding result

$$[H^{-2}(\Omega), L^2(\Omega)]_{s_0,1} \subset [H^{-2}(\Omega), L^2(\Omega)]_{s_0,\infty},$$

and (4.10), we get

$$(4.11) \quad \|u\|_{[H^2(\Omega), H^4(\Omega)]_{s_0,\infty}} \leq c\|f\|_{[H^{-2}(\Omega), L^2(\Omega)]_{s_0,1}}, \quad f \in [H^{-2}(\Omega), L^2(\Omega)]_{s_0,1}.$$

Thus, by introducing the Besov spaces  $B_\infty^{2+s_0}(\Omega) := [H^2(\Omega), H^4(\Omega)]_{s_0,\infty}$  and  $B_1^{-2+s_0}(\Omega) := [H^{-2}(\Omega), L^2(\Omega)]_{s_0,1}$ , the estimate (4.11) becomes (1.3).

Let us consider now the case  $\Omega = S_\omega$ . From the expansion (4.5) and the estimate (4.6), we have

$$(4.12) \quad \|u\|_{H^4(\Omega)} \leq c\|f\|, \quad \text{for all } f \in L^2(\Omega)_\mathcal{K}.$$

Combining (4.12) with the standard estimate

$$(4.13) \quad \|u\|_{H^2(\Omega)} \leq c\|f\|_{H^{-2}(\Omega)}, \quad \text{for all } f \in H^{-2}(\Omega),$$

by interpolation, we obtain

$$(4.14) \quad \|u\|_{[H^2(\Omega), H^4(\Omega)]_{s_0,\infty}} \leq c\|f\|_{[H^{-2}(\Omega), L^2(\Omega)_\mathcal{K}]_{s_0,\infty}}.$$

From (4.14) and the embedding result (4.8) of Lemma 4.1, we conclude that (1.3) holds for  $\Omega = S_\omega$ . Here,  $s_0$  corresponds to the largest inner angle  $\omega$  of the polygon. The function  $\omega \rightarrow 2 + 2s_0(\omega)$  represents the regularity threshold for the biharmonic problem on  $S_\omega$ . From the plot of the function, given in Figure 1, we see that function decreases on the interval  $(0, 2\pi)$ . Thus, by involving interpolation and the reiteration theorem, (see part iii) and part iv) of Proposition 6.2), the estimate (1.3) holds for all sector domains  $S_{\omega_j}$ ,  $j = 1, 2, \dots, m$ .

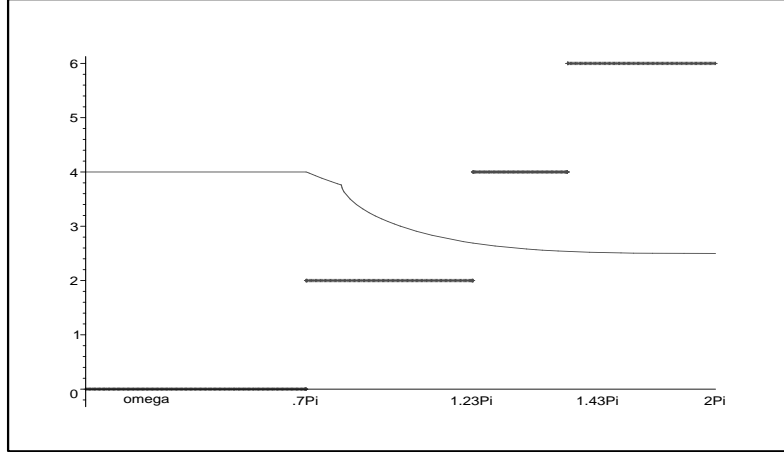


Figure1: Regularity for the biharmonic problem

Therefore, we have proved (1.3) for any polygonal domain with not self intersecting boundary. Next, we prove the standard estimate (1.2). Let  $T : H^{-2}(\Omega) \rightarrow H_0^2(\Omega)$  be defined by  $Tf = u$ , where  $u$  is the solution of (4.1). Then, (1.3) becomes

