Multilevel discretization of Symmetric Saddle Point Systems without the discrete LBB Condition

Constantin Bacuta

University of Delaware, Department of Mathematics, 501 Ewing Hall 19716

Peter Monk

University of Delaware, Department of Mathematics, 501 Ewing Hall 19716

Abstract

Using an inexact Uzawa algorithm at the continuous level, we study the convergence of multilevel algorithms for solving saddle-point problems. The discrete stability Ladyshenskaya-Babušca-Brezzi (LBB) condition does not have to be satisfied. The algorithms are based on the existence of a multilevel sequence of nested approximation spaces for the constrained variable. The main idea is to maintain an accurate representation of the residual associated with the main equation at each step of the inexact Uzawa algorithm at the continuous level. The residual representation is approximated by a Galerkin projection. Whenever a sufficient condition for the accuracy of the representation fails to be satisfied, the representation of the residual is projected on the next (larger) space available in the prescribed multilevel sequence. Numerical results supporting the efficiency of the algorithms are presented for the Stokes equations and a $div - curl$ system.

Key words: inexact Uzawa algorithms, saddle point system, multilevel methods, adaptive methods

Email addresses: bacuta@math.udel.edu (Constantin Bacuta), monk@math.udel.edu (Peter Monk).

[1] The work of C. Bacuta was partially supported by NSF, DMS-0713125.

Dedication: This paper is dedicated to our friend and colleague Prof. George Hsiao on the occasion of his 75-th birthday.

1 Introduction

Currently in the literature, the finite element discretization of Saddle Point Problems (SPP) is mostly done on families of pairs of finite element spaces that satisfy a uniform stability or (LBB) condition with respect to a family parameter h. This classical theory is well developed, see e.g., [3,10,18,20,19,25,27,30,33]. In this paper, we try to depart from the classical theory by using an inexact Uzawa (IU) type algorithm, as introduced for the finite dimensional case in [15,23], and for the infinite dimensional case (or continuous level) in [5,9,22]. Convergence results for such algorithms at the continuous level, combined with standard techniques of discretization and error estimates, [28,32], lead to adaptive type algorithms for solving saddle point systems. The idea is present in [9,22,26], in the (local) adaptive context. In this paper we interpret the IU algorithm as a multilevel algorithm and avoid the use of a posteriori error estimators. In both cases, the main advantage of the resulting algorithms is that the discrete LBB condition is not needed and solvers are required only for simple elliptic problems.

The proposed algorithms are based on the presence of a multilevel structure. Our approach of approximating the solution of a PDE on a fine mesh, by performing significant work on coarser meshes is related to the multigrid and cascadic multilevel or multigrid approaches as presented e.g., in [11,12,17].

The convergence result for the Inexact Uzawa algorithm at the continuous level, presented in [5] suggests that efficient algorithms can be developed based on the control of the *relative error* for representing the residual at each step of the IU algorithm, for the main equation of the system. The principal result of [5] states that for any saddle point problem with a symmetric and positive definite form $a(\cdot,\cdot)$, the IU algorithm converges provided that the inexact process for representing the residual at each step has a relative error smaller than a computable threshold which, in fact, is independent on the SPP solved. This result together with the availability of a sequence of nested approximation subspaces for the constrained variable (which are to be efficiently chosen by the algorithm), are the building blocks for the new multilevel algorithms for solving SPPs. One advantage of such algorithms is that we can choose discrete multilevel sequences of approximation pairs which are not necessarily stable but have good approximation properties. In contrast to local adaptive methodes, no local error estimator or analysis is needed for the proposed algorithms.

The paper is organized as follows. In Section 2, we provide the notation for this paper, review the Schur complement properties and the main convergence result for the Uzawa algorithm at the continuous level. In Section 3 we introduce an abstract double inexact Uzawa algorithm and study its convergence.

In Section 4 we interpret the double inexact Uzawa algorithm as a multilevel algorithm, by replacing the approximate inverse for the first equation with a Galerkin projection on an appropriate finite element subspace, and by replacing the second Riesz representation operator with a projection on a discrete space. Numerical results for the Stokes system are presented in Section 5. An improved version of Babušca's Lemma together with applications to discretization for first order systems of differential equations are presented in Section 6. An appendix that discusses the feasibility of one of the assumption of Section 4 is presented as Section 8.

2 The Schur complement and the Uzawa algorithm

In this section, we introduce the notation and the standard spaces, operators and norms for the general abstract case of the saddle point theory we deal with in this paper. Using our notation, we also review the standard Uzawa algorithm at the continuous level.

We let $\mathbf{V}$ and Q be two Hilbert spaces with inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, with the corresponding induced norms $|\cdot|_{\mathbf{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$. The dual pairings on $\mathbf{V}^* \times \mathbf{V}$ and $Q^* \times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $\mathbf{V}^*$ and Q^* denote the duals of $\mathbf{V}$ and Q, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate operators $\mathcal{A} : V \to V^*$ and $\mathcal{C} : Q \to Q^*$ defined by

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle = a_0(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

and

$$\langle \mathcal{C}p, q\rangle = (p, q) \quad \text{for all } \ p, q \in Q.$$

The operators $\mathcal{A}^{-1} : V^* \to V$ and $\mathcal{C}^{-1} : Q^* \to Q$ are called the Riesz-canonical isometries and satisfy

$$a_0(\mathcal{A}^{-1}\mathbf{u}^*, \mathbf{v}) = \langle \mathbf{u}^*, \mathbf{v}\rangle, \quad |\mathcal{A}^{-1}\mathbf{u}^*|_{\mathbf{V}} = \|\mathbf{u}^*\|_{\mathbf{V}^*}, \mathbf{u}^* \in \mathbf{V}^*, \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

$$(\mathcal{C}^{-1}p^*, q) = \langle p^*, q\rangle, \quad \|\mathcal{C}^{-1}p^*\| = \|p^*\|_{Q^*}, \ p^* \in Q^*, q \in Q. \tag{2.2}$$

Next, we suppose that $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, satisfying the inf-sup condition. More precisely, we assume that

$$\inf_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| \ |\mathbf{v}|} = m > 0 \tag{2.3}$$

and

$$\sup_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| \ |\mathbf{v}|} = M < \infty. \tag{2.4}$$

Here, and throughout this paper, the "inf" and "sup" are taken over nonzero vectors. With the form b, we associate the linear operators $B : V \to Q^*$ and $B^* : Q \to V^*$ defined by

$$\langle B\mathbf{v}, q\rangle = b(\mathbf{v}, q) = \langle B^* q, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V},\ q \in Q.$$

Let $\mathbf{V}_0$ be the kernel of B or $\mathcal{C}^{-1}B$, i.e.,

$$\mathbf{V}_0 = Ker(B) = \{\mathbf{v} \in \mathbf{V} |\ B\mathbf{v} = 0\} = \{\mathbf{v} \in \mathbf{V} |\ \mathcal{C}^{-1}B\mathbf{v} = 0\}.$$

Due to (2.4), $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$. We also denote by $\mathbf{V}_1$ the orthogonal complement of $\mathbf{V}_0$ with respect to the $a_0(\cdot,\cdot)$ inner product. The Schur complement on Q is the operator $S_0 := \mathcal{C}^{-1}B\mathcal{A}^{-1}B^* : Q \to Q$. The operator S_0 is symmetric and positive definite on Q, satisfying

$$(S_0 p, p) = \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)^2}{|\mathbf{v}|^2}. \tag{2.5}$$

Consequently, $m^2, M^2 \in \sigma(S_0)$,

$$\sigma(S_0) \subset [m^2, M^2], \tag{2.6}$$

and

$$M\|p\| \geq \|p\|_{S_0} := (S_0 p, p)^{1/2} = |\mathcal{A}^{-1}B^* p|_{\mathbf{V}} \geq m\|p\| \qquad \text{for all } \ p \in Q. \tag{2.7}$$

A proof of (2.5) be found in [5]. For $f \in \mathbf{V}^*$, $g \in Q^*$, we consider the following variational problem: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v}\rangle \qquad \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) \qquad\qquad &= \langle g, q\rangle \qquad \text{for all } q \in Q. \end{aligned} \tag{2.8}$$

In this paper, we consider the particular case of (2.8) when the symmetric, bounded bilinear form $a(\cdot,\cdot)$ is *coercive on the whole space* $\mathbf{V}$. We assume that the bilinear form $b : \mathbf{V} \times Q \to \mathbb{R}$ satisfies (2.3) and (2.4). If $A : V \to V^*$ is the standard operator associated with the form $a(\cdot,\cdot)$, the the problem (2.8) is equivalent to: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} A\mathbf{u} + B^* p &= \mathbf{f}, \\ B\mathbf{u} \qquad &= g. \end{aligned} \tag{2.9}$$

It is known that the above variational problem or system has a unique solution for any $f \in \mathbf{V}^*$, $g \in Q^*$ (see [18,19,25,30]).

The form $a(\cdot,\cdot)$ might not agree with the a natural inner product on $\mathbf{V}$, but due to the symmetry, boundness, and the coercivity assumptions, $a(\cdot,\cdot)$ induces a norm that is equivalent with the natural norm on $\mathbf{V}$. Thus, without loss of generality, we assume from now on that the inner product $a_0(\cdot,\cdot)$ on $\mathbf{V}$ is the same as $a(\cdot,\cdot)$, and consequently $A = \mathcal{A}$.

The Uzawa algorithm for solving the Stokes system was first introduced in [1]. Given a parameter $\alpha > 0$, called the relaxation parameter, the Uzawa algorithm for approximating the solution $(\mathbf{u}, p)$ of (2.8) can be described as follows.

Algorithm 1 (Uzawa Method (UM)) *Let p_0 be any initial guess for p. For $j = 1, 2, \ldots,$ construct $(\mathbf{u}_j, p_j)$ by*

$$\begin{aligned} \mathbf{u}_j &= A^{-1}(\mathbf{f} - B^* p_{j-1}), \\ p_j &= p_{j-1} + \alpha \mathcal{C}^{-1}(B\mathbf{u}_j - g). \end{aligned} \tag{2.10}$$

The convergence of the **UM** is discussed for particular cases in many publications, see e.g., [10,19,24,25,29,31]. For the continuous level, the convergence factor for the error $\|p - p_j\|$ is $\gamma := \|I - \alpha S_0\| = \max\{|1 - \alpha m^2|, |1 - \alpha M^2|\}$ and $\gamma < 1$ provided $\alpha \in \left(0, \frac{2}{M^2}\right)$. A proof can be found in [4].

