RESIDUAL REDUCTION ALGORITHMS FOR NONSYMMETRIC SADDLE POINT PROBLEMS

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Abstract. In this paper, we introduce and analyze Uzawa algorithms for non-symmetric saddle point systems. Convergence for the algorithms is established based on new spectral results about Schur complements. A new Uzawa type algorithm with optimal relaxation parameters at each new iteration is introduced and analyzed in a general framework. Numerical results supporting the efficiency of the algorithms are presented for finite element discretization of steady state Navier-Stokes equations.

1. Introduction

We consider new Uzawa type anlysis and algorithms for solving non-symmetric saddle point systems. Typical problems where such systems appear are, for example, in finite element or finite difference discretization of steady state Navier-Stokes systems. The convergence results for the Uzawa type algorithms at the continuous level, presented in [4] suggest that more efficient algorithms can be developed based on the spectral properties of the two Schur complements associated with a connecting bilinear form which satisfies a Ladyshenskaya-Babusca-Brezzi (LBB) condition. For the symmetric case recent results about Uzawa and inexact Uzawa algorithms are presented in [3, 4, 14, 26, 27, 29]. In this paper we use spectral properties to prove a sharper result for an inexact Uzawa algorithm proposed in [18] and introduce and analyze a new inexact Uzawa algorithm for non-symmetric saddle point problems (SPP) based on efficient choice of the relaxation parameters.

The paper is organized as follows. In Section 2, we introduce the notation and review properties of the Schur operators. In Section 3 we prove the main convergence results about inexact Uzawa algorithms for non-symmetric SPP introduced in [18]. In Section 4 we introduce the new Inexact Uzawa Algorithm for non-symmetric SPP which has the advantage that the relaxation parameter for the residual of the constrained variable is optimal and the

2000 *Mathematics Subject Classification.* 74S05, 74B05, 65N22, 65N55.

Key words and phrases. Uzawa algorithms, non-symmetric saddle point system, multilevel methods, residual reduction.

The work of C. Bacuta was partially supported by NSF DMS-0713125. The work of B. McCracken was supported by REU NSF DMS-0713125, and the work of L. Shu was supported by University of Delaware's GEMS Project.

other relaxation parameter can be efficiently chosen to minimize the convergence factor of the algorithm. Implementable versions of the two algorithms introduced in Section 3 and Section 4 are discussed in Section 5. Numerical results for finite element discretization of the steady state Navier-Stokes system are presented in Section 6.

2. Schur Complements

In this section, we start with a review of the notation of the classical LBB theory and introduce natural operators and norms for the general abstract case.

We let $\mathbf{V}$ and Q be two Hilbert spaces with inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, with the corresponding induced norms $|\cdot|_{\mathbf{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$. The dual parings on $\mathbf{V}^* \times \mathbf{V}$ and $Q^* \times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $\mathbf{V}^*$ and Q^* denote the duals of $\mathbf{V}$ and Q, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate operators
$A_0 : V \to V^*$ and $C_0 : Q \to Q^*$ defined by

$$\langle A_0\mathbf{u}, \mathbf{v}\rangle = a_0(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

and

$$\langle C_0 p, q\rangle = (p, q) \quad \text{for all } \ p, q \in Q.$$

The operators ${A_0}^{-1} : V^* \to V$ and ${C_0}^{-1} : Q^* \to Q$ are called the Riesz-canonical isometries and satisfy

$$a_0({A_0}^{-1}\mathbf{u}^*, \mathbf{v}) = \langle \mathbf{u}^*, \mathbf{v}\rangle, \quad |{A_0}^{-1}\mathbf{u}^*|_{\mathbf{V}} = \|\mathbf{u}^*\|_{\mathbf{V}^*}, \mathbf{u}^* \in \mathbf{V}^*, \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

$$({C_0}^{-1}p^*, q) = \langle p^*, q\rangle, \quad \|{C_0}^{-1}p^*\| = \|p^*\|_{Q^*}, \ p^* \in Q^*, q \in Q. \tag{2.2}$$

Next, we consider that $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, satisfying the inf-sup condition,[2, 34] . More precisely, we assume that

$$\inf_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| \ |\mathbf{v}|} = m > 0 \tag{2.3}$$

and

$$\sup_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| \ |\mathbf{v}|} = M < \infty. \tag{2.4}$$

Here, and throughout this paper, the "inf" and "sup" are taken over nonzero vectors. With the form b, we associate the linear operators $B : V \to Q^*$ and $B^* : Q \to V^*$ defined by

$$\langle B\mathbf{v}, q\rangle = b(\mathbf{v}, q) = \langle B^* q, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}, \ q \in Q.$$

Let $\mathbf{V}_0$ be the kernel of B or ${C_0}^{-1}B$, i.e.,

$$\mathbf{V}_0 = Ker(B) = \{\mathbf{v} \in \mathbf{V} |\ B\mathbf{v} = 0\} = \{\mathbf{v} \in \mathbf{V} |\ {C_0}^{-1}B\mathbf{v} = 0\}.$$

Due to (2.4), $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$. Before we present the main result of this section, we review a few useful functional analysis results.

For a bounded linear operator $T : X \to Y$ between two Hilbert spaces X and Y, we denote by T^t the Hilbert transpose of T. When $X = Y$ and $T = T^t$, we say that T is symmetric (in the Hilbert sense). The spectrum of the operator $T : X \to X$ is denoted by $\sigma(T)$. The following lemma provides important properties of norms and operators to be used in this paper. A proof can be found in [4].

Lemma 2.1. *(Schur complements). Let A_0, C_0, B, and B^* be the operators associated with the spaces $\mathbf{V}, Q$ and the connecting form $b(\cdot,\cdot)$. Assume that* (2.3) *and* (2.4) *are satisfied.*

i) **The operators $C_0{}^{-1}B : \mathbf{V} \to Q$ and $A_0{}^{-1}B^* : Q \to \mathbf{V}$ are symmetric to each other**, *i.e.,*

$$(C_0{}^{-1}B\mathbf{v}, q) = a_0(\mathbf{v}, A_0{}^{-1}B^*q), \;\; \mathbf{v} \in \mathbf{V}, q \in Q, \tag{2.5}$$

consequently,

$$(C_0{}^{-1}B)^t = A_0{}^{-1}B^* \textit{and} \;\; (A_0{}^{-1}B^*)^t = C_0{}^{-1}B.$$

ii) **The Schur complement on Q** *is the operator $S_0 := C_0{}^{-1}BA_0{}^{-1}B^* : Q \to Q$. The operator S_0 is symmetric and positive definite on Q, satisfying*

$$(S_0p, p) = \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)^2}{|\mathbf{v}|^2}. \tag{2.6}$$

Consequently, $m^2, M^2 \in \sigma(S_0)$ and

$$\sigma(S_0) \subset [m^2, M^2]. \tag{2.7}$$

iii) **An orthogonal decomposition of V.** *The following estimate holds*

$$\|p\|_{S_0} := (S_0p, p)^{1/2} = |A_0{}^{-1}B^*p|_{\mathbf{V}} \geq m\|p\| \qquad \textit{for all} \;\; p \in Q. \tag{2.8}$$