$$(4.15) \quad \|Tf\|_{[H^2, H^4]_{s_0, \infty}} \leq c \|f\|_{[H^{-2}, H^0]_{s_0, 1}},$$

and the standard estimate (4.13) can be written us

$$(4.16) \quad \|Tf\|_{[H^2, H^4]_{0, 2}} \leq c \|f\|_{[H^{-2}, H^0]_{0, 2}}.$$

Let  $0 < s < s_0$ . By Proposition 6.2 part iii) (“By interpolation” Theorem), applied with  $\lambda = \frac{s}{s_0}$  and  $q = 2$ , we obtain

$$(4.17) \quad \|Tf\|_{[[H^2, H^4]_{0, 2}, [H^2, H^4]_{s_0, \infty}]_{\lambda, 2}} \leq c \|f\|_{[[H^{-2}, H^0]_{0, 2}, [H^{-2}, H^0]_{s_0, 1}]_{\lambda, 2}}.$$

By applying the reiteration theorem, we recover the classical estimate (1.2). The proof is completed.  $\square$

**Remark 4.3.** Figure 1 gives the graph of the function  $\omega \rightarrow 2 + 2s_0(\omega)$  which represents also the regularity threshold for the biharmonic problem in terms of the largest inner angle of the polygon  $\omega \in (0, \pi) \cup (\pi, 2\pi)$ . From (4.2) and (4.7), it follows that  $s_0(\omega) \rightarrow 1/2$  for  $\omega \rightarrow \pi$  and  $s_0(\omega) \rightarrow 1/4$  for  $\omega \rightarrow 2\pi$ . On the same graph we represent the number of singular (dual singular) functions as a function of  $\omega \in (0, \pi) \cup (\pi, 2\pi)$ . Note that, if  $\omega_0 \in (0, 2\pi)$  is the solution of  $\tan \omega = \omega$  ( $\omega_0 \approx 1.43\pi$ ) and  $\omega > \omega_0$ , then, the the dimension of the space  $\mathcal{K}$  is six.

**Remark 4.4.** If the coefficient of the most singular function in the expansion (4.5) is not zero, then the solution  $u$  of (1.1) does not belong to  $B_q^{2+2s_0}(\Omega)$  for any  $q < \infty$ . Thus, any possible improvement of the shift estimate (1.3) would be in the form:

$$(4.18) \quad \|u\|_{B_\infty^{2+2s_0}(\Omega)} \leq c \|f\|_{B_q^{-2+2s_0}(\Omega)}, f \in B_q^{-2+2s_0}(\Omega), q > 1.$$

The validity of (4.18) remains an open problem.

5. REGULARITY FOR THE STOKES AND NAVIER-STOKES SYSTEMS ON CONVEX POLYGONAL DOMAINS

As a consequence of the regularity for the biharmonic problem we prove shift estimates for the Stokes and the Navier-Stokes systems. Let  $\Omega$  be a *convex* polygonal domain in  $R^2$  with boundary  $\partial\Omega$  and let  $\omega$  be the measure of the largest angle of  $\partial\Omega$ . In this case, we have that  $s_0 = s_0(\omega) \in (1/2, 1)$ , (see Remark 4.3). Let  $\gamma_0 := 2s_0 - 1$ . Note that  $\gamma_0 \in (0, 1)$ . Then, according to the estimate (1.2), the solution  $u$  of the biharmonic problem (1.1) satisfies

$$(5.1) \quad \|u\|_{H^{3+\gamma}} \leq c\|f\|_{H^{-1+\gamma}}, \quad \text{for all } f \in H^{-1+\gamma}(\Omega), \quad -1 \leq \gamma < \gamma_0.$$

The parameter  $\gamma_0$  can be viewed as the exact amount of *extra regularity* due to the *convexity* of the domain  $\Omega$ .