3 A Double Inexact Uzawa Algorithm

Next, following the ideas in [15,23], we introduce an abstract double inexact Uzawa algorithm. First, the exact solve of the elliptic problem or the action of A^{-1} in the standard Uzawa algorithm, is replaced by an approximation process acting on the residual $\mathbf{r}_{j-1} := \mathbf{f} - A\mathbf{u}_{j-1} - B^* p_{j-1}$. The approximate process is described as a map Ψ defined on a subset of $\mathbf{V}^*$ which for $\phi \in \mathbf{V}^*$ it returns an approximation of $\xi = A^{-1}\phi$. If $\mathbf{V}$ and Q are finite dimensional spaces, then Ψ can be considered as a linear or non-linear preconditioner for A (see e.g., [15]). If $\mathbf{V}$ and Q are not finite dimensional spaces, then $\Psi(\phi)$ can be considered as a discrete Galerkin approximation of the elliptic problem $A\xi = \phi$, see [4,5]. Second, the exact action of $\mathcal{C}^{-1}$ in the Uzawa algorithm, is also replaced by a linear or non-linear approximation (**or** smoothing) process acting on the residual $q_j := B\mathbf{u}_j - g$. The approximate process is described as a map Φ defined on a subset of Q^*, which for $q^* \in Q^*$, returns an approximation of $\eta := \mathcal{C}^{-1}q^*$. As an example of a linear process $\Phi(q^*)$ one can take the projection of $\mathcal{C}^{-1}q^*$ on a finite dimensional subspace of Q. The role of Φ is to provide a good approximation of the natural representation operator $\mathcal{C}^{-1}$ or, in other concrete cases, to smooth the residual associated with the constrained variable. By using a smooth approximation $\Phi(q^*)$ of $\mathcal{C}^{-1}q^*$ we get that the p_j iterations becomes smoother. Consequently, in the presence of elliptic regularity for A, the representation $A^{-1}\mathbf{r}_j$ can be better approximated by $\Psi(\mathbf{r}_j)$. The improvement can be seen in numerical experiments (see Section

5).

Let $\alpha > 0$ be a given relaxation parameter. The "*Inexact*^{2}" Uzawa ($\mathbf{I^2U}$) algorithm for approximating the solution $(\mathbf{u}, p)$ of (2.8) is as follows.

Algorithm 2 *:* **Inexact^2 Uzawa** ($\mathbf{I^2U}$). *Let* $(\mathbf{u}_0, p_0)$ *be any approximation for* $(\mathbf{u}, p)$. *For* $j = 1, 2, \cdots,$ *construct* $(\mathbf{u}_j, p_j)$ *by*

$$\begin{aligned} \mathbf{u}_j &= \mathbf{u}_{j-1} + \Psi(\mathbf{f} - A\mathbf{u}_{j-1} - B^* p_{j-1}), \\ p_j &= p_{j-1} + \alpha\Phi(B\mathbf{u}_j - g). \end{aligned}$$

For $j = 0, 1, \ldots$, let $e_j^{\mathbf{u}} := \mathbf{u} - \mathbf{u}_j$, $e_j^p := p - p_j$, $\mathbf{r}_j = \mathbf{f} - A\mathbf{u}_j - B^* p_j$, $q_j = B\mathbf{u}_j - g$. To state the main result of this section, we will also need the convergence factor for the Uzawa algorithm $\gamma := \|I - \alpha S_0\| = max\{|1 - \alpha m^2|, |1 - \alpha M^2|\}$.

Theorem 1 *Assume that* ϵ, δ, *and* $\alpha < 2/M^2$ *are positive numbers chosen such that* Ψ *and* Φ *satisfy*

$$\left|\Psi(\mathbf{r}_j) - A^{-1}\mathbf{r}_j\right|_{\mathbf{V}} \le \delta \left|A^{-1}\mathbf{r}_j\right|_{\mathbf{V}}, \quad j = 0, 1, \ldots, \tag{3.1}$$

$$\left\|\Phi(q_j) - \mathcal{C}^{-1}q_j\right\| \le \epsilon \left\|\mathcal{C}^{-1}q_j\right\|, \quad j = 0, 1, \ldots, \tag{3.2}$$

with

$$\epsilon < \frac{1-\gamma}{\alpha M^2}, \tag{3.3}$$

and

$$\delta < \delta^1 := \frac{1 - \gamma - \alpha M^2 \epsilon}{1 - \gamma + 2\alpha M^2 + \alpha M^2 \epsilon}. \tag{3.4}$$

Then, the I^2U *algorithm converges.*

Remark 1 *For* $\epsilon = 0$, *we have that* $\Phi = \mathcal{C}^{-1}$ *and the Algorithm 2 reduces to Algorithm 4.3 of [5]. ln this case the algorithm has only one inexact process and we will refer to it as the* **Inexact Uzawa** (**IU**) *algorithm. It was proven in [5], that the* (**IU**) *algorithm converges under a better (optimal) threshold for* δ, *more precisely for* $\delta < \delta^0 := \frac{2-\alpha M^2}{2+\alpha M^2}$. *An improvement on the threshold* δ^1 *such that, when* $\epsilon = 0$, *we get* $\delta^1 = \delta^0$ *for all* $0 < \alpha < 2/M^2$ *remains to be investigated.*

Proof: From the first equation of (2.9) and the first equation of Algorithm 2 we have

$$\begin{aligned} e_j^{\mathbf{u}} &= e_{j-1}^{\mathbf{u}} - \Psi(Ae_{j-1}^{\mathbf{u}} + B^* e_{j-1}^p) \\ &= (A^{-1} - \Psi)(Ae_{j-1}^{\mathbf{u}} + B^* e_{j-1}^p) - A^{-1}B^* e_{j-1}^p \\ &= (A^{-1} - \Psi)(\mathbf{r}_{j-1}) - A^{-1}B^* e_{j-1}^p. \end{aligned} \tag{3.5}$$

From the second equation of (2.9) and the second equation of Algorithm 2, we get

$$\begin{aligned} e_j^p = e_{j-1}^p - \alpha\Phi q_j &= e_{j-1}^p + \alpha(\mathcal{C}^{-1} - \Phi)q_j - \alpha\mathcal{C}^{-1}q_j = \\ &= e_{j-1}^p + \alpha(\mathcal{C}^{-1} - \Phi)q_j + \alpha\mathcal{C}^{-1}Be_j^{\mathbf{u}}. \end{aligned} \tag{3.6}$$

If we substitute $e_j^{\mathbf{u}}$ from (3.5) in (3.6), then

$$e_j^p = (I - \alpha S_0)e_{j-1}^p + \alpha(\mathcal{C}^{-1} - \Phi)q_j + \alpha\mathcal{C}^{-1}B(A^{-1} - \Psi)(\mathbf{r}_{j-1}). \tag{3.7}$$

From (3.5) and (3.7) by the triangle inequality and the assumption (3.1), we obtain

$$|e_j^{\mathbf{u}}| \le \delta(|e_{j-1}^{\mathbf{u}}| + M\|e_{j-1}^p\|) + M\|e_{j-1}^p\| = \delta|e_{j-1}^{\mathbf{u}}| + M(1+\delta)\|e_{j-1}^p\|.$$

From (3.7) by the triangle inequality and the assumptions (3.1), (3.2), we obtain

$$\|e_j^p\| \le \gamma\|e_{j-1}^p\| + \alpha\epsilon\|\mathcal{C}^{-1}q_j\| + \alpha M\delta(|e_{j-1}^{\mathbf{u}}| + M\|e_{j-1}^p\|).$$

Combining the above two estimate with

$$\|\mathcal{C}^{-1}q_j\| = \| - \mathcal{C}^{-1}Be_j^{\mathbf{u}}| \le M|e_j^{\mathbf{u}}|,$$

we get

$$\|e_j^p\| \le \alpha M\delta(1+\epsilon)|e_{j-1}^{\mathbf{u}}| + \left(\gamma + \alpha M^2(\delta + \epsilon + \epsilon\delta)\right)\|e_{j-1}^p\|.$$

Let $E_j = \begin{pmatrix} |e_j^{\mathbf{u}}| \\ \|e_j^p\| \end{pmatrix}$, and define the 2×2 matrix

$$\mathbf{M} := \begin{pmatrix} \delta & M(1+\delta) \\ \alpha M\delta(1+\epsilon) & \gamma + \alpha M^2(\delta + \epsilon + \epsilon\delta) \end{pmatrix}.$$

On $\mathbf{R}^2$ we introduce the inner product $[\cdot,\cdot]_w$ defined by

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]_w = w_1x_1y_1 + w_2x_2y_2,$$

where w_1, w_2 are any two positive numbers such that

$$\frac{w_1}{w_2} = \frac{\alpha\delta(1+\epsilon)}{1+\delta},$$

We note that $\mathbf{M}$ is symmetric with respect to the $[\cdot,\cdot]_w$ inner product. We will denote the norm induced by $[\cdot,\cdot]_w$ with $\|\cdot\|_w$. With the above notation the errors at step j and $j-1$ can be written

$$E_j \leq \mathbf{M}\ E_{j-1}, \tag{3.8}$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ means $x_1 \leq y_1$ and $x_2 \leq y_2$. From (3.8), we deduce

$$E_j \leq \mathbf{M}^j\ E_0. \tag{3.9}$$

Since $\mathbf{M}$ is symmetric with respect to $[\cdot,\cdot]_w$-inner product, we have

$$\|E_j\|_w = [E_j, E_j]_w^{1/2} \leq (\rho(\delta))^j\ \|E_0\|_w,$$

where $\rho(\delta)$ is the spectral radius of M as a symmetric matrix with respect to $[\cdot,\cdot]_w$-inner product. To complete the proof, we just have to show that $\rho(\delta) \in (0,1)$ provided that $0 < \alpha < 2/M^2$ and (3.4) holds. The characteristic equation of the matrix $\mathbf{M}$ is

$$\lambda^2 - \lambda(\gamma + \delta + \alpha M^2(\delta + \epsilon + \epsilon\delta)) + \delta(\gamma - \alpha M^2) = 0.$$

Since $\mathbf{M}$ has positive entries, the characteristic equation has real roots and the largest (positive) root agrees with the spectral radius of $\mathbf{M}$. Consequently,

$$\rho(\delta) = \frac{1}{2}\left(t + \sqrt{t^2 - 4\delta(\gamma - \alpha M^2)}\right),$$

where

$$t = \gamma + \delta + \alpha M^2(\delta + \epsilon + \epsilon\delta).$$

Using that $\gamma = \max\{|1-\alpha m^2|, |1-\alpha M^2|\}$ and $\alpha \in (0, 2/M^2)$, it is easy to verify that $\rho(\delta) = 1$ for

$$\delta = \delta^1 := \frac{1 - \gamma - \alpha M^2\epsilon}{1 - \gamma + 2\alpha M^2 + \alpha M^2\epsilon}, \tag{3.10}$$

and that the function $\delta \to r = r(\delta)$ is an increasing function on $[0, \delta^1]$.