Consequently, $A_0{}^{-1}B^* : Q \to \mathbf{V}$ has closed range, $\mathbf{V}_1 := A_0{}^{-1}B^*(Q)$ is a closed subspace of $\mathbf{V}$ and $A_0{}^{-1}B^* : Q \to \mathbf{V}_1$ is an isomorphism. Moreover, $\mathbf{V}_0 = Ker(C_0{}^{-1}B) = A_0{}^{-1}B^*(Q)^{\perp}$ and

$$\mathbf{V} = Ker(C_0{}^{-1}B) \oplus A_0{}^{-1}B^*(Q) = \mathbf{V}_0 \oplus \mathbf{V}_1.$$

iv) **The Schur complement on V** *is defined as the operator $S := A_0{}^{-1}B^*C_0{}^{-1}B : \mathbf{V} \to \mathbf{V}$. The operator S is symmetric and non-negative definite on $\mathbf{V}$, with $Ker(S) = \mathbf{V}_0$, $S(\mathbf{V}) = \mathbf{V}_1$, and satisfies*

$$a_0(S\mathbf{u}, \mathbf{v}) = (C_0{}^{-1}B\mathbf{u}, C_0{}^{-1}B\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}. \tag{2.9}$$

v) **The Schur complement on $\mathbf{V}_1 = \mathbf{V}_0^{\perp}$** *is the restriction of S to $\mathbf{V}_1$, i.e., $S_1 := A_0{}^{-1}B^*C_0{}^{-1}B : \mathbf{V}_1 \to \mathbf{V}_1$. The operator S_1 is symmetric and positive definite on $\mathbf{V}_1$, satisfying*

$$\sigma(S_1) = \sigma(S_0) \subset [m^2, M^2]. \tag{2.10}$$

vi) ${C_0}^{-1}B$ **is a double isometry.** *The following statements hold*

$$\|{C_0}^{-1}B\mathbf{u}_1\| = a_0(S_1\mathbf{u}_1, \mathbf{u}_1)^{1/2} := |\mathbf{u}_1|_{S_1} \geq m\ |\mathbf{u}_1|, \quad \mathbf{u}_1 \in \mathbf{V}_1, \tag{2.11}$$

and

$$\|{C_0}^{-1}B\mathbf{u}_1\|_{S_0^{-1}} := (S_0^{-1}{C_0}^{-1}B\mathbf{u}_1, {C_0}^{-1}B\mathbf{u}_1)^{1/2} = |\mathbf{u}_1|,\ \mathbf{u}_1 \in \mathbf{V}_1. \tag{2.12}$$

Consequently, ${C_0}^{-1}B$ *is an isometry from* $(\mathbf{V}_1, |\cdot|_{S_1})$ *to* Q*, and from* $\mathbf{V}_1$ *to* $(Q, \|\cdot\|_{S_0^{-1}})$.

vii) ${A_0}^{-1}B^*$ **is a double isometry.** *The following identity holds*

$$|{A_0}^{-1}B^*p|_{S_1^{-1}} := a_0(S_1^{-1}{A_0}^{-1}B^*p, {A_0}^{-1}B^*p)^{1/2} = \|p\|, \quad p \in Q. \tag{2.13}$$

Consequently, ${A_0}^{-1}B^*$ *is an isometry from* Q *to* $(\mathbf{V}_1, |\cdot|_{S_1^{-1}})$*, and from* $(Q, \|\cdot\|_{S_0})$ *to* $\mathbf{V}_1$.

3. Schur Complements and Uzawa Algorihms

The Uzawa algorithm for solving the Stokes system was first introduced in [1]. In this section, we present the relation between the Schur operators and the Uzawa type algorithms. Next, we consider that a bilinear form $a(\cdot,\cdot)$ is defined and bounded on $\mathbf{V}\times\mathbf{V}$ and that the form coerces the norm induced by the inner product $a_0(\cdot,\cdot)$ on $\mathbf{V}$, more precisely we assume,

$$|a(\mathbf{u},\mathbf{v})| \leq M_a|\mathbf{u}|\ |\mathbf{v}|,\ , \quad \text{for all } \mathbf{u} \in \mathbf{V}, \tag{3.1}$$

for a constant $M_a \geq 1$ and that

$$a(\mathbf{u},\mathbf{u}) \geq\ a_0(\mathbf{u},\mathbf{u}), \quad \text{for all } \mathbf{u} \in \mathbf{V}. \tag{3.2}$$

The form $a(\cdot,\cdot)$ might not be symmetric. With the form a, we associate the linear operator $A : V \to V^*$ defined by

$$\langle A\mathbf{u}, \mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{V}.$$

Let $b : \mathbf{V} \times Q \to \mathbb{R}$ be a bilinear form satisfying (2.3) and (2.4). For $f \in \mathbf{V}^*$, $g \in Q^*$, we consider the following variational problem:

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{array}{lcll} a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) & & & = \langle g,q\rangle \quad \text{for all } q \in Q. \end{array} \tag{3.3}$$

The problem (3.3) is equivalent to the following reformulation:

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{array}{lcll} A\mathbf{u} & + & B^*p & = \mathbf{f}, \\ B\mathbf{u} & & & = g. \end{array} \tag{3.4}$$

It is known that the above variational problem or system has a unique solution for any $f \in \mathbf{V}^*$, $g \in Q^*$ (see [20, 21, 25, 30, 31]).

The use of Schur complements turns out to be of practical interest in designing and analyzing Arrow-Hurvitz-Uzawa type algorithms for saddle point systems. In this section, we present the connection between the Schur complements and the Uzawa iterations.

Given a parameter $\alpha > 0$, called the relaxation parameter, the Uzawa algorithm for approximating the solution $(\mathbf{u}, p)$ of (3.3) can be described as follows.

Algorithm 3.1 (Uzawa Method (UM))**.** *Let p_0 be any approximation for p, and for $k = 1, 2, \ldots,$ construct $(\mathbf{u}_k, p_k)$ by*

$$\begin{aligned} \mathbf{u}_k &= A^{-1}(\mathbf{f} - B^* p_{k-1}), \\ p_k &= p_{k-1} + \alpha {C_0}^{-1}(B\mathbf{u}_k - g). \end{aligned} \tag{3.5}$$

The convergence of the **UM** is discussed for particular cases in many publications, see e.g., [14, 17, 20, 24, 25, 22, 23, 32]. The following theorem relates the convergence of the method and the Schur complements S_0 and S. A proof of the theorem can be found in [3].

Theorem 3.2. *Assume that $A = A_0$ and let $(\mathbf{u}, p)$ be the solution of* (3.3)*. If $(\mathbf{u}_k, p_k)$ is the sequence of approximations built by the UM* (3.5)*. Then, the following holds.*

(i) *The sequences $\mathbf{u} - \mathbf{u}_k$ and $p - p_k$ satisfy*

$$\mathbf{u} - \mathbf{u}_k = -A^{-1}B^*(p - p_{k-1}), \tag{3.6}$$

$$p - p_k = (I - \alpha S_0)(p - p_{k-1}), \tag{3.7}$$

$$\mathbf{u} - \mathbf{u}_k = (I - \alpha S)(\mathbf{u} - \mathbf{u}_{k-1}). \tag{3.8}$$

(ii) *For $\alpha < \frac{2}{M^2}$, the UM is convergent and the convergence factor is*

$$\|I - \alpha S_0\| = \max\{|1 - \alpha m^2|, |1 - \alpha M^2|\} < 1.$$

The optimal convergence factor is achieved for

$$\alpha_{opt} = \frac{2}{M^2 + m^2} \quad \text{and} \quad \|I - \alpha_{opt} S_0\| = \frac{M^2 - m^2}{M^2 + m^2}.$$

Next, we will introduce the abstract one step Uzawa algorithm for the non-symetric case. Let $(\mathbf{u}, p)$ be the solution of (3.3), and let α, β be arbitrary positive (relaxation) parameters. As presented in [18], for the finite dimensional case, an extension of the above Uzawa algorithm to the case when the form a is non-symmetric (but coercive on $\mathbf{V}$) is as follows.

Algorithm 3.3. Non-Symmetric Uzawa Method (NSUM). *Let $(\mathbf{u}_0, p_0)$ be any approximation for $(\mathbf{u}, p)$, and for $k = 1, 2, \cdots,$ construct $(\mathbf{u}_k, p_k)$ by*

$$\begin{aligned} \mathbf{u}_k &= \mathbf{u}_{k-1} + \beta {A_0}^{-1}(\mathbf{f} - A\mathbf{u}_{k-1} - B^* p_{k-1}), \\ p_k &= p_{k-1} + \alpha {C_0}^{-1}(B\mathbf{u}_k - g). \end{aligned} \tag{3.9}$$

The following lemma will be used in analyzing the **(NSUM)**.