Next, we consider the steady-state Stokes problem in the velocity-pressure formulation. More precisely we consider the following problem:

Given  $\mathbf{F} \in (H^\gamma)^2$  with  $0 \leq \gamma < \gamma_0$ , find the vector-valued function  $\mathbf{u}$  and the scalar-valued function  $p$  satisfying

$$(5.2) \quad \begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{F} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} p = 0. \end{cases}$$

The first two equations are considered in the standard weak sense. According to a well known Kellogg-Osborn result, [20], we have that (5.2) has a unique solution  $(\mathbf{u}, p) \in ((H^2(\Omega))^2, H^1(\Omega))$ . Since  $\nabla \cdot \mathbf{u} = 0$  in  $\Omega$ , and  $\mathbf{u} = 0$  on  $\partial\Omega$ , one can find  $w \in H_0^2$ , (see for example I.3.1 in [15] or [2]), such that

$$\mathbf{u} = (u_1, u_2) = \mathbf{curl} w := \left( \frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right).$$

If we substitute  $\mathbf{u} = \mathbf{curl} w$  in (5.2), we get

$$(5.3) \quad \begin{cases} -\Delta \left( \frac{\partial w}{\partial x_2} \right) + \frac{\partial p}{\partial x_1} = f_1 & \text{in } \Omega, \\ \Delta \left( \frac{\partial w}{\partial x_1} \right) + \frac{\partial p}{\partial x_2} = f_2 & \text{in } \Omega, \end{cases}$$

where  $(f_1, f_2) = \mathbf{F}$ . Next, we apply the differential operators  $-\frac{\partial}{\partial x_2}$  and  $\frac{\partial}{\partial x_1}$  to the first and second equations of (5.3), respectively, and sum up the two new equations. Thus, we have that  $w \in H_0^2$  and

$$\Delta^2 w = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \quad \text{in } \Omega.$$

Consequently, for a fixed  $\gamma \in (0, \gamma_0)$ , from (5.1), we have that

$$\|w\|_{H^{3+\gamma}} \leq c \left\| \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right\|_{H^{-1+\gamma}},$$

where  $c$  is a constant independent of  $\mathbf{F}$ . It follows that

$$\|\mathbf{u}\|_{(H^{2+\gamma})^2} \leq c\|\mathbf{F}\|_{(H^\gamma)^2}, \quad \text{for all } \mathbf{F} \in (H^\gamma(\Omega))^2.$$

From the first part of (5.2), we have  $\nabla p = \Delta \mathbf{u} + \mathbf{F}$ . Hence,

$$\|\nabla p\|_{(H^\gamma)^2} \leq \|\Delta \mathbf{u}\|_{(H^\gamma)^2} + \|\mathbf{F}\|_{(H^\gamma)^2} \leq \|\mathbf{u}\|_{(H^{2+\gamma})^2} + \|\mathbf{F}\|_{(H^\gamma)^2} \leq c\|\mathbf{F}\|_{(H^\gamma)^2}.$$

In conclusion we obtain:

**Theorem 5.1.** *Let  $\Omega$  be a convex polygonal domain in  $\mathbb{R}^2$  with  $\omega$  the measure of the largest angle. Let  $\gamma_0 = 2s_0(\omega) - 1$  and let  $(\mathbf{u}, p)$  be the solution of (5.2). Then for any  $\gamma \in (0, \gamma_0)$ , there exist a constant  $c$  such that*

$$(5.4) \quad \|\mathbf{u}\|_{(H^{2+\gamma})^2} + \|p\|_{H^{1+\gamma}} \leq c\|\mathbf{F}\|_{(H^\gamma)^2} \quad \text{for all } \mathbf{F} \in (H^\gamma)^2.$$

We conclude this section by a similar regularity result for solutions of the Navier-Stokes equations

$$(5.5) \quad \begin{cases} -\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{G} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} p = 0. \end{cases}$$