End of the proof.

4 Multilevel Algorithms

In this section we interpret Algorithm 2 as a multilevel algorithm, by defining the approximate inverse $\Psi(\mathbf{r}_j)$ as a Galerkin projection of $A^{-1}(\mathbf{r}_j)$ on an appropriate finite element subspace of $\mathbf{V}$ that changes when j increases, and by defining a computable approximation $\Phi(q_j)$ of $\mathcal{C}^{-1}q_j$. The approximation $\mathbf{u}_j$ of $\mathbf{u}$ is updated by solving on larger and larger subspaces of $\mathbf{V}$ a simple *elliptic, symmetric and positive definite* problem.

In what follows, we consider that (2.8) is the variational formulation of a boundary value problem on a fixed domain Ω, and assume that two sequences of nested finite element spaces

$$\mathbf{V}_1 \subset \mathbf{V}_2 \subset \cdots \subset \mathbf{V}, \text{ and } M_1 \subset M_2 \subset \cdots \subset Q,$$

are given. We note that a discrete stability condition for the family $\{(\mathbf{V}_k, M_k)\}$ is not required, but we assume that the sequence $\{\mathbf{V}_k\}$ is **dense** in $\mathbf{V}$. The subspaces $\mathbf{V}_k$ and M_k could be standard multilevel spaces of functions on Ω associated with uniform or non-uniform meshes $\{\mathcal{T}_k\}$ on Ω. We will refer to the index k of $\mathbf{V}_k$ as the *iteration level.* Assume also that $\alpha \in (0, 2/M^2)$, $\epsilon > 0$, and $\delta > 0$ are chosen such that the conditions (3.3), (3.4) are satisfied. A natural choice for $\Phi(q_j)$ is the orthogonal projection of $C^{-1}q_j$ onto M_k, where k is the level used to compute $\mathbf{u}_j$. More precisely we define $\Phi(q_j)$ by:

$$(\Phi(q_j), q) := (\mathcal{C}^{-1}q_j, q) = b(\mathbf{u}_j, q) - \langle g, q\rangle, \qquad \text{for all } q \in M_k.$$

If we denote by $\mathcal{R}_k : \mathcal{C}^{-1}(B\mathbf{V}_k - g) \to \mathbf{M}_k \subset Q$ the orthogonal projection onto M_k, then we have $\Phi(q_j) = \mathcal{R}_k\mathcal{C}^{-1}(q_j)$.

Further, we assume that two logical conditions **(SUF1)** and **(SUF2)** are defined such that whenever each one of them is true we have that (3.1) is satisfied for some $\delta < \delta^1$. We will provide a concrete example of such logical conditions shortly, but the general assumptions we make about **(SUF1)** and **(SUF2)** are as follows: The logical condition **(SUF1)** is a *same level iteration* sufficient condition and is based only on computed quantities in the current and the previous step on a fixed level k. It guaranties that the relative error for representing the residual, when iterating on the same fixed subspace $\mathbf{V}_k$, stays under a prescribed threshold. If **(SUF1)** is not true then we change the iteration process from level k to level $k+1$ (project the residual representation on the next subspace $\mathbf{V}_{k+1}$). The logical condition **(SUF2)** is an *inter-level iteration* sufficient condition. It is checked whenever the residual representation is approximated on a higher level. If **(SUF2)** is true then the relative error for representing the residual falls under an even smaller threshold, in order to allow again same level iteration. Having **(SUF1)** and **(SUF2)** available, we interpret the $(\mathbf{I^2U})$ algorithm as a multilevel algorithm by taking $\Psi(\mathbf{r}_j) = \mathbf{w}_{j+1}$ the $a_0(\cdot,\cdot)$ orthogonal projection of $A^{-1}(\mathbf{r}_j)$ onto an appropriate subspace $\mathbf{V}_k$, and by replacing $\Phi(q_j)$ with $\mathcal{R}_k\mathcal{C}^{-1}(q_j)$.

Algorithm 3 Multilevel (Inexact)^2 Uzawa (MI^{2}U)

Set $\mathbf{u}_0 = 0 \in \mathbf{V},\ p_0 = 0 \in Q,\ j = 1,\ k = 1.$

Step 0. *Compute* $\mathbf{w}_1 \in \mathbf{V}_k$ *as the solution of*

$a(\mathbf{w}_1, \mathbf{v}) =< \mathbf{f}, \mathbf{v} > -a(\mathbf{u}_0, \mathbf{v}) - b(\mathbf{v}, p_0),\ \ \mathbf{v} \in \mathbf{V}_k$
and compute $\mathbf{u}_1 := \mathbf{u}_0 + \mathbf{w}_1,\ p_1 := p_0 + \alpha \mathcal{R}_k \mathcal{C}^{-1}(B\mathbf{u}_1 - g).$

Step 1. *Compute* $\mathbf{w}_{j+1} \in \mathbf{V}_k$ *as the solution of*

$$a(\mathbf{w}_{j+1}, \mathbf{v}) =< \mathbf{f}, \mathbf{v} > -a(\mathbf{u}_j, \mathbf{v}) - b(\mathbf{v}, p_j),\ \ \mathbf{v} \in \mathbf{V}_k.$$

If (SUF1)

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \mathbf{w}_{j+1}, \quad p_{j+1} = p_j + \alpha \mathcal{R}_k \mathcal{C}^{-1}(B\mathbf{u}_{j+1} - g)$$

j=j+1, **Go To Step 1**

Else, k=k+1, End

Step 2. *Compute* $\mathbf{w}_{j+1} \in \mathbf{V}_k$ *as the solution of*

$$a(\mathbf{w}_{j+1}, \mathbf{v}) =< \mathbf{f}, \mathbf{v} > -a(\mathbf{u}_j, \mathbf{v}) - b(\mathbf{v}, p_j),\ \ \mathbf{v} \in \mathbf{V}_k.$$

If (SUF2)

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \mathbf{w}_{j+1}, \quad p_{j+1} = p_j + \alpha \mathcal{R}_k \mathcal{C}^{-1}(B\mathbf{u}_{j+1} - g)$$

j=j+1, **Go To Step 1**

Else

k=k+1, **Go To Step 2**

End

In the particular case when $\mathcal{R}_k$ is the orthogonal projection onto a subspace $M_k \subset Q$ that contains $\mathcal{C}^{-1}(B\mathbf{V}_k)$ we have $\Phi(q_j) = \mathcal{C}^{-1}(B\mathbf{u}_j) - \mathcal{R}_k \mathcal{C}^{-1}(g)$ and the above multilevel algorithm will be called the **(MIU)** algorithm. If $g = 0$ we simply take $M_k := \mathcal{C}^{-1}(B\mathbf{V}_k)$, $\Phi(q_j) = \mathcal{C}^{-1}(B\mathbf{u}_j)$ for the **(MIU)** algorithm. In order to completely define the general (**MI²U**) algorithm, we need to come up with a concrete pair of logical conditions **(SUF1)** and **(SUF2)**. To do so in this general context, we define first $F(\delta) := \frac{1}{\sqrt{1-\delta^2}}$, $\delta \in (0,1)$ and let $0 < \delta_0 < \delta_{00} < \delta^1 = \frac{1-\gamma-\alpha M^2\epsilon}{1-\gamma+2\alpha M^2+\alpha M^2\epsilon}$ be fixed.

Let $\mathbf{w}_{j+1}^c = A^{-1}\mathbf{r}_j \in \mathbf{V}$ be the representation at the **continuous** level of $\mathbf{r}_j$, i.e., the solution of the problem

$$a(\mathbf{w}_{j+1}^c, \mathbf{v}) = < \mathbf{f}, \mathbf{v} > -a(\mathbf{u}_j, v) - b(\mathbf{v}, p_j), \ \mathbf{v} \in \mathbf{V}. \tag{4.1}$$

Then condition (3.1) is equivalent to

$$|\mathbf{w}_j - \mathbf{w}_j^c| \leq \delta \ |\mathbf{w}_j^c|, \tag{4.2}$$

or using that $|\mathbf{w}_j - \mathbf{w}_j^c|^2 + |\mathbf{w}_j|^2 = |\mathbf{w}_j^c|^2$, it is equivalent to

$$|\mathbf{w}_j^c| \leq F(\delta) \ |\mathbf{w}_j|. \tag{4.3}$$

which is also equivalent to

$$|\mathbf{w}_j - \mathbf{w}_j^c| \leq \delta \ F(\delta) \ |\mathbf{w}_j|. \tag{4.4}$$

To define **(SUF1)**, let us assume that both $\mathbf{w}_j$ and $\mathbf{w}_{j+1}$ are computed on $\mathbf{V}_k$. We have that,

$$\begin{aligned} a(\mathbf{w}_j, \mathbf{v}) &= < f, \mathbf{v} > -a(\mathbf{u}_{j-1}, v) - b(\mathbf{v}, p_{j-1}), \ \mathbf{v} \in \mathbf{V}_k, \ \text{and} \\ a(\mathbf{w}_{j+1}, \mathbf{v}) &= < f, \mathbf{v} > -a(\mathbf{u}_j, v) - b(\mathbf{v}, p_j), \ \mathbf{v} \in \mathbf{V}_k. \end{aligned} \tag{4.5}$$

Subtracting term by term the above equations and using that $\mathbf{u}_{j+1} = \mathbf{u}_j + \mathbf{w}_{j+1}$, we get

$$a(\mathbf{w}_{j+1}, \mathbf{v}) = -b(\mathbf{v}, p_j - p_{j-1}), \ \mathbf{v} \in \mathbf{V}_k. \tag{4.6}$$

Combining (4.1) with a similar relation for $\mathbf{w}_j^c$ we get that

$$a(\mathbf{w}_{j+1}^c - \mathbf{w}_j^c + \mathbf{w}_j, \mathbf{v}) = -b(\mathbf{v}, p_j - p_{j-1}), \ \mathbf{v} \in \mathbf{V}. \tag{4.7}$$

From (4.6) and (4.7) we have that $\mathbf{w}_{j+1} \in \mathbf{V}_k$ is the Galerkin projection of $\mathbf{w}_{j+1}^c - \mathbf{w}_j^c + \mathbf{w}_j$ and

$$\mathbf{w}_{j+1}^c = \mathbf{w}_j^c - \mathbf{w}_j - A^{-1}B^*(p_j - p_{j-1}). \tag{4.8}$$