Lemma 3.4. *Assume that the form $a(\cdot,\cdot)$ satisfies* (3.1)*and* (3.2)*. Then,*

$$|(I-\beta {A_0}^{-1}A)\mathbf{v}|^2_{\mathbf{V}} \le (1-2\beta+\beta^2 M_a^2)|\mathbf{v}|^2_{\mathbf{V}}, \qquad \text{for all } \mathbf{v}\in\mathbf{V}. \tag{3.10}$$

If β satisfies the following restriction,

$$0<\beta<\frac{2}{M_a^2}, \tag{3.11}$$

then,

$$\|I-\beta {A_0}^{-1}A\| \le (1-2\beta+\beta^2 M_a^2)^{1/2} < 1. \tag{3.12}$$

Proof. For (3.10) we observe that

$$a_0({A_0}^{-1}A\mathbf{v},\mathbf{v}) = \langle A\mathbf{v},\mathbf{v}\rangle = a(\mathbf{v},\mathbf{v}) \ge \ a_0(\mathbf{v},\mathbf{v}),$$

and

$$a_0({A_0}^{-1}A\mathbf{v},{A_0}^{-1}A\mathbf{v}) = |{A_0}^{-1}A\mathbf{v}|^2_{\mathbf{V}} = |A\mathbf{v}|^2_{\mathbf{V}^*} \le \ M_a^2\ a_0(\mathbf{v},\mathbf{v}).$$

The estimate (3.12) follows from (3.10) and the assumption (3.11). □

Before we study the stability and convergence rate of Algorithm 3.3, we shall introduce the following notation. For $k=0,1,\ldots$, let $e_k^{\mathbf{u}} := \mathbf{u}-\mathbf{u}_k$, and $e_k^p := p-p_k$. We also denote the operator $(I-\beta {A_0}^{-1}A):\mathbf{V}\to\mathbf{V}$ by R and let $\gamma := \|R\| = \|I-\beta {A_0}^{-1}A\|$.

Theorem 3.5. *Assume that β satisfies* (3.11) *and α satisfies*

$$0<\alpha<\frac{1-\gamma}{\beta}\ \frac{2}{M^2}. \tag{3.13}$$

Then, the **(NSUM)** *converges. There exists $\rho=\rho(\alpha,\beta,\gamma,m,M)\in(0,1)$ such that*

$$\left(\frac{\gamma}{\beta}|e_k^{\mathbf{u}}|^2+\frac{1}{\alpha}\|e_k^p\|^2\right)^{1/2} \le \rho^k\ \left(\frac{\gamma}{\beta}|e_0^{\mathbf{u}}|^2+\frac{1}{\alpha}\|e_0^p\|^2\right)^{1/2},\quad k=1,2,\ldots. \tag{3.14}$$

The optimal convergence factor is achieved for

$$\alpha_{opt} := \frac{1-\gamma}{\beta}\ \frac{2}{m^2+M^2},$$

and defining $f(x)=\frac{1}{2}(x+\sqrt{x^2+4\gamma})$, we have that

$$\rho_{opt}(T) = f(x_{opt}),\ \text{where } x_{opt} := (1-\gamma)\frac{M^2-m^2}{M^2+m^2}.$$

Proof. From the first equation of (3.4) and the first equation of (3.9) we have

$$\begin{aligned} e_k^{\mathbf{u}} &= e_{k-1}^{\mathbf{u}} - \beta {A_0}^{-1}(Ae_{k-1}^{\mathbf{u}} + B^* e_{k-1}^p)\\ &= (I-\beta {A_0}^{-1}A)e_{k-1}^{\mathbf{u}} - {A_0}^{-1}B^*e_{k-1}^p. \end{aligned} \tag{3.15}$$

From the second equation of (3.4) and the second equation of (3.9), we get

$$e^p_k = e^p_{k-1} + \alpha {C_0}^{-1} B e^{\mathbf{u}}_k. \tag{3.16}$$

If we substitute $e^{\mathbf{u}}_k$ from (3.15) in (3.16), then

$$e^p_k = \alpha {C_0}^{-1} B (I - \beta {A_0}^{-1} A) e^{\mathbf{u}}_{k-1} + (I - \alpha\beta {C_0}^{-1} B {A_0}^{-1} B^*) e^p_{k-1} \tag{3.17}$$

and

$$\begin{aligned} {A_0}^{-1} B^* e^p_k = & \alpha {A_0}^{-1} B^* {C_0}^{-1} B (I - \beta {A_0}^{-1} A) e^{\mathbf{u}}_{k-1} + \\ & (I - \alpha\beta {A_0}^{-1} B^* {C_0}^{-1} B) {A_0}^{-1} B^* e^p_{k-1}. \end{aligned} \tag{3.18}$$

With the notation of Section 2, we have ${A_0}^{-1} B^* {C_0}^{-1} B = S$, $\mathbf{V}_1 = A^{-1} B^*(Q)$, and $S_1 : \mathbf{V}_1 \to \mathbf{V}_1$ is the restriction of S to $\mathbf{V}_1$. We will need also $S_{12} : \mathbf{V}_1 \to \mathbf{V}$, $S_{12}\mathbf{v}_1 = S\mathbf{v}_1$ and $S_{21} : \mathbf{V} \to \mathbf{V}_1$, $S_{21}\mathbf{v} = S\mathbf{v}$.

Then, from (3.15) and (3.18), we obtain

$$\begin{pmatrix} e^{\mathbf{u}}_k \\ {A_0}^{-1} B^* e^p_k \end{pmatrix} = \begin{pmatrix} I & \beta I_{12} \\ \alpha S_{21} & -I_1 + \alpha\beta\ S_1 \end{pmatrix} \begin{pmatrix} (I - \beta {A_0}^{-1} A) e^{\mathbf{u}}_{k-1} \\ -{A_0}^{-1} B^* e^p_{k-1} \end{pmatrix},$$

where $I_{12} : \mathbf{V}_1 \to \mathbf{V}$ is the inclusion operator. Applying S_1^{-1} to the second component, and using just elementary manipulation we get

$$\begin{pmatrix} \frac{1}{\beta} e^{\mathbf{u}}_k \\ \frac{1}{\alpha} S_1^{-1} {A_0}^{-1} B^* e^p_k \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} I & I_{12} \\ S_1^{-1} S_{21} & -\frac{1}{\alpha} S_1^{-1} + \beta\ I_1 \end{pmatrix} \begin{pmatrix} R e^{\mathbf{u}}_{k-1} \\ -{A_0}^{-1} B^* e^p_{k-1} \end{pmatrix},$$

or

$$\begin{pmatrix} \frac{1}{\beta} I & 0 \\ 0 & \frac{1}{\alpha} S_1^{-1} \end{pmatrix} \begin{pmatrix} \gamma^{1/2} e^{\mathbf{u}}_k \\ {A_0}^{-1} B^* e^p_k \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{\beta} I & \gamma^{1/2} I_{12} \\ \gamma^{1/2} S_1^{-1} S_{21} & -\frac{1}{\alpha} S_1^{-1} + \beta\ I_1 \end{pmatrix} \begin{pmatrix} \gamma^{-1/2} R e^{\mathbf{u}}_{k-1} \\ -{A_0}^{-1} B^* e^p_{k-1} \end{pmatrix}.$$

On the Hilbert space $\mathbf{V} \times \mathbf{V}_1$ we consider, besides the standard inner product, the inner product defined by the symmetric and positive definite operator $\mathcal{N} := \begin{pmatrix} \frac{1}{\beta} I & 0 \\ 0 & \frac{1}{\alpha} S_1^{-1} \end{pmatrix}$ viewed as an operator on $\mathbf{V} \times \mathbf{V}_1$, i.e.,

$$\left\langle \begin{pmatrix} \mathbf{u} \\ \mathbf{u}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \end{pmatrix} \right\rangle_{\mathcal{N}} := \left\langle \mathcal{N} \begin{pmatrix} \mathbf{u} \\ \mathbf{u}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \end{pmatrix} \right\rangle_{\mathbf{V} \times \mathbf{V}_1} = \frac{1}{\beta} a_0(\mathbf{u}, \mathbf{v}) + \frac{1}{\alpha} a_0(S_1^{-1} \mathbf{u}_1, \mathbf{v}_1).$$