**Theorem 5.2.** *Let  $\Omega$  be a convex polygonal domain in  $\mathbb{R}^2$  with  $\omega$  the measure of the largest angle. Let  $\gamma_0 = 2s_0(\omega) - 1$ ,  $\mathbf{G} \in (H^\gamma)^2$  with  $0 \leq \gamma < \gamma_0$  and let  $(\mathbf{u}, p) \in (H_0^1)^2 \times L^2$  be a solution of (5.5). Then,*

$$(5.6) \quad \mathbf{u} \in (H^{2+\gamma})^2 \quad \text{and } p \in H^{1+\gamma}.$$

*Proof.* The main idea of the proof is due to Temam. The proof given here was inspired by the proof presented by Kellogg and Osborn in [20]. Let  $(\mathbf{u}, p) \in (H_0^1)^2 \times L^2$  be a solution of (5.5) and denote  $\mathbf{F} = \mathbf{G} - (\mathbf{u} \cdot \nabla) \mathbf{u}$ . According to [20], we have that  $\mathbf{u} \in (H^{5/3})^2$ . In particular we get that  $\nabla \mathbf{u} \in (H^{2/3})^2$  and  $\mathbf{u}$  is bounded. Thus  $\mathbf{F} \in (H^{\min\{\gamma, 2/3\}})^2$ , and by using (5.4), we have that  $\mathbf{u} \in (H^{2+\min\{\gamma, 2/3\}})^2$  and  $\nabla \mathbf{u} \in (H^{1+\min\{\gamma, 2/3\}})^2$ . Then, we deduce that, in fact,  $\mathbf{F} \in (H^\gamma)^2$ . Using (5.4) again, we conclude that (5.6) holds.  $\square$

**Remark 5.3.** *In the light of Theorem 4.2, we can extend the last two results to hold for the threshold value  $\gamma = \gamma_0$  by considering the interpolation norms with the second index of interpolation  $q \neq 2$ . More precisely, we have to take  $q = 1$  for the norm of  $\mathbf{F}$  or  $\mathbf{G}$  and  $q = \infty$  for the norm of  $(\mathbf{u}, p)$ . In other words, the last two theorems hold if the spaces of type  $H^{k+\gamma}$  are replaced by  $B_q^{k+\gamma_0}$  with the above mentioned choice for  $q$ .*

## 6. APPENDIX

Here, we present the postponed proofs and some auxiliary results.

### 6.1. The Postponed Proofs.

*Proof of Theorem 2.6.* Let  $u$  be fixed in  $Y$  and set

$$(6.1) \quad w := (I + t^2 S^{-1})^{-1} u \quad \text{and} \quad w_\varphi := (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u.$$

Then, according to (2.12), (2.6) and (2.13), we have

$$\|u\|_{[Y, X_{\mathcal{K}}]_{s, 2}}^2 = \int_0^\infty t^{2s} (w_{\mathcal{K}}, u)_Y \frac{dt}{t}, \quad \|u\|_{[Y, X]_{s, 2}}^2 = \int_0^\infty t^{2s} (w, u)_Y \frac{dt}{t},$$

and

$$\|u\|_{[Y, X_{\mathcal{K}}]_{s, \infty}}^2 = \sup_{t>0} (t^{2s} (w_{\mathcal{K}}, u)_Y),$$

respectively. Thus, (2.18) and (2.19) would follow provided we establish

$$(6.2) \quad (w_{\mathcal{K}}, u)_Y = (w, u)_Y + t^2 \langle M_t^{-1} y_t, y_t \rangle.$$



From (6.1), we get that

$$(6.3) \quad (I + t^2 S_{\mathcal{K}}^{-1})w_{\mathcal{K}} = u.$$

Combining (2.15) and (6.3) we obtain

$$(I + t^2 S^{-1})w_{\mathcal{K}} = u + t^2 \sum_{k=1}^n \alpha_k \varphi_k,$$

where  $\alpha_k$  are constants determined by  $Q_{\mathcal{K}}(S^{-1}w_{\varphi}) = \sum_{k=1}^n \alpha_k \varphi_k$ . Equivalently, applying  $(I + t^2 S^{-1})^{-1}$  to both sides, we have

$$(6.4) \quad w_{\mathcal{K}} = w + t^2 \sum_{k=1}^n \alpha_k (I + t^2 S^{-1})^{-1} \varphi_k.$$