Using (2.6) and (2.7) , we have

$$|\mathbf{w}_{j+1}^c| \leq |\mathbf{w}_j^c - \mathbf{w}_j| + |A^{-1}B^*(p_j - p_{j-1})| \leq |\mathbf{w}_j^c - \mathbf{w}_j| + M\|p_j - p_{j-1}\|. \tag{4.9}$$

Let $j_0 = j_0(k)$ be defined as $\min\{j| \ \mathbf{w}_j$ is computed and accepted on$\mathbf{V}_k\}$. We assume that $\mathbf{w}_{j_0}$ satisfies (4.2) with $\delta = \delta_{j_0} = \delta_0$. The assumption about $\mathbf{w}_{j_0}$

is always satisfied if a good approximation space $\mathbf{V}_1$ is chosen to start the algorithm with, or if **(SUF2)**, to be introduced shortly, is satisfied. Thus, assuming that $\delta_j \le \delta_{00}$ with $j \ge j_0$ is defined such that

$$|\mathbf{w}_j - \mathbf{w}_j^c| \le \delta_j \; |\mathbf{w}_j^c|, \tag{4.10}$$

using the estimate (4.9) and that (4.2) and (4.4) are equivalent, we get

$$\frac{|\mathbf{w}_{j+1}^c|}{|\mathbf{w}_{j+1}|} \le \delta_j \; F(\delta_j)\frac{|\mathbf{w}_j|}{|\mathbf{w}_{j+1}|} + \frac{M\|p_j - p_{j-1}\|}{|\mathbf{w}_{j+1}|}.$$

Since F is injective, we can define δ_{j+1} by

$$F(\delta_{j+1}) := \delta_j \; F(\delta_j)\frac{|\mathbf{w}_j|}{|\mathbf{w}_{j+1}|} + \frac{M\|p_j - p_{j-1}\|}{|\mathbf{w}_{j+1}|},$$

with the understanding that if $|\mathbf{w}_{j+1}| = 0$, then $\delta_{j+1} = 1$. Our **(SUF1)** condition for accepting $\mathbf{w}_{j+1}$, hence $\mathbf{u}_{j+1}$, is simply defined by

$$(\mathbf{SUF1}) \quad F(\delta_{j+1}) = \delta_j \; F(\delta_j)\frac{|\mathbf{w}_j|}{|\mathbf{w}_{j+1}|} + \frac{M\|p_j - p_{j-1}\|}{|\mathbf{w}_{j+1}|} \le F(\delta_{00}).$$

which is equivalent to $\delta_{j+1} \; \le \delta_{00}$. Thus, in the light of the fact that (4.2) and (4.3) are equivalent, **(SUF1)** implies

$$|\mathbf{w}_{j+1} - \mathbf{w}_{j+1}^c| \le \delta_{j+1} \; |\mathbf{w}_{j+1}^c|, \tag{4.11}$$

with $\delta_{j+1} \; \le \delta_{00} < \delta^1$.

Next, in order to define **(SUF2)**, we introduce further notation. Let $\mathbf{u}^\mathbf{f} \in \mathbf{V}$, and $\mathbf{u}_k^f \in \mathbf{V}_k$ be defined as the solutions of the following variational problems:

$$a(\mathbf{u}^\mathbf{f}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \qquad \text{for all } \mathbf{v} \in \mathbf{V}, \tag{4.12}$$

$$a(\mathbf{u}_k^\mathbf{f}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \qquad \text{for all } \mathbf{v} \in \mathbf{V}_k. \tag{4.13}$$

For $k = 1, 2, \ldots$, we let c_k be positive constants such that:

$$|\mathbf{u}_k^\mathbf{f} - \mathbf{u}^\mathbf{f}| \le c_k |\mathbf{u}^\mathbf{f}|. \tag{4.14}$$

For any $q \in Q$ we define $\mathbf{u}^q \in \mathbf{V}$, and $\mathbf{u}_k^q \in \mathbf{V}_k$ as the solutions of the following variational problems:

$$a(\mathbf{u}^q, \mathbf{v}) = -b(\mathbf{v}, q), \qquad \text{for all } \mathbf{v} \in \mathbf{V}, \tag{4.15}$$

$$a(\mathbf{u}_k^q, \mathbf{v}) = -b(\mathbf{v}, q), \qquad \text{for all } \mathbf{v} \in \mathbf{V}_k. \tag{4.16}$$

For a fixed $k_0 \in \{1, 2, \ldots\}$, we let $j = j(k_0)$ be the smallest iteration step for which **(SUF1)** fails to be satisfied on $\mathbf{V}_{k_0}$. This means that $\mathbf{w}_j$ (hence $\mathbf{u}_j$) is computed (and accepted) on $\mathbf{V}_{k_0}$ and the accepted $\mathbf{w}_{j+1}$ belongs to $\mathbf{V}_k$ with $k > k_0$. Next, for $k > k_0$, we let $c_{k_0,k}$ be positive constants such that:

$$|\mathbf{u}_k^{p_j} - \mathbf{u}^{p_j}| \leq c_{k_0,k} \, |\mathbf{u}^{p_j}|. \tag{4.17}$$

We introduce now the following assumptions:

(A1) $\quad c_k \to 0, \text{ as } k \to \infty.$
(A2) $\quad c_{k_0,k} \to 0, \text{ as } k \to \infty, \text{ for any } k_0 \geq 1.$
(A3) $\quad |\mathbf{w}_{j+1}|$ computed at **Step 2** is nonzero if k is large enough.

We notice here that the assumptions **(A1), (A2)** follow from Cea's theorem and the density assumption for the nested sequence of approximation subspaces $\{\mathbf{V}_k\}$.

Remark 2 *Evidently, c_k and $c_{k_0,k}$ depend on the data $\mathbf{f}$ and g. In addition, in the general abstract case, $c_{k_0,k}$ also depends on $p_j \in M_{k_0}$ for $j = j(k_0)$. Nevertheless, as proven in the appendix, for special cases, e.g. if elliptic regularity for A^{-1} is assumed, due to the fact that $p_j \in M_{k_0}$- a finite dimensional space, and the fact that the sequence of* **nested** *subspaces $\{M_k\}$ is a priori specified, we can prove the existence of estimates for $c_{k_0,k}$ that are independent of p_j, the iteration order j and also independent of $\mathbf{f}$ and g.*

We assume now that $\mathbf{w}_j$, $\mathbf{u}_j$ are computed on $\mathbf{V}_{k_0}$, that $p_j = p_{j-1} + \alpha \mathcal{R}_{k_0} \mathcal{C}^{-1}(B\mathbf{u}_j - g)$, and that $\mathbf{w}_{j+1}$ is computed as the solution of (4.5) (second equation) on $\mathbf{V}_k$, with $k > k_0$. Thus, we get

$$\mathbf{w}_{j+1} = \mathbf{u}_k^{\mathbf{f}} - \mathbf{u}_j + \mathbf{u}_k^{p_j}, \quad \text{and}$$
$$\mathbf{w}_{j+1}^c = \mathbf{u}^{\mathbf{f}} - \mathbf{u}_j + \mathbf{u}^{p_j},$$

and using (4.14), (4.17), (2.6) and (2.7) we have

$$\begin{aligned} |\mathbf{w}_{j+1}^c - \mathbf{w}_{j+1}| \leq & |\mathbf{u}^{\mathbf{f}} - \mathbf{u}_k^{\mathbf{f}}| + |\mathbf{u}^{p_j} - \mathbf{u}_k^{p_j}| \\ \leq & c_k \, |\mathbf{u}^{\mathbf{f}}| + c_{k_0,k} \, |\mathbf{u}^{p_j}| \leq c_k \, F(c_k)|\mathbf{u}_k^{\mathbf{f}}| + c_{k_0,k} M \|p_j\|. \end{aligned}$$

Using the above estimate, and the equivalence between (4.2) and (4.4) our sufficient condition **(SUF2)** is designed such that, when satisfied, we have that (4.11) holds with $\delta_{j+1} = \delta_0 < \delta^1$. Thus, we define

$$(\textbf{SUF2}) \qquad c_k \, F(c_k) \, |\mathbf{u}_k^{\mathbf{f}}| + c_{k_0,k} \, M\|p_j\| \leq \delta_0 \, F(\delta_0) \, |\mathbf{w}_{j+1}|,$$

and emphasize that $p_j \in Q$ is computed by the algorithm at **Step1** for $j = j(k_0)$, and that $\mathbf{u}^{\mathbf{f}}_k, \mathbf{w}_{j+1} \in \mathbf{V}_k$ are computed in **Step 2** on the level k with $k > k_0$. The logical condition **(SUF2)** can be false when $|\mathbf{w}_{j+1}| = 0$ for all values $k \geq k_0$. In this case we have that $\mathbf{w}^c_{j+1} := A^{-1}\mathbf{r}_j = 0$, and a reduction of the error $(\mathbf{u}_j - u, p_j - p)$ can be proved in this case as in the standard Uzawa algorithm error analysis. To avoid this situations in the convergence analysis of $\mathbf{MI^2U}$, we will assume that **(A3)** holds. Then, **(A1), (A2), (A3)** imply that **(SUF2)** is true for k large enough.

Theorem 2 *Let $\alpha \in (0, 2/M^2)$ and $\epsilon > 0$ be chosen such that the condition* (3.3) *is satisfied and assume that $\Phi(q_j) = \mathcal{R}_k\mathcal{C}^{-1}(q_j)$ satisfies* (3.2). *Let δ_0, δ_{00} be such that $0 < \delta_0 < \delta_{00} < \delta^1$ and assume that* **(A1), (A2), (A3)** *hold. In addition assume that the space $\mathbf{V}_1$ is chosen such that the Galerkin approximation $\mathbf{w}_1 \in \mathbf{V}_1$ of $\mathbf{u}^{\mathbf{f}} = A^{-1}\mathbf{f}$ satisfies $|\mathbf{u}^{\mathbf{f}} - \mathbf{w}_1| \leq \delta_0 \; |\mathbf{u}^{\mathbf{f}}|$. Then, the sequence $(\mathbf{u}_j, p_j)$ produced by* $\mathbf{MI^2U}$ *converges to $(\mathbf{u}, p)$ the solution of* (2.8).