Let

$$\mathcal{M} := \begin{pmatrix} \frac{\gamma}{\beta} I & \gamma^{1/2} I_{12} \\ \gamma^{1/2} S_1^{-1} S_{21} & -\frac{1}{\alpha} S_1^{-1} + \beta\ I_1 \end{pmatrix},\ E_k := \begin{pmatrix} \gamma^{1/2} e^{\mathbf{u}}_k \\ {A_0}^{-1} B^* e^p_k \end{pmatrix},$$

and $F_{k-1} := \begin{pmatrix} \gamma^{-1/2} R e^{\mathbf{u}}_{k-1} \\ -{A_0}^{-1} B^* e^p_{k-1} \end{pmatrix}$. The above identity becomes

$$\mathcal{N}\ E_k = \mathcal{M}\ F_{k-1}. \tag{3.19}$$

Note that $S_1^{-1} S_{21} : \mathbf{V} \to \mathbf{V}_1$ agrees with the orthogonal projection $P_{21} : \mathbf{V} \to \mathbf{V}_1$, and that P_{21} is the transpose of the inclusion $I_{12} : \mathbf{V}_1 \to \mathbf{V}$, i.e.,

$$a_0(P_{21}\mathbf{v}, \mathbf{v}_1) = a_0(\mathbf{v}, I_{12}\mathbf{v}_1),\ \mathbf{v} \in \mathbf{V}, \mathbf{v}_1 \in \mathbf{V}_1.$$

This leads to the fact that $T := \mathcal{N}^{-1}\mathcal{M}$ is a symmetric operator on $\mathbf{V} \times \mathbf{V}_1$ with respect to the $\mathcal{N}$-inner product. Then, using (3.19) we have

$$\begin{aligned}\|E_k\|_{\mathcal{N}}^2 &= \langle \mathcal{N} E_k, E_k\rangle_{\mathbf{V}\times\mathbf{V}_1} = \langle \mathcal{M} F_{k-1}, E_k\rangle_{\mathbf{V}\times\mathbf{V}_1} \\ &= \left\langle \mathcal{N}^{-1}\mathcal{M}\ F_{k-1}, E_k\right\rangle_{\mathcal{N}} \le \rho(T)\|F_{k-1}\|_{\mathcal{N}}\|E_k\|_{\mathcal{N}},\end{aligned}$$

which leads to

$$\|E_k\|_{\mathcal{N}} \le \rho(T)\|F_{k-1}\|_{\mathcal{N}}. \tag{3.20}$$

On the other hand, using Lemma 2.1 (iv),

$$\|E_k\|_{\mathcal{N}}^2 = \left\|\begin{pmatrix}\gamma^{1/2}\ e_k^{\mathbf{u}} \\ A_0{}^{-1}B^* e_k^p\end{pmatrix}\right\|_{\mathcal{N}}^2 = \frac{\gamma}{\beta}|e_k^{\mathbf{u}}|_{\mathbf{V}}^2 + \frac{1}{\alpha}|A_0{}^{-1}B^* e_k^p|_{S_1^{-1}}^2 = \frac{\gamma}{\beta}|e_k^{\mathbf{u}}|^2 + \frac{1}{\alpha}\|e_k^p\|^2,$$

and

$$\begin{aligned}\|F_{k-1}\|_{\mathcal{N}}^2 &= \left\|\begin{pmatrix}\gamma^{-1/2} R\ e_{k-1}^{\mathbf{u}} \\ -A_0{}^{-1}B^* e_{k-1}^p\end{pmatrix}\right\|_{\mathcal{N}}^2 = \frac{\gamma^{-1}}{\beta}|R\ e_{k-1}^{\mathbf{r}}|_{\mathbf{V}}^2 + \frac{1}{\alpha}|A_0{}^{-1}B^* e_{k-1}^p|_{S_1^{-1}}^2 \\ &\le \frac{\gamma^{-1}}{\beta}\|R\|^2|e_{k-1}^{\mathbf{u}}|_{\mathbf{V}}^2 + \frac{1}{\alpha}\|e_k^p\|^2 = \frac{\gamma}{\beta}|e_{k-1}^{\mathbf{u}}|^2 + \frac{1}{\alpha}\|e_{k-1}^p\|^2.\end{aligned}$$

Thus, (3.20) gives

$$\left(\frac{\gamma}{\beta}|e_k^{\mathbf{u}}|^2 + \frac{1}{\alpha}\|e_k^p\|^2\right)^{1/2} \le \rho(T)\ \left(\frac{\gamma}{\beta}|e_{k-1}^{\mathbf{u}}|^2 + \frac{1}{\alpha}\|e_{k-1}^p\|^2\right)^{1/2},$$

where $\rho(T)$ is the spectral radius of $T = \mathcal{N}^{-1}\mathcal{M} = \begin{pmatrix}\gamma I & \gamma^{1/2}\beta I_{12} \\ \gamma^{1/2}\alpha S_{21} & -I_1 + \alpha\beta\ S_1\end{pmatrix}$.

To complete the proof, we have to show that $\rho(T) < 1$ provided that (3.11) and (3.13) hold. First we will prove that any eigenvalue $\rho \in \sigma(T)$, the spectrum of T, corresponds to a value $\lambda \in [m^2, M^2]$ and a relation between ρ and λ holds, see (3.22). Then, as done in the proof of Theorem 4.4 of [4], one can prove that the same relation remains valid in the general case when $\rho \in \sigma(T)$ and ρ is not necessarily an eigenvalue. Finally, we will show that, under the assumptions of the theorem, (3.22) implies $\rho(T) < 1$.

Let us consider now that ρ is an eigenvalue of T, and let $\begin{pmatrix}x \\ y\end{pmatrix} \in \mathbf{V} \times \mathbf{V}_1$ be a corresponding eigenvector. Then,

$$T\begin{pmatrix}x \\ y\end{pmatrix} = \rho\begin{pmatrix}x \\ y\end{pmatrix},$$

which lads to

$$\begin{aligned}\gamma\ x \quad &+ \quad \beta\gamma^{1/2} y &= \rho x, \\ \gamma^{1/2}\alpha S x \quad &+ \quad (-I + \alpha\beta S) y &= \rho y.\end{aligned}$$

Equivalently,

$$\begin{aligned}\beta\gamma^{1/2}\ y \quad &= (\rho - \gamma)\ x, \\ S(\gamma^{1/2}\alpha\ x + \alpha\beta\ y) \quad &= (\rho + 1) y.\end{aligned} \tag{3.21}$$

One can easily see from the above system that if $x \in \mathbf{V}_0, x \neq 0$, then $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an eigenvector for T corresponding to $\rho = \gamma$. Thus, $\gamma \in \sigma(T)$. If $\rho \neq \gamma$, then from (3.21) we have that

$$x = \frac{\beta\gamma^{1/2}}{\rho - \gamma}\ y, \text{ with } y \neq 0, \text{ and } S_1\ y = \frac{(\rho-\gamma)(\rho+1)}{\rho\alpha\beta}\ y.$$

From Lemma 2.1 (iv), we deduce that

$$\lambda = \frac{(\rho-\gamma)(\rho+1)}{\rho\alpha\beta} \in [m^2, M^2].$$

or,

$$\rho^2 - (\alpha\beta\lambda + \gamma - 1)\rho - \gamma = 0, \text{ with } \lambda \in [m^2, M^2]. \tag{3.22}$$

Using the real functions $f(x) = \frac{1}{2}(x + \sqrt{x^2+4\gamma})$ and $g(x) = \frac{1}{2}(x - \sqrt{x^2+4\gamma}) = -f(-x)$, the roots of (3.22) are

$$\rho_1(\lambda) := f(\alpha\beta\lambda + \gamma - 1), \quad \rho_2(\lambda) := g(\alpha\beta\lambda + \gamma - 1),$$

and we have

$$\rho(T) \leq \max\left\{\gamma, \sup_{\lambda\in[m^2,M^2]} |\rho_1(\lambda)|, \sup_{\lambda\in[m^2,M^2]} |\rho_2(\lambda)|\right\}. \tag{3.23}$$

Since f is increasing and positive function on $\mathbb{R}$, and g is increasing and negative function on $\mathbb{R}$, we get

$$\sup_{\lambda\in[m^2,M^2]} |\rho_1(\lambda)| = f(\alpha\beta M^2 + \gamma - 1),$$

and

$$\sup_{\lambda\in[m^2,M^2]} |\rho_2(\lambda)| = -g(\alpha\beta m^2 + \gamma - 1).$$