We calculate the coefficients  $\alpha_k$  by taking the  $(\cdot, \cdot)_X$  inner product with  $\varphi$  on both sides of (6.4), i.e.,

$$(w_{\mathcal{K}}, \varphi_j)_X = (w, \varphi_j)_X + t^2 \sum_{k=1}^n \alpha_k ((I + t^2 S^{-1})^{-1} \varphi_k, \varphi_j)_X = (w, \varphi_j)_X + t^2 \sum_{k=1}^n \alpha_k (\varphi_k, \varphi_j)_{X,t}.$$

From (6.3), since  $S_{\mathcal{K}}^{-1}w_{\varphi_k} \in X_{\mathcal{K}}$ , one sees that  $(w_{\mathcal{K}}, \varphi_j)_X = (u, \varphi_j)_X$ . With the notation adopted in (2.16) and (6.1) we obtain the following  $n \times n$  system:

$$\sum_{k=1}^n \alpha_k (\varphi_k, \varphi_j)_{X,t} t = t^{-2}((u, \varphi_j)_X - (u, \varphi_j)_{X,t}).$$

Using (2.4) and a simple manipulation of the operator  $S$  we get

$$(6.5) \quad t^2(u, \varphi_j)_{Y,t} = (u, \varphi_j)_X - (u, \varphi_j)_{X,t}.$$

Thus

$$\sum_{k=1}^n \alpha_k (\varphi_k, \varphi_j)_{X,t} = (u, \varphi_j)_{Y,t} = (y_t)_j.$$

Let  $\alpha \in \mathbb{C}^n$  be the the vector with components  $\alpha_1, \alpha_2, \dots, \alpha_n$ , then

$$\alpha = M_t^{-1} y_t.$$

Now, going back to (6.4), we get

$$(w_{\mathcal{K}}, u)_Y = (w, y) + t^2 \sum_{k=1}^n \alpha_k (\varphi_k, u)_{Y,t}.$$

By substituting the vector  $\alpha$  we obtain (6.2) and complete the proof.

Next, we present a general subspace interpolation lemma used for the proof of Theorem 4.2. Let  $(X, Y), (\tilde{X}, \tilde{Y})$  be two pairs of Hilbert spaces satisfying (2.1) and let  $(Y^*, X^*), (\tilde{Y}^*, \tilde{X}^*)$  be the corresponding dual pairs. Then, the dual pairs satisfy (2.1) also, see e.g., [18]. We further identify  $Y^*$  with  $Y$  and  $\tilde{Y}^*$  with  $\tilde{Y}$ , and assume that there are linear operators  $E$  and  $R$  such that

$$(6.6) \quad E : Y \rightarrow \tilde{Y}, \quad E : X \rightarrow \tilde{X} \quad \text{are bounded operators,}$$

$$(6.7) \quad R : \tilde{Y} \rightarrow Y, \quad R : \tilde{X} \rightarrow X, \quad \text{are bounded operators,}$$

$$(6.8) \quad REu = u \quad \text{for all } u \in Y.$$

Let  $\mathcal{K} \subset Y = Y^*$ ,  $\tilde{\mathcal{K}} = E(\mathcal{K}) \subset \tilde{Y} = \tilde{Y}^*$  be closed subspaces of  $Y$  and  $\tilde{Y}$ , respectively, and denote the corresponding orthogonal complements by  $Y_{\mathcal{K}}$  and  $\tilde{Y}_{\tilde{\mathcal{K}}}$ . Let  $\theta_0 \in (0, 1)$  be such that

$$(6.9) \quad [\tilde{X}^*, \tilde{Y}]_{\theta_0, 1} \subset [\tilde{X}^*, \tilde{Y}_{\tilde{\mathcal{K}}}]_{\theta_0, \infty}.$$