Proof: The proof is based on Theorem 1. The ($\mathbf{MI^2U}$) algorithm is as a particular case of the ($\mathbf{I^2U}$) algorithm with the approximate inverse $\Psi(\mathbf{r}_j)$ being the Galerkin projection of $A^{-1}(\mathbf{r}_j)$ onto $\mathbf{V}_k$, with the dependence $k = k(j)$ uniquely determined by **(SUF1)** or **(SUF2)**, and with $\Phi(q_j) = \mathcal{R}_k\mathcal{C}^{-1}(q_j)$. Using Theorem 1, we have only to prove that the condition (3.1) is satisfied. Under the assumptions **(A1), (A2), (A3)**, we have that **(SUF2)** holds for k large enough, and this implies that infinitely many iterations $(\mathbf{u}_j, p_j)$ are computed. We will prove now by induction over $j \geq 1$ that the condition (3.1) of Theorem 1 is satisfied with $\delta \leq \delta_{00} < \delta^1$. With the notation used to define **(SUF1)** and **(SUF2)**, it is enough to show that for $j \geq 1$ we have (4.2) satisfied with some $\delta = \delta_j \leq \delta_{00}$. The assumption $|\mathbf{u}^{\mathbf{f}} - \mathbf{w}_1| \leq \delta_0 \; |\mathbf{u}^{\mathbf{f}}|$ imply that (4.2) is satisfied for $j = 1$ with $\delta = \delta_1 := \delta_0 < \delta_{00}$. Next, we assume that (4.2) holds for $j \geq 1$ and prove it for $j + 1$. From the **MIU** algorithm, each new iteration $(\mathbf{u}_{j+1}, p_{j+1})$ is accepted if and only if either **(SUF1)** or **(SUF2)** is true. If **(SUF1)** is satisfied, then both $\mathbf{w}_j$ and $\mathbf{w}_{j+1}$ are computed on the same subspace $\mathbf{V}_k$. From the induction hypothesis, there exists $\delta_j \leq \delta_{00}$ such that (4.10) is satisfied. Then, from the definition of **(SUF1)** we get (4.11), hence (4.2) is satisfied with $\delta = \delta_{j+1}$ of **(SUF1)**. If **(SUF2)** is satisfied, then (4.2) is satisfied with $\delta = \delta_{j+1} = \delta_0$ (see the statement above the definition of **(SUF2)**). *This completes the proof of the Theorem.*

Remark 3 *The* ($\mathbf{MI^2U}$) *algorithm is introduced for theoretical purposes as a first step in developing a practical implementattion. First we notice that the conditions* (3.3) *and* (3.4) *of Theorem 1 are not difficult to satisfy. If we choose e.g., $\alpha \in \left(\frac{2}{m^2+M^2}, \frac{2}{M^2}\right)$, then $\gamma = \alpha M^2 - 1$ and the conditions* (3.3) *and* (3.4) *become $\epsilon < \frac{2-\alpha M^2}{\alpha M^2}$ and $\delta < \delta^1 = \frac{2-(\alpha+\epsilon)M^2}{2+(\alpha+\epsilon)M^2}$, respectively. Thus, the condition* (3.2) *can be easily satisfied if we have approximability for $\mathcal{R}_k$ and the initial space M_1 has good approximation properties.*

Remark 4 *Concerning the simplification of* **(SUF1)**, *the pair of assumptions* **(SUF-SUF1)** *defined by:*

$$\begin{array}{ll} \textbf{(SUF-SUF1a)} & M\ \|p_j - p_{j-1}\| \leq R_0\ |\mathbf{w}_{j+1}| \\ \textbf{(SUF-SUF1b)} & |\mathbf{w}_j| \leq r_0\ |\mathbf{w}_{j+1}|, \end{array}$$

where $r_0 \geq 1$ and $R_0 \geq 1$ are fixed numbers, is simpler than **(SUF1)**. *It is easy to see that* **(SUF-SUF1)** *implies* **(SUF1)** *if $r_0 - 1$ and $R_0 - 1$ are small enough and δ_{00}, is large enough. More precisely, let us take $\delta_{01} \in (\delta_0, \delta_{00})$ be arbitrary. Then, directly from the definition of $F(\delta_{j+1})$, we can see that* **(SUF-SUF1)** *implies* **(SUF1)** *provided that*

$$\delta_{01} F(\delta_{01})\ r_0 + M\ R_0 \leq F(\delta_{00}). \tag{4.18}$$

Here we are using that δ_j in the definition of $F(\delta_{j+1})$ satisfies $\delta_j < \delta_{01} < \delta^1$ (by induction), and the fact that the function $\delta \to \delta F(\delta)$ is increasing on $(0, 1)$. Notice that for $r_0 \to 1$ and $R_0 \to 1$ the condition (4.18) can be satisfied (for any M) provided δ_{00}, (hence δ^1) approaches 1. According to Remark 3, this is feasible if the the initial space M_1 has good approximation properties.

Condition **(SUF2)** is needed for the proof of convergence of the algorithm in this general abstract case. In practice, if elliptic regularity for A^{-1} is present, and the first approximation subspace $\mathbf{V}_1$ corresponds to a fine enough initial mesh, the condition **(SUF2)** can be satisfied by changing the iteration one level up, i.e., for $k = k_0+1$. Based on these assumptions, a simplified multilevel version of the Algorithm 2 can be formulated as follows:

Algorithm 4 Multilevel (Inexact)^2 Uzawa Simplified (MI²US).
Let $\alpha > 0$, and $r_0 > 1, R_0 > 1$ be fixed numbers.

Set $\mathbf{u}_0 = 0 \in \mathbf{V}$, $p_0 = 0 \in Q$, $j = 1$, $k = 1$.

Step 1. *Solve for $\mathbf{w}_j \in \mathbf{V}_k$ and find $\mathbf{u}_j \in \mathbf{V}_k$, $p_j \in M_k$:*

$$\begin{array}{l} a(\mathbf{w}_j, \mathbf{v}) = < f, \mathbf{v} > - a(\mathbf{u}_{j-1}, v) - b(\mathbf{v}, p_{j-1}),\ \ \mathbf{v} \in \mathbf{V}_k, \\ \mathbf{u}_j = \mathbf{u}_{j-1} + \mathbf{w}_j, \quad p_j = p_{j-1} + \alpha \mathcal{R}_k\ \mathcal{C}^{-1}(B\mathbf{u}_j - g), \end{array}$$

Step 2. *Solve for $\mathbf{w}_{j+1} \in \mathbf{V}_k$:*

$$a(\mathbf{w}_{j+1}, \mathbf{v}) = < f, \mathbf{v} > - a(\mathbf{u}_j, v) - b(\mathbf{v}, p_j),\ \ \mathbf{v} \in \mathbf{V}_k,$$

If $(M\|p_j - p_{j-1}\| < R_0 |\mathbf{w}_{j+1}|$ **and** $|\mathbf{w}_j| < r_0 |\mathbf{w}_{j+1}|)$,

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \mathbf{w}_{j+1}, \quad p_{j+1} = p_j + \alpha \mathcal{R}_k\ \mathcal{C}^{-1}(B\mathbf{u}_{j+1} - g),$$

j=j+1, Go To Step 2

Else: $\mathbf{k = k + 1}$, $\mathbf{j = j + 1}$, Go To Step 1

Again, for the particular case when $\mathcal{R}_k$ is the orthogonal projection onto a subspace $M_k \subset Q$ that contains $\mathcal{C}^{-1}(B\mathbf{V}_k)$, we have $\mathcal{R}_k\ \mathcal{C}^{-1}(B\mathbf{u}_j - g) = \mathcal{C}^{-1}(B\mathbf{u}_j) - \mathcal{R}_k\mathcal{C}^{-1}(g)$ and the above multilevel algorithm will be called the **(MIUS)** algorithm. As a direct consequence of Theorem 2 and Remark 4 we have the following convergence result.

Proposition 1 *Let $\alpha \in (0, 2/M^2)$ and assume that $\epsilon > 0$ is chosen such that the condition* (3.3) *is satisfied, and $\Phi(q_j) = \mathcal{R}_k\mathcal{C}^{-1}(q_j)$ satisfies* (3.2)*. Let $\delta_0, \delta_{01}, \delta_{00}$ be such that $0 < \delta_0 < \delta_{01} < \delta_{00} < \delta^1$ and assume r_0, R_0 are chosen such that* (4.18) *is satisfied. We assume in addition that the condition* **(SUF2)** *is satisfied for any k_0 by doing one level refinement (i.e., for $k = k_0 + 1$), and that the space $\mathbf{V}_1$ is chosen such that the Galerkin approximation $\mathbf{w}_1 \in \mathbf{V}_1$ of $\mathbf{u^f} = A^{-1}\mathbf{f}$ satisfies $|\mathbf{u^f} - \mathbf{w}_1| \leq \delta_0\ |\mathbf{u^f}|$. Then, the sequence $(\mathbf{u}_j, p_j)$ produced by* ($\mathbf{MI^2US}$) *converges to $(\mathbf{u}, p)$ the solution of* (2.8).

We notice here that (4.18) is a sufficient condition for **(SUF1)** which is a sufficient condition for (3.1). In implementing **(MIUS)** and ($\mathbf{MI^2US}$) the restrictions for r_0 and R_0 can be considerably relaxed. It is easy to see that when iterating on a fixed level k, the iterations of ($\mathbf{MI^2US}$) coincide with standard Uzawa iterations for the pair $(\mathbf{V}_k, M_k)$. For the numerical experiments we performed the sequence $\{R_j\} := \left\{\frac{\|p_j - p_{j-1}\|}{|\mathbf{w}_{j+1}|}\right\} = \left\{\frac{\|p_j - p_{j-1}\|}{\|p_j - p_{j-1}\|_{S_0^{(k)}}}\right\}$, where $S_0^{(k)}$ is the discrete Schur complement associated with the pair $(\mathbf{V}_k, \mathcal{R}_k\ \mathcal{C}^{-1}(B\mathbf{V}_k))$, turned out to be an increasing sequence for the same level iteration. In the light of the estimate (2.7), the condition **(SUF-SUF1a)** can be viewed as an *enforced discrete stability*. If the exact solution $(\mathbf{u}, p)$ is available then, the user can choose $R_0 \approx \frac{1}{M}\ R_{j_0}$ where j_0 is the first index for which the condition $\|p_j - p\| < \|p_{j-1} - p\|$ fails to be satisfied. We note that j_0 could depend on the level k, but for the numerical experiments we performed, the choice of R_0 could be made independent of the level k and of the data $(\mathbf{f}, g)$. The computable ratio $\{r_j\} := \left\{\frac{|\mathbf{w}_j|}{|\mathbf{w}_{j+1}|}\right\}$ had larger values for the first two iterations on the same level k and then become a decreasing sequence. Nevertheless, in the definition of $F(\delta_{j+1})$ we have that the ratio r_j is multiplied with $\delta_j F(\delta_j)$. Thus $F(\delta_{j+1})$, hence δ_{j+1}, can remain small for the first few iterations because $\delta_j \approx \delta_0$ is small. Large values of r_j for the first level iteration can indicate that the the algorithm should start on a space $\mathbf{V}_1$ that corresponds to a more refined mesh.