Moreover, elementary calculations show that

$$0 \leq \gamma < f(\alpha\beta M^2 + \gamma - 1),$$

and for any eigenvalue $\lambda \in [m^2, M^2]$ of S_1 we have that $\rho_1(\lambda), \rho_2(\lambda)$ are eigenvalues for T. In particular, $f(\alpha\beta M^2 + \gamma - 1)$ and $g(\alpha\beta m^2 + \gamma - 1)$ are eigenvalues of T. Therefore

$$\rho(T) = \max\left\{f(\alpha\beta M^2 + \gamma - 1), -g(\alpha\beta m^2 + \gamma - 1)\right\}. \tag{3.24}$$

Using the monotonicity of the two functions f and g, it is easy to verify the following:

$$0 < -g(\alpha\beta m^2 + \gamma - 1) < 1, \quad \text{for all } \alpha > 0,$$

$$0 < f(\alpha\beta M^2 + \gamma - 1) < 1, \text{ iff } \alpha < \frac{1-\gamma}{\beta}\ \frac{2}{M^2}.$$

We note here that, due to the assumption (3.11), from Lemma 3.4, we have $0 < 1-\gamma \leq 1$. By studding the optimality of $\rho(T)$ as function of α we obtain that $\rho(T)(\alpha)$ is optimal (minimal) for

$$\alpha_{opt} := \frac{1-\gamma}{\beta} \, \frac{2}{m^2+M^2}.$$

For $0 < \alpha \leq \alpha_{opt}$, we have $\rho(T) = -g(\alpha\beta m^2 + \gamma - 1)$, and for $\alpha_{opt} \leq \alpha < \frac{1-\gamma}{\beta} \, \frac{2}{M^2}$, we have $\rho(T) = f(\alpha\beta M^2 + \gamma - 1)$.

Since $g(x) = -f(-x)$ on $[\gamma-1, 1-\gamma]$, it is easy to also notice that

$$\rho_{opt}(T) = f(x_{opt}), \text{ where } x_{opt} := (1-\gamma)\, \frac{M^2-m^2}{M^2+m^2}, \tag{3.25}$$

and that for $\gamma = 0$ we have $\rho_{opt}(T) = x_{opt} = \frac{M^2-m^2}{M^2+m^2}$. This completes the proof of the theorem. □

Using the spectral properties of the Schur complement S_1, we were able to compute the spectral radius of the error operator T. In addition, we can see that for $A_0 = A$ and $\beta = 1$ we have $\gamma = 0$ and the convergence result of Theorem 3.2 is recovered. Thus, if comparing with the symmetric case, the convergence result of the above theorem is optimal. Another advantage of **NSUM** is that only two SPD operators have to be inverted at each step. To insure convergence of **NSUM**, the parameters α and β can be chosen independent of the inf-sup constant m which might be difficult to estimate. One natural choice for β is $\beta := \frac{1}{M_a^2}$. Then, for $\gamma_0 := (1-2\beta+\beta^2 M_a^2)^{1/2}$, a convenient choice for α is $\alpha_0 := \frac{1-\gamma_0}{\beta_0} \, \frac{1}{M^2}$. The theorem can be applied for finite or infinite dimensional spaces. For the finite dimensional case, a similar result with a different estimate for the convergence factor is studied in [18] by Bramble Pasciak and Vassilev. The similar algorithm presented in [18] replaces ${A_0}^{-1}$ and ${C_0}^{-1}$ by preconditioners for A_0 and C_0, respectively. Their case can be reduced to our analysis presented in the above proof by redefining the inner products on the two discrete spaces $\mathbf{V} = \mathbf{V}_h$ and $Q = Q_h$.

4. A Residual Reduction Algorithm

In the previous section we presented an optimal result for the convergence of non-symmetric Uzawa algorithm. By the best knowledge of the authors, the result is new for the infinite dimensional case. Concerning the applicability of Theorem 3.5, we note that if A is not symmetric, and $M_a \approx 1/\nu$ with ν a small positive number, then the best estimate for $(1-2\beta+\beta^2 M_a^2)^{1/2}$ is obtained for $\beta = \nu^2$ and in this this case $\gamma \approx \sqrt{1-\nu^2}$. Choosing the best choice for alpha $\alpha = \alpha_{opt}$, leads to a convergence factor $\rho_{opt}(T)$ strictly bigger than γ, which is close to one, see (3.25). For example, for $\nu = 0.1$ we get $\rho_{opt}(T) > \gamma \approx 0.995$. Such situations appear, for example, in approximating the solution of the steady state Navier Stokes equations with viscosity ν by Picard iteration. Thus, the algorithm might not be efficient for the case when M_a is large. In this section we will come up with a way to improve the estimate for the convergence factor of the algorithm. The main idea is to analyze the **one step** reduction of the residuals produced by the **NSUM** ((3.9)) and choose optimal values $\beta = \beta_k$ and $\alpha = \alpha_k$ such that a certain residual reduction factor is minimal.

Let us consider again the algorithm (3.9), and for $k = 0, 1, 2\cdots$, define

$$\mathbf{w}_k := A_0^{-1}(\mathbf{f} - A\mathbf{u}_{k-1} - B^* p_{k-1}) \quad \text{and} \quad q_k := {C_0}^{-1}(B\mathbf{u}_k - g).$$

Before presenting the main result of this section we introduce the following notation. For any non-zero $\mathbf{w} \in \mathbf{V}$ we let $\mathbf{u_w} := A_0^{-1} A\mathbf{w}$, i.e., $\mathbf{u_w}$ satisfies

$$a_0(\mathbf{u_w}, \mathbf{v}) = a(\mathbf{w}, \mathbf{v}), \qquad \text{for all } \mathbf{v} \in \mathbf{V}.$$

Using (3.2), we have that

$$a_0(\mathbf{u_w}, \mathbf{w}) = a(\mathbf{w}, \mathbf{w}) \geq a_0(\mathbf{w}, \mathbf{w}). \tag{4.1}$$

We also let $\beta = \beta(\mathbf{w})$ be the minimizer of $|\mathbf{w} - \beta\ \mathbf{u_w}|$, over all real numbers β. It is easy to see that $\beta = \frac{a_0(\mathbf{w},\mathbf{u_w})}{a_0(\mathbf{u_w},\mathbf{u_w})}$, and that

$$\gamma = \gamma_{\mathbf{w}} = \frac{|(I - \beta A_0^{-1} A)\mathbf{w}|}{|\mathbf{w}|} = \frac{|\mathbf{w} - \beta\ \mathbf{u_w}|}{|\mathbf{w}|} = \sqrt{1 - \frac{a_0(\mathbf{w}, \mathbf{u_w})^2}{a_0(\mathbf{w}, \mathbf{w})\ a_0(\mathbf{u_w}, \mathbf{u_w})}} < 1.$$

Theorem 4.1. *Assume that $\mathbf{u}_{k+1}$ and p_{k+1} are computed via (3.9) with*

$$\beta = \beta_k := \frac{a_0(\mathbf{w}_k, \mathbf{u}_{\mathbf{w}_k})}{a_0(\mathbf{u}_{\mathbf{w}_k}, \mathbf{u}_{\mathbf{w}_k})}, \ \text{ and } \ \alpha = \alpha_k < \frac{1-\gamma}{\beta_k}\ \frac{2}{M^2}, \tag{4.2}$$

where

$$\gamma = \gamma_k := \frac{|\mathbf{w}_k - \beta_k\ \mathbf{u}_{\mathbf{w}_k}|}{|\mathbf{w}_k|} = \sqrt{1 - \frac{a_0(\mathbf{w}_k, \mathbf{u}_{\mathbf{w}_k})^2}{a_0(\mathbf{w}_k, \mathbf{w}_k)\ a_0(\mathbf{u}_{\mathbf{w}_k}, \mathbf{u}_{\mathbf{w}_k})}} < 1. \tag{4.3}$$