**Lemma 6.1.** *Assume that (6.6)-(6.9) are satisfied. Then,*

$$(6.10) \quad [X^*, Y]_{\theta_0, 1} \subset [X^*, Y_{\mathcal{K}}]_{\theta_0, \infty}$$

*Proof.* Using the duality, from (6.6)-(6.8), we obtain linear operators  $E^*$ ,  $R^*$  such that

$$(6.11) \quad E^* : \tilde{Y} \rightarrow Y, \quad E^* : \tilde{X}^* \rightarrow X^*, \quad \text{are bounded operators,}$$

$$(6.12) \quad R^* : Y \rightarrow \tilde{Y}, \quad R^* : X^* \rightarrow \tilde{X}^* \quad \text{are bounded operators,}$$

$$(6.13) \quad E^* R^* u = u \quad \text{for all } u \in Y,$$

$$(6.14) \quad E^* \text{ maps } \tilde{Y}_{\tilde{\mathcal{K}}} \text{ to } Y_{\mathcal{K}}.$$

From (6.11) and (6.14), by interpolation, we obtain

$$(6.15) \quad \|E^* v\|_{[X^*, Y_{\mathcal{K}}]_{\theta_0, \infty}} \leq c \|v\|_{[\tilde{X}^*, \tilde{Y}_{\tilde{\mathcal{K}}}]_{\theta_0, \infty}} \quad \text{for all } v \in \tilde{Y}.$$

For any  $u \in Y$ , we take  $v := R^* u$  in (6.15), and use (6.13) to get

$$(6.16) \quad \|u\|_{[X^*, Y_{\mathcal{K}}]_{\theta_0, \infty}} \leq c \|R^* u\|_{[\tilde{X}^*, \tilde{Y}_{\tilde{\mathcal{K}}}]_{\theta_0, \infty}} \quad \text{for all } u \in Y.$$

Also, from the hypothesis (6.9), we deduce that

$$(6.17) \quad \|R^* u\|_{[\tilde{X}^*, \tilde{Y}_{\tilde{\mathcal{K}}}]_{\theta_0, \infty}} \leq c \|R^* u\|_{[\tilde{X}^*, \tilde{Y}]_{\theta_0, 1}} \quad \text{for all } u \in Y.$$

From (6.12), again by interpolation, we have in particular

$$(6.18) \quad \|R^* u\|_{[\tilde{X}^*, \tilde{Y}]_{\theta_0, 1}} \leq c \|u\|_{[X^*, Y]_{\theta_0, 1}} \quad \text{for all } u \in Y.$$

Combining (6.16)-(6.18), it follows that

$$(6.19) \quad \|u\|_{[X^*, Y_{\mathcal{K}}]_{\theta_0, \infty}} \leq c \|u\|_{[X^*, Y]_{\theta_0, 1}} \quad \text{for all } u \in Y.$$

Since  $Y$  is dense in both  $[X^*, Y]_{\theta_0, 1}$  and  $[X^*, Y_{\mathcal{K}}]_{\theta_0, \infty}$  we obtain (6.10).  $\square$

*Proof of Lemma 4.1.* In [4], it was proven that there are bounded operators  $E$  and  $R$ ,

$$\begin{aligned} E : L^2(\Omega) &\longrightarrow L^2(R), \quad E : H_0^2(\Omega) \longrightarrow H^2(\mathbb{R}^2), \\ R : L^2(\mathbb{R}^2) &\longrightarrow L^2(\Omega), \quad R : H^2(\mathbb{R}^2) \longrightarrow H_0^2(\Omega) \end{aligned}$$