5 Numerical results for the Stokes system

In this section we consider application of the multilevel algorithms developed in the previous sections to finite element discretization for the standard Stokes system:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \text{ in } \Omega \\ \operatorname{div} \mathbf{u} \quad\quad &= g, \text{ in } \Omega, \end{aligned} \tag{5.1}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = 0$ on $\partial\Omega$ and $\int_\Omega g \; dx = 0$, where Ω is the unit square and Δ is the componentwise Laplace operator. Define $\mathbf{V} := (H_0^1(\Omega))^2$, and $Q = L_0^2(\Omega) := \{h \in L^2(\Omega) | \int_\Omega h \; dx = 0\}$ and assume that $\mathbf{f} \in (L^2(\Omega))^2$ and $g \in L_0^2(\Omega)$. The variational formulation of the problem is:

Find $\mathbf{u} \in \mathbf{V}, p \in Q$ such that

$$\begin{aligned} \textstyle\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \; - \int_\Omega p \; \operatorname{div} \mathbf{v} &= \textstyle\int_\Omega \mathbf{f} \cdot \mathbf{v}, \;\; \mathbf{v} \in \mathbf{V}. \\ \textstyle\int_\Omega q \; \operatorname{div} \mathbf{u} \quad\quad\quad\quad &= \textstyle\int_\Omega g \; q, \quad q \in Q. \end{aligned}$$

We introduce $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ the bilinear forms:

$$a(\mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \sum_{i=1}^{2} \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i,$$

and

$$b(\mathbf{v}, q) := -\int_\Omega q \; \operatorname{div} \mathbf{v}, \;\; \mathbf{v} \in \mathbf{V}, q \in Q.$$

Let $\langle \mathbf{f}, \mathbf{v} \rangle := \int_\Omega \mathbf{f} \cdot \mathbf{v}$, and $\langle g, q \rangle := \int_\Omega g \; q$. The corresponding spaces and operators for the Stokes system are $\mathbf{V} := (H_0^1(\Omega))^2$, and $Q = Q^* = L_0^2(\Omega)$, $A : V \to V^*$ is $(-\Delta)^2 : (H_0^1(\Omega))^2 \to (H^{-1}(\Omega))^2$, $\mathcal{C} = \mathcal{C}^{-1} = I$ on $L_0^2(\Omega)$. $B : V \to Q^*$ and $B^* : Q \to V^*$ are defined by

$$\langle B\mathbf{v}, q \rangle = -\int_\Omega q \; \operatorname{div} \mathbf{v} = \langle B^* q, \mathbf{v} \rangle \quad \text{ for all } \mathbf{v} \in \mathbf{V}, \; q \in Q.$$

Thus, $B = -div$ and $B^* = grad$.

We consider an original mesh $\mathcal{T}_0$ on Ω by splitting the square into two triangles using the unit slope diagonal of the square. The family of uniform meshes $\{\mathcal{T}_k\}_{k \geq 0}$ is defined by a uniform refinement strategy, i.e., $\mathcal{T}_k$ is obtained from $\mathcal{T}_{k-1}$ by splitting each triangle of $\mathcal{T}_{k-1}$ in four similar triangles. We define $\mathbf{V}_k$ as the space of functions which vanish on $\partial\Omega$ and are continuous piecewise linear functions with respect to the mesh $\mathcal{T}_k$ and let M_k be the space of discontinuous piecewise linear functions w.r.t. $\mathcal{T}_k$. We apply first the Algorithm **MIUS** for the Stokes system. The exact solution is $u_1 = \frac{1}{5\pi^2}\sin(\pi x)\sin(2\pi y)$, $u_2 = \frac{1}{5\pi^2}\sin(2\pi x)\sin(\pi y)$, and $p = 2/3 - x^2 - y^2$. We start the algorithm on the fourth level of refinement, with the driving parameters: $\alpha = \mathbf{2/3}$, $\mathbf{r_0} = \mathbf{2}$, and $\mathbf{R_0} = \mathbf{2}$. In Table 1 we record the error for the velocity and the pressure for the last iteration $j = j(k)$ before leaving the level k. We also record on the last column of the table the number of iterations performed by the algorithm on each level k. We used three finite element spaces on a triangular mesh which we denote as follows:

$(P1)^2c$ - Continuous piecewise linear vector functions.
$P0d$ - Discontinuous piecewise constant scalar functions.
$P1c$ - Continuous piecewise linear scalar functions.

$h = 1/2^k$	$\lvert \mathbf{u} - \mathbf{u}_j \rvert$	ratio	$\lVert p - p_j \rVert$	ratio	# iter
k=4	0.027274		0.060033		11
k=5	0.014572	1.87	0.036219	1.66	5
k=6	0.006844	2.13	0.021119	1.71	6
k=7	0.003269	2.09	0.012347	1.71	6
k=8	0.001580	2.07	0.007149	1.72	6

Table 1
A summary of results for (5.1) using **MIU** ($(P1)^2c$-$P0d$). We see convergence despite the lack of an LBB condition as predicted by the theory.

Next, we apply the **MI²US** for solving the same problem. For any residual $q_j := B\mathbf{u}_j - g$, with $\mathbf{u}_j \in \mathbf{V}_k$, we define $\Phi(q_j)$ as the the L^2 projection onto the space M_k of continuous piecewise linear functions w.r.t. $\mathcal{T}_k$. We start the algorithm on the fourth level of refinement, with the driving parameters: $\alpha = \mathbf{0.6}$, $\mathbf{r_0} = \mathbf{2}$, and $\mathbf{R_0} = \mathbf{3}$. We record the error for the velocity and the pressure iteration at the time of leaving the level $\mathbf{V}_k$ and also the number of iterations performed by the algorithm on each level (see Table 2).

Remark 5 *In terms of discrete spaces for velocity and pressure, when iterating on a fixed level,* **MIUS** *corresponds to the standard Uzawa algorithm for the discrete* $(P1c)^2 - \mathrm{div}((P1c)^2)$ *pairs which are known to be unstable pairs. The* **MI²US** *corresponds to* $(P1c)^2 - \Phi(\mathrm{div}(P1c)^2)$*, which are also known to be unstable pairs. We compare our numerical results with the discretization on the stable pairs* $(P1c)^2(h/2) - P0d(h)$*, where we expect* $|\mathbf{u} - \mathbf{u}_h| = O(h^1)$*, and*

$h = 1/2^k$	$\|\mathbf{u} - \mathbf{u}_j\|$	ratio	$\|p - p_j\|$	ratio	# iter
k=4	0.015159		0.014074		29
k=5	0.007433	2.04	0.004290	3.28	15
k=6	0.003682	2.02	0.001540	2.79	15
k=7	0.000118	2.01	0.000750	2.05	16
k=8	0.000917	2.00	0.000460	1.63	17

Table 2
A summary of results for (5.1) using $\mathbf{MI^2U}$ ($(P1)^2c$-$P1c$). See Remark 5 for a discussion of the results.

$\|p-p_h\| = O(h^1)$. By averaging the error for the five levels of computations we have: For the **MIUS** *algorithm $|\mathbf{u}-\mathbf{u}_h| = O(h^{1.03})$, and $\|p-p_h\| = O(h^{0.77})$. For the* $\mathbf{MI^2U}$ *algorithm, $|\mathbf{u}-\mathbf{u}_h| = O(h^{1.01})$, and $\|p-p_h\| = O(h^{1.24})$. We notice inter-level error reduction for both algorithms in spite of the absence of the classical discrete stability. This is due to the global convergence of the algorithms to the continuous solution.*

6 Applications to Discretization for (first order) systems of Differential Equations

Using the notation of Section 2, assume that $b : \mathbf{V} \times Q \to \mathbb{R}$ is a bilinear form satisfying (2.3) and (2.4), and let $F \in \mathbf{V}^*$ be given.

Many variational formulations of first order systems of partial differential equations can be written in the form:

Find $p \in Q$ such that

$$b(\mathbf{v}, p) = \langle F, \mathbf{v} \rangle, \qquad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{6.1}$$

The existence and the uniqueness of (6.1) was first studied by Aziz and Babušca in [2] and is known as the Babušca's Lemma. We present next the Lemma in the light of the Schur Complement approach introduced in Section 2, with an additional Schur stability estimate. The subspaces $\mathbf{V}_0$ and $\mathbf{V}_1$ of $\mathbf{V}$ are defined in Section 2.

Lemma 1 *(Babušca) Let $b : \mathbf{V} \times Q \to \mathbb{R}$ be a bilinear form satisfying* (2.3) *and* (2.4)*, and let $F \in \mathbf{V}^*$.*

i) The problem: Find $p \in Q$ such that

$$b(\mathbf{v}, p) = \langle F, \mathbf{v} \rangle, \qquad \textit{for all } \mathbf{v} \in \mathbf{V} \tag{6.2}$$

has a unique solution if and only if

$$\langle F, \mathbf{v}\rangle = 0, \qquad \textit{for all } \mathbf{v} \in \mathbf{V}_0. \tag{6.3}$$

If (6.3) *holds and* p *is the solution of* (6.2), *then*

$$m\|p\| \leq \|p\|_{S_0} = \|F\|_{\mathbf{V}^*} = \|F\|_{\mathbf{V}_1^*} \leq M\|p\|. \tag{6.4}$$

ii) Let $F \in \mathbf{V}_1^*$. *The problem: Find* $p \in Q$ *such that*

$$b(\mathbf{v}, p) = \langle F, \mathbf{v}\rangle, \;\; \mathbf{v} \in \mathbf{V}_1 \tag{6.5}$$

has a unique solution p, *and*

$$\|p\|_{S_0} = \|F\|_{\mathbf{V}_1^*}. \tag{6.6}$$

iii) Assume that the form b, *in addition, satisfies the condition*

$$b(\mathbf{v}, p) = 0 \qquad \textit{for all } p \in Q \quad \textit{implies} \quad \mathbf{v} = 0. \tag{6.7}$$

Then, the problem (6.2) *has a unique solution* p *which satisfies* (6.4).

A proof of the above version of Babušca's Lemma can be found in [5]. Next, we present a simple but very useful note which allows us to solve (6.1) by using the algorithms developed in the previous sections.