Then, there exists $\rho_k = \rho_k(\alpha_k, \beta_k, \gamma_k, m, M) \in (0, 1)$ such that

$$\left(\frac{\gamma}{\alpha}|\mathbf{w}_{k+1}|^2 + \frac{1}{\beta}\|q_{k+1}\|^2\right)^{1/2} \leq \rho_k\ \left(\frac{\gamma}{\alpha}|\mathbf{w}_k|^2 + \frac{1}{\beta}\|q_k\|^2\right)^{1/2}. \tag{4.4}$$

The optimal residual reduction factor (in the above weighted norm) is achieved for

$$\alpha_{opt} := \frac{1-\gamma_k}{\beta_k}\ \frac{2}{m^2 + M^2}.$$

Proof. From the first equation of (3.4), the first equation of (3.9) and the definition of $\mathbf{w}_k$, we have

$$\mathbf{w}_{k+1} = (I - \beta {A_0}^{-1} A)\mathbf{w}_k - \alpha {A_0}^{-1} B^* q_k. \tag{4.5}$$

From the second equation of (3.4), the second equation of (3.9), and the definition of q_k, we get

$$q_{k+1} = q_k + \beta\ {C_0}^{-1} B\mathbf{w}_{k+1}. \tag{4.6}$$

If we substitute $\mathbf{w}_{k+1}$ from (4.5) in (4.6), then

$$q_{k+1} = \beta {C_0}^{-1} B(I - \beta {A_0}^{-1} A)\mathbf{w}_k + (I - \alpha\beta {C_0}^{-1} B {A_0}^{-1} B^*) q_k, \tag{4.7}$$

and

$$\begin{aligned} {A_0}^{-1} B^* q_{k+1} =& \beta {A_0}^{-1} B^* {C_0}^{-1} B(I - \beta {A_0}^{-1} A)\mathbf{w}_k + \\ & (I - \alpha\beta {A_0}^{-1} B^* {C_0}^{-1} B) {A_0}^{-1} B^* q_k. \end{aligned} \tag{4.8}$$

With the notation of the previous section, from (4.5) and (4.8), we obtain

$$\begin{pmatrix} \mathbf{w}_{k+1} \\ {A_0}^{-1} B^* q_{k+1} \end{pmatrix} = \begin{pmatrix} I & \alpha I_{12} \\ \beta S_{21} & -I_1 + \alpha\beta\ S_1 \end{pmatrix} \begin{pmatrix} (I - \beta {A_0}^{-1} A)\mathbf{w}_k \\ -{A_0}^{-1} B^* q_k \end{pmatrix},$$

Applying S_1^{-1} to the second component, and using just elementary manipulation we get

$$\begin{pmatrix} \frac{1}{\alpha}\mathbf{w}_{k+1} \\ \frac{1}{\beta} S_1^{-1} {A_0}^{-1} B^* q_{k+1} \end{pmatrix} = \begin{pmatrix} 1/\alpha\ I & I_{12} \\ S_1^{-1} S_{21} & -\frac{1}{\beta} S_1^{-1} + \alpha\ I_1 \end{pmatrix} \begin{pmatrix} (I - \beta {A_0}^{-1} A)\mathbf{w}_k \\ -{A_0}^{-1} B^* q_k \end{pmatrix}.$$

By noticing that $|(I - \beta {A_0}^{-1} A)\mathbf{w}_k| = \gamma\ |\mathbf{w}_k|$, and that the operator relating the residuals at consecutive steps is similar (the rolls of α and β is swapped) with the error operator of Theorem 3.5, the conclusion of the theorem follows immediately by similar arguments that are presented in details in the proof of Theorem 3.5. □

The following multistep algorithm takes advantage of optimal residual reduction at each new step.

Algorithm 4.2. Residual Reduction Method (RRM)

Let $\mathbf{u}_0 \in \mathbf{V}$, $p_0 \in Q$, *be any initial guess.*

Step 0. *Compute* $\mathbf{w}_0 \in \mathbf{V}$ *as the solution of*

$$a_0(\mathbf{w}_0, \mathbf{v}) = < f, \mathbf{v} > - a(\mathbf{u}_0, v) - b(\mathbf{v}, p_0),\ \ \mathbf{v} \in \mathbf{V}$$

For $k = 1, 2, \cdots, K_{max}$

Step 1. *Compute* $\mathbf{u}_{\mathbf{w}_0} \in \mathbf{V}$ *as the solution of*

$$a_0(\mathbf{u}_{\mathbf{w}_0}, \mathbf{v}) = a(\mathbf{w}_0, v),\ \ \mathbf{v} \in \mathbf{V}.$$

Step 2. *Compute*

$$\beta = \frac{a_0(\mathbf{w}_0, \mathbf{u}_{\mathbf{w}_0})}{a_0(\mathbf{u}_{\mathbf{w}_0}, \mathbf{u}_{\mathbf{w}_0})},\ \ \gamma = \sqrt{1 - \beta\ \frac{a_0(\mathbf{w}_0, \mathbf{u}_{\mathbf{w}_0})}{a_0(\mathbf{w}_0, \mathbf{w}_0)}}.$$

and choose positive α *s.t.* $\alpha < \frac{1-\gamma}{\beta}\ \frac{2}{M^2}$.

Step 3. *Compute* $\mathbf{u}_1 = \mathbf{u}_0 + \beta\mathbf{w}_0$, $q_1 = {C_0}^{-1}(B\mathbf{u}_1 - g)$, $p_1 = p_0 + \alpha q_1$.

Step 4. *Compute* $\mathbf{u}_{q_1} \in \mathbf{V}$ *as the solution of*

$$a_0(\mathbf{u}_{q_1}, \mathbf{v}) = b(\mathbf{v}, q_1), \ \mathbf{v} \in \mathbf{V}.$$

Step 5. *Compute the* $\mathbf{w}$*- residual for the next iteration:*

$$\mathbf{w}_1 = \mathbf{w}_0 - \beta \mathbf{u}_{\mathbf{w}_0} - \alpha \mathbf{u}_{q_1}.$$

Step 6. *Set* $\mathbf{u}_0 = \mathbf{u}_1, \ p_0 = p_1, \ \mathbf{w}_0 = \mathbf{w}_1$.
End

Based on Theorem 4.1, if $\alpha > \alpha_0 > 0$ and $\beta > \beta_0 > 0$ at each step of **RRM** we can conclude that the algorithm converges if K_{max} tends to ∞. We notice that in the particular case that $a(\mathbf{v}, \mathbf{v}) = a_0(\mathbf{v}, \mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$, due to (4.1), we have that $\gamma = \sqrt{1-\beta}$ at each step of the Algorithm 4.2. If information about m is available, then we can choose at each step $\alpha = \alpha_{opt} = \frac{1-\gamma}{\beta} \frac{2}{m^2+M^2}$. Other convenient choices for α are $\alpha = \frac{1-\gamma}{\beta} \frac{1}{M^2}$ or $\alpha = \gamma\beta$.

5. Discretization

The two algorithms for approximating the solution of a non-symmetric saddle point problem that we introduced and analyzed in the previous sections are presented in the abstract case, which also includes the infinite dimensional (or continuous case). The two algorithms are applicable if the actions of ${A_0}^{-1}$ and ${C_0}^{-1}$ are available. In this section we will modify the two algorithms by replacing ${A_0}^{-1}$ and ${C_0}^{-1}$ by discrete approximation and computable operators. Even though, we can apply the abstract theory to the case of finite dimensional spaces $\mathbf{V} = \mathbf{V}_h$ and $Q = Q_h$, we prefer to present the new algorithms as modifications of the corresponding algorithms for the continuous case. The advantage of such approach is that a discrete inf-sup condition is *automatically satisfied.*

Let $\mathbf{V}_h \subset \mathbf{V}$ be a good approximation discrete space and let $Q_h := R_h {C_0}^{-1} B \mathbf{V}_h$, where R_h is the orthogonal projection onto a discrete subspace $\bar{Q}_h$ of Q, i.e., $R_h : Q \to \bar{Q}_h$ satisfies

$$(R_h \ p_h, q_h) = (p_h, q_h), \qquad \text{for all } q_h \in \bar{Q}_h.$$

We consider the restrictions of the forms a_0, a and b to the corresponding pairs of discrete spaces and on each discrete subspace we consider the corresponding induced inner product. The discrete operators $A_{h,0}, C_{h,0}, B_h$, and A_h are defined similarly. For example, $A_{h,0}$ is the discrete version of A_0, and $A_{h,0} : \mathbf{V}_h \to \mathbf{V}_h^*$ is defined by

$$\langle A_{h,0} \ \mathbf{u}_h, \mathbf{v}_h \rangle = a_0(\mathbf{u}_h, \mathbf{v}_h), \quad \text{for all } \mathbf{u}_h \in \mathbf{V}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

The operator A_h, the discrete version of A, is defined in a similar way.