such that

$$REu = u, \quad \text{for all } u \in L^2(\Omega).$$

Next, let  $\phi_j$  be the Fourier transform of  $E\varphi_j$ ,  $j = 1, \dots, n$ . Using the asymptotic expansion of integrals estimates presented in [4], (see also [11, 19, 27]), we have that the functions  $\{\phi_j, j = 1, \dots, n\}$  satisfy for some positive constants  $c$  and  $\epsilon$ ,

$$(6.20) \quad \begin{cases} |\phi_j(\xi) - \tilde{\phi}_j(\xi)| < c\rho^{-1+(-2+s_j)-\epsilon} \text{ for } |\xi| > 1 \\ -2 < -2 + s_i < 0, \quad i = 1, \dots, n, \end{cases}$$

where  $s_j = \text{Re}(z_j)$  and

$$\tilde{\phi}_j(\xi) = b_i(\omega)\rho^{-1+(-2+s_j)}, \quad \xi = (\rho, \omega) \text{ in polar coordinates,}$$

and  $b_j(\cdot)$  is a bounded measurable function on the unit circle, which is non zero on a set of positive measure. Let  $s_0 = s_0(\omega)$  be defined by (4.7). Thus, we have that the functions  $\{E\varphi_j, j = 1, \dots, n\}$  satisfy the hypothesis (3.16) of Theorem 3.4 with  $N = 2$ ,  $\beta = 0$ ,  $\alpha = -2$  and  $\gamma_j = -2 + s_j$ ,  $j = 1, \dots, n$ . Let us define now  $\mathcal{L} := \text{span}\{E\varphi_j, j = 1, \dots, n\}$ . By Theorem 3.4 applied with  $\theta_0 = s_0$ , we have that

$$(6.21) \quad [H^{-2}(\mathbb{R}^2), L^2(\mathbb{R}^2)]_{s_0,1} \subset [H^{-2}(\mathbb{R}^2), L^2(\mathbb{R}^2)]_{s_0,\infty}.$$

Finally, using (6.21) and Lemma 6.1, we conclude that (4.8) holds.

**6.2. Appendix B. Interpolation Results.** Using the notation of Section 2, we review the classical interpolation results involved in this paper. The proofs can be found, for example, in [9, 10, 23].

**Proposition 6.2.** *Let  $(X, Y)$  be a pair of Banach spaces satisfying (2.1). The following hold.*

i) *The relative strength of the smoothness index  $s$ :*

$$[Y, X]_{s,q} \subset [Y, X]_{\tilde{s},\tilde{q}}, \quad 0 < \tilde{s} < s < 1, \quad \text{and } 1 \leq q, \tilde{q} \leq \infty.$$

ii) *The relative strength of the second index of interpolation:*

$$[Y, X]_{s,q} \subset [Y, X]_{s,\tilde{q}}, \quad 1 \leq q \leq \tilde{q} \leq \infty.$$

iii) *“By Interpolation” Theorem: Let  $(\tilde{X}, \tilde{Y})$  be another pair of Banach spaces satisfying (2.1). If  $T : Y \rightarrow \tilde{Y}$  and  $T : X \rightarrow \tilde{X}$  satisfy*

$$\|Tf\|_{\tilde{Y}} \leq c_0 \|f\|_Y, \quad \text{and } \|Tf\|_{\tilde{X}} \leq c_1 \|f\|_X,$$

*then,*

$$\|Tf\|_{[\tilde{Y}, \tilde{X}]_{\lambda,q}} \leq c_0^{1-\lambda} c_1^\lambda \|f\|_{[Y, X]_{\lambda,q}}, \quad 0 < \lambda < 1, \quad \text{and } 1 \leq q \leq \infty.$$

iv) *The Reiteration Theorem: For any  $0 \leq s_0 < s_1 \leq 1$ ,  $q_0, q_1, q \in [1, \infty]$ , and  $0 < \lambda < 1$ ,*

$$[[Y, X]_{s_0,q_0}, [Y, X]_{s_1,q_1}]_{\lambda,q} = [Y, X]_{(1-\lambda)s_0 + \lambda s_1, q}.$$

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