Remark 6 *p is the solution of* (6.1) **if and only if** $(\mathbf{u} = 0, p)$ *is the solution of*

$$\begin{aligned} a_0(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{F}, \mathbf{v}\rangle \qquad &&\textit{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 &&\textit{for all } q \in Q. \end{aligned} \tag{6.8}$$

As a direct application of the above note we provide a new way of discretization for the div − curl systems. Our approach is related with the *dual formulation* of Bramble-Pasciak [13]. Following [13], let us consider the problem

$$\begin{aligned} \operatorname{curl} \mathbf{h} &= f, \text{ in } \Omega \\ \operatorname{div}(\mu \mathbf{h}) &= g, \text{ in } \Omega \\ (\mu \mathbf{h}) \cdot \mathbf{n} &= \sigma, \text{ on } \; \textit{on } \Gamma = \partial\Omega. \end{aligned} \tag{6.9}$$

Let $\mathbf{V} = H_0^1(\Omega) \times H^1(\Omega)/\mathbb{R}$, $\mathbf{Q} = (L^2(\Omega))^2$ and define $b(\cdot, \cdot)$ by

$$b((w, \phi), \mathbf{h}) := (\operatorname{curl} w, \mathbf{h}) + (\operatorname{grad} \phi, \mu \mathbf{h}), \qquad \text{for all } (w, \phi) \in \mathbf{V}, \;\mathbf{h} \in \mathbf{Q},$$

and

$$\langle \mathbf{F}, (w, \phi) \rangle = (f, w) - (g, \phi) + \langle \sigma, \phi \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the L^2 inner product or duality on Γ. By multiplying with appropriate functions and integrating by parts, the variational formulation of the above system becomes: Find $\mathbf{h} \in \mathbf{Q}$ such that

$$b((w, \phi), \mathbf{h}) = \langle \mathbf{F}, \mathbf{v} \rangle \qquad \text{for all } (w, \phi) \in \mathbf{V}. \tag{6.10}$$

It is proved in [13], that the functional $\mathbf{F}$ satisfies the compatibility condition (6.3) and that the form b satisfies the LBB condition (2.3) independent of μ if on $\mathbf{V}$ the following inner product is considered:

$$a_0\left((u, \psi), (w, \phi)\right) := \left(\mu^{-1} \operatorname{grad} u, \operatorname{grad} w\right) + \left(\mu \operatorname{grad} \psi, \operatorname{grad} \phi\right).$$

Thus, (6.10) has a unique solution, and using Remark 6, we can approximate the unique solution $\mathbf{h}$ by solving a corresponding saddle point system of type (6.8). For finding the solution of (6.10), we applied both multilevel algorithms **MIUS** and **MI²US** to the corresponding system (6.8). Numerical results were performed first for the unit square. The data is calculated such that the exact solution for $\mu = 1$ is the vector function
$\mathbf{h} = (\sin(\pi x)\cos(\pi y), \cos(\pi x)\sin(\pi y))/(2\pi)$. Thus, $f = 0$, and
$g = \cos(\pi x) cos(\pi y)$. The discrete subspaces of $\mathbf{V}$ are chosen to be standard spaces of continuous piecewise linear functions. The driving parameters are $\alpha = \mathbf{1}, \mathbf{R_0} = \mathbf{2}, \mathbf{r_0} = \mathbf{3}$. For the **MI²US** the smoothing process Φ is taken the L^2 projection onto M_k the space of continuous piecewise linear functions as subspace of $\mathbf{Q} = (L^2(\Omega))^2$. The number of iterations on each level for the **MIUS** is always one because the **MIUS** algorithm converges in one iteration for our choice of *conforming subspaces* and $\alpha = 1$. Table 3 records the number of iterations on each level for both **MIUS** and **MI²US** algorithms. It is interesting to note the behavior of the error for the two algorithms in spite of the absence of a discrete stability condition. We used two finite element spaces on a triangular mesh which we denote as follows:

$P1c$ - Continuous piecewise linear scalar functions.
$P0d$ - Discontinuous piecewise constant scalar functions.

Numerically, we obtained that $\|\mathbf{h} - \mathbf{h}_h\| = O(h)$ for the **MIU** algorithm $(((P1c)^2 - B((P1c)^2))$, and $\|\mathbf{h} - \mathbf{h}_h\| \approx O(h^2)$ for the (**MI²U**) algorithm $((P1c)^2 - \Psi(B((P1c)^2))$.

In addition, we performed numerical tests for an L-shaped domain Ω. In this non-convex case the data is calculated such that, for $\mu = 1$, the exact solution is $\mathbf{h} = \operatorname{grad}(r^\beta \cos(\theta\beta))$, with $\beta = 2/3$. Thus, $f = g = 0$. The driving parameters were chosen $\alpha = 1$, and $R_0 = 2, r_0 = 3$. We note here that $\mathbf{h} \notin H^{2/3}(\Omega)$,

lev	$\|\mathbf{h}-\mathbf{h}_k\|, IU$	ratio	# iter	$\|\mathbf{h}-\mathbf{h}_k\|, I^2U$	ratio	# iter
k=3	0.016113		1	0.0041658		3
k=4	0.008178	1.972	1	0.0014506	2.87	3
k=5	0.004109	1.988	1	0.0003967	3.66	3
k=6	0.002057	1.997	1	0.0001012	3.92	3
k=7	0.001029	1.999	1	0.0000254	3.98	3
k=8	0.000514	2.002	1	0.0000063	4.03	3

Table 3
A summary of results for (6.9) on the unit square using both **MIUS** $(((P1c)^2 - ((P0d)^2))$ and **MI²US** $(((P1c)^2 - (P1c)^2))$. Despite not having a classical statable pairs of discrete spaces, the error reduction for consecutive levels tends to be optimal.

and $2^{2/3} \approx 1.5874$.

lev	$\|\mathbf{h}-\mathbf{h}_k\|, IU$	ratio	# iter	$\|\mathbf{h}-\mathbf{h}_k\|, I^2U$	ratio	# iter
k=3	0.080541		1	0.051180		3
k=4	0.051764	1.556	1	0.032793	1.5607	3
k=5	0.033011	1.568	1	0.020595	1.5923	3
k=6	0.020953	1.575	1	0.012968	1.5881	3
k=7	0.013261	1.580	1	0.008168	1.5877	3
k=8	0.008378	1.583	1	0.005145	1.5874	3

Table 4
A summary of results for (6.9) on a L shaped domain using quasi-uniform meshes. Again we observe optimal convergence rate from level to level.

Numerically, we obtained that $\|\mathbf{h}-\mathbf{h}_h\| = O(h^{2/3})$ for $P^1 - B(P^1)$ (the **MIUS** algorithm), and $\|\mathbf{h}-\mathbf{h}_h\|$ is slightly better than $O(h^{2/3})$ for $P^1 - \Psi(B(P^1))$, the **MI²US** algorithm, (see Table 4). Even though we got a better (smaller) error for the second algorithm, the order of convergence seems to be just sligthly higher. Nevertheless, if graded meshes are used, see e.g., [6–8], and we refine by dividing all the edges that contains the singular point (the vertex of the optuse angle of the domain) under a fixed ratio κ such that the segment containing the singular point is κ-times the other segment, then the numerical results are improved. The results using graded meshes with $\kappa = 0.1$ are given in Table 5.

Numerically, we obtained that the error $\|\mathbf{h}-\mathbf{h}_h\|$ is of order $O(h^1)$ for $P^1 - B(P^1)$ (the **MIUS** algorithm), and close to $O(h^{1+2/3})$ for $P^1 - Q(B(P^1))$ (the

lev	$\|\mathbf{h}-\mathbf{h}_k\|, IU$	ratio	# iter	$\|\mathbf{h}-\mathbf{h}_k\|, I^2U$	ratio	# iter
k=3	0.076293		1	0.030103		3
k=4	0.040883	1.867	1	0.011187	2.691	3
k=5	0.021074	1.940	1	0.003888	2.877	3
k=6	0.010658	1.977	1	0.001259	3.088	3
k=7	0.005348	1.993	1	0.000386	3.262	3
k=8	0.002677	1.996	1	0.000113	3.410	3

Table 5
A summary of results for (6.9) on a L shaped domain using graded meshes. The rate of convergence is improved compared with Table 4.

(**MI²US**) algorithm). It is also interesting that in both situations, uniform meshes as well as graded meshes, the order of the error seems to improve as the level k increases while the number of iterations on each level remains constant. The numerical behavior for the L shaped domain demonstrates again that the convergence of the multilevel inexact Uzawa algorithms is driven by the accuracy of the residual representation for the elliptic problem and the discrete stability condition is not essential or even required for the (fast) convergence of the algorithms.

7 Conclusion

We presented a multilevel algorithm for discretizing symmetric saddle point problems for the particular case when the form $a(\cdot,\cdot)$ is symmetric and coercive. The algorithm is based on the Inexact Uzawa method at the continuous level and on the existence of multilevel sequences of nested approximation spaces. The convergence of the algorithm is driven by the accuracy of the residual representation for a simple elliptic problem at each iteration step. The main advantage of the algorithm is that the discrete Ladyshenskaya-Babušca-Brezzi stability condition is not essential or even required for (a fast) convergence. We introduced a new way to discretize systems of differential equations whose variational formulations fit the general framework of Babušca's Lemma. The convergence rates seen in Tables 3-5 are not explained by our analysis so far. This good behavior needs to be further investigated. The approach introduced in this paper can be easily modified to deal with the case when the form $a(\cdot,\cdot)$ is not coercive but a modified form $a_r(\mathbf{u},\mathbf{v}) := a(\mathbf{u},\mathbf{v}) + r(\mathcal{C}^{-1}B\mathbf{u},\mathcal{C}^{-1}B\mathbf{v})$ is coercive for a positive real value r. The advantages and applicability of the new multilevel algorithm to discretization of other PDEs, including the Maxwell equations, are to be investigated in a future work.

8 Appendix

We present in this section an estimate for the constants $c_{k_0,k}$ that appear in defining **(SUF2)** for a concrete case that is presented in Section 4. For simplicity, let us assume that Ω is a polygonal domain in $\mathbb{R}^2$ and that the spaces and the operators are as in the Stokes problem (Section 6), i.e., $\mathbf{V} := (H_0^1(\Omega))^2$, and $Q = L_0^2(\Omega)$, $A : V \to V^*$ is $(-\Delta)^2 : (H_0^1(\Omega))^2 \to (H^{-1}(\Omega))^2$, $\mathcal{C} = \mathcal{C}^{-1} = I$ on $L_0^2(\Omega)$, $B : V \to Q^*$ and $B^* : Q \to V^*$ are defined by $B = -div$, and $B^* = grad = \nabla$.