If we identify the dual of Q_h with itself, then $C_{h,0}$ is the identity on Q_h and for any $q_h \in Q_h, \mathbf{v}_h \in \mathbf{V}_h$ we have

$$(B_h \mathbf{v}_h, q_h) = b(\mathbf{v}_h, q_h) = ({C_0}^{-1} B \mathbf{v}_h, q_h) = (R_h {C_0}^{-1} B \mathbf{v}_h, q_h).$$

Thus, $B_h \mathbf{v}_h = R_h {C_0}^{-1} B \mathbf{v}_h$ for all $\mathbf{v}_h \in \mathbf{V}_h$. This implies that B_h is onto Q_h and, using that $\mathbf{V}_h$ and Q_h are finite dimensional spaces, a discrete inf-sup condition holds, i.e., there exists $m_h > 0$ such that

$$\inf_{p_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|p_h\| \, |\mathbf{v}_h|} = m_h > 0. \tag{5.1}$$

We also assume that

$$\sup_{p_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|p_h\| \, |\mathbf{v}_h|} = M_h \le M. \tag{5.2}$$

Thus, the problem: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{array}{llll} a(\mathbf{u}_h, \mathbf{v}_h) & + \quad b(\mathbf{v}_h, p_h) & = \langle \mathbf{f}, \mathbf{v}_h \rangle & \text{for all } \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}, q_h) & & = \langle g, q_h \rangle & \text{for all } q_h \in Q_h. \end{array} \tag{5.3}$$

has unique solution $(\mathbf{u}_h, p_h)$ and, under further assumptions for the discrete spaces, the discrete solution $(\mathbf{u}_h, p_h)$ approaches the solution $(\mathbf{u}, p)$ of the continuous problem (3.3).

Next, we present the discrete versions of **NSUM** and **RRM**. For the first algorithm we assume that β and α are chosen such that $\beta \in (0, \frac{2}{M_a^2})$ and $\alpha \in \left(0, \frac{1-\gamma}{\beta} \frac{2}{M_h^2}\right)$, where $\gamma := \|I - \beta A_{h,0}^{-1} A_h\|$. A discrete (implementable) version of **NSUM** is:

Algorithm 5.1. NSUM-h.

Let $\mathbf{u}_0 \in \mathbf{V}_h$, $p_0 \in Q_h$, *be any initial guess for* $(\mathbf{u}_h, p_h)$.

For $k = 1, 2, \cdots, K_{max}$

Step 1. *Compute* $\mathbf{w}_0 \in \mathbf{V}_h$ *as the solution of*

$$a_0(\mathbf{w}_0, \mathbf{v}_h) =< f, \mathbf{v}_h > -a(\mathbf{u}_0, v_h) - b(\mathbf{v}_h, p_0), \ \mathbf{v}_h \in \mathbf{V}_h.$$

Step 2. *Compute* $\mathbf{u}_1 = \mathbf{u}_0 + \beta \mathbf{w}_0 \in \mathbf{V}_h$.

Step 3. *Compute* $q_1, p_1 \in Q_h$ *by*

$$(q_1, q_h) = b(\mathbf{u}_1, q_h) - \langle g, q_h \rangle, \quad \textit{for all } q_h \in \bar{Q}_h, \ \textit{and } p_1 = p_0 + \alpha q_1.$$

Step 4. *Set* $\mathbf{u}_0 = \mathbf{u}_1, \ p_0 = p_1$.

End

Note that **NSUM-h** coincides with **NSUM** with $\mathbf{V} = \mathbf{V}_h$ and $Q = Q_h$. Using the Theorem 3.5 for $\mathbf{V} = \mathbf{V}_h$ and $Q = Q_h$, we have that the iterations $(\mathbf{u}_1, p_1)$ of **(NSUM-h)** converge to $(\mathbf{u}_h, p_h)$ as $K_{max} \to \infty$.

Next, we present the discrete version of **RRM**. The new algorithm has the advantage that the relaxation parameter β does not have to be prescribed. It is computed automatically by the algorithm.

Algorithm 5.2. (RRM-h)

Let $\mathbf{u}_0 \in \mathbf{V}_h$, $p_0 \in Q_h$, *be any initial guess.*

Step 0. *Compute* $\mathbf{w}_0 \in \mathbf{V}_h$ *as the solution of*

$$a_0(\mathbf{w}_0, \mathbf{v}) =< f, \mathbf{v}_h > -a(\mathbf{u}_0, v_h) - b(\mathbf{v}_h, p_0),\ \ \mathbf{v} \in \mathbf{V}_h$$

For $k = 1, 2, \cdots, K_{max}$

Step 1. *Compute* $\mathbf{u}_{\mathbf{w}_0} \in \mathbf{V}_h$ *as the solution of*

$$a_0(\mathbf{u}_{\mathbf{w}_0}, \mathbf{v}_h) = a(\mathbf{w}_0, v_h),\ \ \mathbf{v}_h \in \mathbf{V}_h.$$

Step 2. *Compute*

$$\beta = \frac{a_0(\mathbf{w}_0, \mathbf{u}_{\mathbf{w}_0})}{a_0(\mathbf{u}_{\mathbf{w}_0}, \mathbf{u}_{\mathbf{w}_0})},\ \gamma = \sqrt{1 - \beta\ \frac{a_0(\mathbf{w}_0, \mathbf{u}_{\mathbf{w}_0})}{a_0(\mathbf{w}_0, \mathbf{w}_0)}}.$$

and choose a positive α *s.t.* $\alpha < \frac{1-\gamma}{\beta}\ \frac{2}{M_h^2}$.

Step 3. *Compute* $\mathbf{u}_1 = \mathbf{u}_0 + \beta \mathbf{w}_0$*, and* $q_1, p_1 \in Q_h$ *by*

$$(q_1, q_h) = b(\mathbf{u}_1, q_h) - \langle g, q_h \rangle, \quad \textit{for all } q_h \in \bar{Q}_h, \quad p_1 = p_0 + \alpha q_1.$$

Step 4. *Compute* $\mathbf{u}_{q_1} \in \mathbf{V}_h$ *as the solution of*

$$a_0(\mathbf{u}_{q_1}, \mathbf{v}_h) = b(\mathbf{v}, q_1),\ \ \mathbf{v}_h \in \mathbf{V}_h.$$

Step 5. *Compute the* $\mathbf{w}$*- residual for the next iteration:*

$$\mathbf{w}_1 = \mathbf{w}_0 - \beta \mathbf{u}_{\mathbf{w}_0} - \alpha \mathbf{u}_{q_1}.$$

Step 6. *Set* $\mathbf{u}_0 = \mathbf{u}_1,\ p_0 = p_1,\ \mathbf{w}_0 = \mathbf{w}_1$.
End

Note that **RRT-h** coincides with **RRT** with $\mathbf{V} = \mathbf{V}_h$ and $Q = Q_h$. We assume that the parameters β, α computed at each step, satisfy $\beta > \beta_0 > 0$ and $\alpha > \alpha_0 > 0$ for some a priori fixed thresholds values β_0, α_0. Using the Theorem 4.1 we can conclude that that the iterations $(\mathbf{u}_1, p_1)$ of **(RRT-h)** converge to $(\mathbf{u}_h, p_h)$ as $K_{max} \to \infty$.