We consider a coarse quasi-uniform mesh $\mathcal{T}_H = \mathcal{T}_{k_0}$ and a uniformly refined mesh $\mathcal{T}_h = \mathcal{T}_k$ on Ω, with $0 < h < H$. We define $\mathbf{V}_H$ as the space of functions which vanish on $\partial\Omega$ and are continuous piecewise linear functions with respect to the mesh $\mathcal{T}_H$. The space $\mathbf{V}_h$ is defined in a similar way. Let $p_H \in M_{k_0} = M_H \subset L_0^2(\Omega)$, where M_H is the space of all piecewise constant function with respect to $\mathcal{T}_H = \mathcal{T}_{k_0}$. Next, we let $\mathbf{u} \in \mathbf{V}$, $\mathbf{u}_h \in \mathbf{V}_h$ be defined as the solutions of the following variational problems:

$$a(\mathbf{u}, \mathbf{v}) = -b(\mathbf{v}, p_H) = \langle \nabla p_H, \mathbf{v} \rangle \qquad \text{for all } \mathbf{v} \in \mathbf{V}, \tag{8.1}$$

$$a(\mathbf{u}_h, \mathbf{v}) = -b(\mathbf{v}, p_H), \qquad \text{for all } \mathbf{v} \in \mathbf{V}_h. \tag{8.2}$$

The key observation is that since p_H is a piecewise polynomial function, $p_H \in H^\alpha(\Omega)$ for any $\alpha < 1/2$, and so we get that $\nabla p_H \in H^{-1+\alpha}(\Omega)$. From the classical regularity theory for elliptic boundary problems on polygonal domains we have that $u \in H^{1+\alpha}$. Using standard error estimate results we have

$$|u - u_h| \le c_1 h^\alpha \|\mathbf{u}\|_{H^{1+\alpha}} \le c_2 h^\alpha \|\nabla p_H\|_{H^{-1+\alpha}(\Omega)}, \tag{8.3}$$

with c_2 independent p_H. Using the continuity of the operator ∇ we have

$$\|\nabla p_H\|_{H^{-1+\alpha}(\Omega)} \le c_2 \|p_H\|_{H^\alpha(\Omega)} \tag{8.4}$$

From the fact that p_H belongs to the finite dimensional space $\mathcal{C}^{-1} B \mathbf{V}_H \cap L_0^2(\Omega)$, we shall shortly prove that the following inverse inequality holds:

$$\|p_H\|_{H^\alpha(\Omega)} \le c_3 H^{-\alpha} \|p_H\|_{L^2(\Omega)}. \tag{8.5}$$

Using that the operator $\nabla : L_0^2(\Omega) \to H^{-1}(\Omega)$ has closed range (see e.g. Corollary 2.1 in [25]), we get that

$$\|p_H\|_{L^2(\Omega)} \le c_4 \|\nabla p_H\|_{H^{-1}(\Omega)} = c_4 |\mathbf{u}|. \tag{8.6}$$

From (8.3) - (8.6) we get that

$$|u - u_h| \le c \left(\frac{h}{H}\right)^\alpha |\mathbf{u}|, \tag{8.7}$$

with $c = c_2c_3c_4$ independent of p_H. Assuming that $H = 1/2^{k_0}$ and $h = 1/2^k$ with $k > k_0$, (8.7) gives the following estimate for $c_{k_0,k}$:

$$c_{k_0,k} \leq c\frac{1}{2^{\alpha(k-k_0)}}.$$

We will now close the appendix by justifying the inverse inequality (8.5). We assume that p_H is constant equal to p_K on each triangle $K \in \mathcal{T}_H$. Then,

$$\begin{aligned}\|p_H\|^2_{H^\alpha(\Omega)} &= \int_\Omega\int_\Omega \frac{|p_H(x)-p_H(y)|^2}{|x-y|^{2+2\alpha}}\,dxdy = \sum_{K,T\in\mathcal{T}_H}\int_K\int_T\frac{|p_K-p_T|^2}{|x-y|^{2+2\alpha}}\,dxdy\\ &= \sum_{K\neq T\in\mathcal{T}_H}|p_K-p_T|^2\int_K\int_T\frac{1}{|x-y|^{2+2\alpha}}\,dxdy.\end{aligned}$$

If $K, T \in \mathcal{T}_H$ are not sharing an edge or a vertex, using that $|x-y| \geq H$, we get

$$\int_K\int_T\frac{1}{|x-y|^{2+2\alpha}}\,dxdy \leq c_5H^{2-2\alpha},$$

with c_5 independent of $\mathcal{T}_H$ and $K, T \in \mathcal{T}_H$. The same estimate (with a different constant say c_6) holds if $K, T \in \mathcal{T}_H$ do share an edge or a vertex. In this case, we can just extend the above integrals to integrals on two adjacent rectangle of size H containing the triangles K and T. Then, a change of variables that reduces the integration on a pair of reference adjacent (unit) squares shows that the estimate remains valid in this case as well. Thus, we get that

$$\begin{aligned}\|p_H\|^2_{H^\alpha(\Omega)} &\leq \max\{c_5,c_6\}H^{2-2\alpha}\sum_{K\neq T\in\mathcal{T}_H}|p_K-p_T|^2 \leq c_7H^{-2\alpha}H^2\sum_{K\in\mathcal{T}_H}|p_K|^2\\ &\leq c_8H^{-2\alpha}\|p_H\|^2_{L^2(\Omega)}.\end{aligned}$$

This implies the inequality (8.5) and completes the proof of the estimate for $c_{k_0,k}$.

Thanks: We would like to thank to the two referee who made very valuable comments, suggestions, and corrections towards improving the content of the original manuscript.

References

[1] K. Arrow, L. Hurwicz and H. Uzawa. *Studies in Nonlinear Programming*, Stanford University Press, Stanford, CA, 1958.

[2] A. Aziz and I. Babušca, Survey lectures on mathematical foundations of the finite element method. In A. Aziz, editor, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Academic Press, 1-362, New York, NY, 1972.

[3] I. Babušca, Error bounds for the finite element method, *Numer. Math.*, Vol. 16, pp. 322-333, 1971.

[4] C. Bacuta, A Unified Approach for Uzawa Algorithms, SIAM Journal of Numerical Analysis, Volume 44, Issue 6, 2006, pp. 2245-2649.

[5] C. Bacuta, Schur Complements on Hilbert Spaces and Saddle Point Systems, *Journal of Computational and Applied Mathematics*, Volume 225, Issue 2, 2009, pp 581-593.

[6] C. Bacuta, V. Nistor and L. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polyhedra I: apriori estimates: Mesh Refinements and Interpolation, with V. Nistor and L. Zikatanov, *Numerical Functional Analysis and Optimization*, Vol. 26, No 6, 2005, pp. 613-639.

[7] C. Bacuta, V. Nistor and L. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polyhedra II: Mesh Refinements and Interpolation, Numerical Functional Analysis and Optimization, Vol. 28, No 7-8, 2007, pp. 775-824.

[8] C. Bacuta, V. Nistor and L. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps, *Numerische Mathematik*, Vol. 100, No 2, 2005, pp. 165 -184

[9] E. Bänsch, P. Morin and R.H. Nochetto. An adaptive Uzawa FEM for the Stokes problem: Convergence without the inf-sup condition. *Siam J. Numer. Anal.*, 40:1207-1229, 2002.

[10] M. Benzi, G.H. Golub and J. Liesen, Numerical Solutions of Saddle Point Problems. *Acta Numerica*, 14 (2005), pp. 1-137.

[11] F. A. Bornemann and P. Deuflhard. The cascadic multigrid method for elliptic problems. *Numer. Math.*, 75:135152, 1996.

[12] D. Braess, and W. Dahmen, A cascadic multigrid algorithm for the Stokes problem, *Numer. Math.*, (1999), 82 :179-191.

[13] J.H Bramble and J.E. Pasciak, A new approximation technique for div-curl systems, *Math. Comp.*, 73(2004), 1739-1762.

[14] J.H Bramble, J.E. Pasciak and P.S. Vassilevski, Computational scales of Sobolev norms with application to preconditioning, *Math. Comp.*, 69 (2000) 463-480.

[15] J. H. Bramble, J. Pasciak and A.T. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problems. *Siam J. Numer. Anal.*, 34:1072-1092, 1997.

[16] J. H. Bramble, J. Pasciak and A.T. Vassilev. Uzawa type algorithm for nonsymetric saddle point problems. *Math. Comp.*, 69 (1999) 667-689.

[17] J. H. Bramble and X. Zhang. The analysis of multigrid methods, in: *Handbook for Numerical Analysis*, Vol. VII, 173-415, P. Ciarlet and J.L. Lions, eds., North Holland, Amsterdam, 2000.

[18] S. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.

[19] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York. 1991.

[20] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *RAIRO Anal. Numer.*, Vol. 8, No. R-2, pp. 129-151, 1974.

[21] X. Cheng. On the nonlinear inexact Uzawa algorithm for saddle-point problems. *Siam J. Numer. Anal.*, 37:1930-1934, 2000.

[22] S. Dahlke, W. Dahmen and K. Urban. Adaptive Wavelet methods for saddle point problems-Optimal convergence rates. *Siam J. Numer. Anal.*, 40:1230-1262, 2002.

[23] H.C. Elman and G.H. Golub. Inexact and preconditioned Uzawa Algorithms for saddle point problems. *Siam J. Numer. Anal.*, 31:1645-1661, 1994.

[24] M. Fortin and R. Glowinski *Augmented Lagrangian Methods: Applications to the numerical solutions of boundary value problems*. Studies in Mathematics and Applications, Vol 15, North-Holland (1983).

[25] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, Berlin, 1986.

[26] Y. Kondratyuk and R. Stevenson. An Optimal Adaptive Finite Element Method for the Stokes Problem. *Siam J. Numer. Anal.*, 46:746-775, 2008.

[27] P. Monk. *Finite element methods for Maxwell's Equations*, Clarendon Press, Oxford, 2003.

[28] P. Morin, R.H. Nochetto and G. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal.*38(2):466-488, 2000.

[29] R.H. Nochetto and J. Pyo. Optimal relaxation parameter for the Uzawa Method. *Numer. Math.*, 98:695-702, 2004.

[30] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*, Springer, Berlin, 1994.

[31] R. Temam. *Navier-Stokes Equations*, North-Holland, 1984.

[32] R. Verfürth. *A Review of A Posterirori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1996.

[33] J. Xu and L. Zikatanov. Some observations on Babuška and Brezzi theories. *Numer. Math.*, 94(1):195-202, 2003.