We notice that q_1 computed in **Step 3** (for both algorithms) is in fact $R_h {C_0}^{-1}(B\mathbf{u}_1 - g)$. Thus, the role of R_h could be two fold. First, it can ensure that (5.1) holds with m_h independent of h, and second, it can smooth the residual ${C_0}^{-1}(B\mathbf{u}_1 - g)$. The fact that in practice, p_0, q_1 are smooth functions will affect the regularity of the elliptic problems solved in **Step 1** (and **Step 4** of **(RRT-h)**), [5, 8, 9, 10, 11, 12]. This also suggests that the algorithm can be easily modified to an adaptive or multilevel algorithm, which changes the space $\mathbf{V}_h$ to a better one if the one step residual reduction is not significant, [4, 13, 28, 33].

6. Numerical results for the steady state Navier-Stokes

In this section we consider an application of the algorithms developed in the previous sections to finite element discretization for the steady state Navier-Stokes equations, for Newtonian, incompressible viscous fluid.

$$(6.1)\qquad \begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u_\Gamma} & \text{on } \Gamma = \partial\Omega, \end{cases}$$

We assume that Ω is a polygonal domain in $\mathbb{R}^2$ and the problem is the calculation of the flow in Ω caused by a boundary velocity field $\mathbf{u}_\Gamma$. We assume also that there are no body forces ($\mathbf{f} = \mathbf{0}$). The variable $\mathbf{u}$ is the vector valued function representing the the fluid velocity, p is a scalar function representing the pressure. The pressure is determined only up to an additive constant, so for uniqueness of the pressure we require that $\int_\Omega p = 0$. The constant ν is the kinematic viscosity of the flow. We also assume here the existence and uniqueness of the solution $(\mathbf{u}, p)$ of (6.1), with $\mathbf{u} \in (H^1_\Gamma(\Omega))^2 := \{\mathbf{u} \in (H^1(\Omega))^2 | \mathbf{u} = \mathbf{u}_\Gamma \text{ on } \Gamma = \partial\Omega\}$ and $p \in Q := L^2_0(\Omega)$.

By rescaling the pressure p and using Picard iteration to approximate the solution of (6.1) we are lead to a typical Oseen problem:

$$(6.2)\qquad \begin{cases} -\Delta\mathbf{u} + \frac{1}{\nu}(\mathbf{w}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u_\Gamma} & \text{on } \partial\Omega. \end{cases}$$

where $\mathbf{w}$ is an approximation of $\mathbf{u}$ that satisfies $\nabla\cdot\mathbf{w} = 0$.

Let $\mathbf{V} := (H^1_0(\Omega))^2$ and define the forms $a_0(\cdot,\cdot)$, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ by:

$$a_0(\mathbf{u},\mathbf{v}) := \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \sum_{i=1}^{2} \nabla\mathbf{u}_i \cdot \nabla\mathbf{v}_i,$$

$$b(\mathbf{v}, q) := -\int_\Omega q \ \operatorname{div}\mathbf{v}, \ \ \mathbf{v}\in\mathbf{V}, q \in Q,$$

and

$$a(\mathbf{u},\mathbf{v}) = \ a_0(\mathbf{u},\mathbf{v}) + \frac{1}{\nu}((\mathbf{w}\cdot\nabla)\mathbf{u}, \mathbf{v}).$$

Then, the solution of of (6.2) is a pair $(\mathbf{u}, p) \in \mathbf{u} \in (H^1_\Gamma(\Omega))^2 \times L^2_0(\Omega)$ that satisfies

$$(6.3)\qquad \begin{array}{llll} a(\mathbf{u},\mathbf{v}) & + \ \ b(\mathbf{v}, p) & = 0, & \text{for all } \mathbf{v}\in\mathbf{V}, \\ b(\mathbf{u}, q) & & = 0 & \text{for all } q \in Q. \end{array}$$

To compensate for the fact that in practice we might not have $\nabla\cdot\mathbf{w} = 0$ the form a is modified (see [32]) to

$$a(\mathbf{u},\mathbf{v}) = \ a_0(\mathbf{u},\mathbf{v}) + \frac{1}{\nu}\left(((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}) + 1/2(\operatorname{div}\mathbf{w}, \mathbf{u}\cdot\mathbf{v})\right).$$

Next, we write the solution $\mathbf{u}$ of (6.3) as a sum of a function in $\mathbf{V}$, still denoted $\mathbf{u}$ but with zero values on the boundary, and a smooth function on Ω that agrees with $\mathbf{u}_\Gamma$ on Γ, and let $\mathbf{v} \to \langle \mathbf{f}, \mathbf{v}\rangle$ and $q \to \langle g, q\rangle$ be the functionals that appear in the variational formulation of (6.3) due to the boundary part of the solution. With the above notation the *anti-symmetrized* variational formulation of (6.2), is reduced to the abstract formulation (3.3) with the forms and spaces just defined above. The operator $A_0 : V \to V^*$, is $(-\Delta)^2 : (H_0^1(\Omega))^2 \to (H^{-1}(\Omega))^2$, and by taking $C_0 = {C_0}^{-1}$ the identity on $L_0^2(\Omega)$, we have that $B : V \to Q^*$ and $B^* : Q \to V^*$ are in fact $B = -div$ and $B^* = grad$.

For a concrete example, we consider the classical driven cavity problem with $\nu = 0.01$. The flow in a unit square cavity is caused by a tangential velocity field at the top side and in the absence of body forces. For the discretization of (6.2) in the variational form (3.3), we consider an original mesh $\mathcal{T}_0$ on Ω by splitting the domain into triangles using the unit slope diagonal of the square. Next, a family of meshes $\{\mathcal{T}_k\}_{k\geq 0}$ is defined by uniform refinement strategy as presented in the standard multilevel theory, [6, 19]. This means that, $\mathcal{T}_k$ is obtained from $\mathcal{T}_{k-1}$ by splitting each triangle of $\mathcal{T}_{k-1}$ in four similar triangles. We define $\mathbf{V}_h = \mathbf{V}_k$ as the space of (vector) functions which vanish on $\partial\Omega$ and are continuous piecewise linear functions with respect to the mesh $\mathcal{T}_k$. We consider both **NSUM-h** and **RRM-h** and for each algorithm consider two types of pressure projector R_h. For **RRM-h** β is chosen optimally by the algorithm. For a fair comparison, at each step of the two algorithms we choose $\alpha = 1.4\,\frac{1-\gamma}{\beta}$, where $\gamma = \sqrt{1-\beta}$ (see the end of Section 4). We start the algorithms on level $k = 4$ and use the solution of the corresponding Stokes equations as the initial guess. We fixed the number of Oseen iterations on each level to 8.

First, we take the pressure projector $R_h = R_{k-1}$ as the L^2 orthogonal projection onto the space $\bar{Q}_h$ of *scalar* discontinuous continuous piecewise linear functions with respect to the mesh $\mathcal{T}_{k-1}$. The family of pairs $\{(\mathbf{V}_h, \bar{Q}_h)\}_h$ it is known to be stable. Since $Q_h \subset \bar{Q}_h$ we also have that the family of pairs $\{(\mathbf{V}_h, Q_h)\}_h$ it is known to be stable. For each fixed Oseen problem we use the following stopping criteria: $\frac{|\mathbf{w}_k|^2+\|q_k\|^2}{|\mathbf{w}_0|^2+\|q_0\|^2} < 10^{-6}$. For each fixed level k we record the residual norms $|\mathbf{w}_j|$ and $\|q_j\|$ for the the last iteration of the last Oseen problem and the number of iteration for each of the eight Oseen problems. For the **NSUM-h** algorithm, numerical results for $\beta = 0.1$ are shown in Table 1. For the **RRM-h** numerical results are shown in Table 2.

lev	$\lvert\mathbf{w}_j\rvert$	rat	$\|q_j\|$	rat	# iter /Ossen problem
k=4	4.88e-08		2.30e-08		128 130 127 127 131 125 130 125
k=5	2.92e-08	1.67	1.08e-08	2.13	130 132 129 134 128 133 128 133
k=6	1.22e-08	2.39	5.10e-09	2.12	131 134 130 137 131 133 131 131
k=7	3.50e-09	3.48	1.30e-09	3.92	131 135 132 137 132 133 133 130

TABLE 1. A summary of results for the Driven Cavity problem using **NSUM-h**, stable pairs. For the last Oseen Problem on level $k = 7$ we have $|\mathbf{w} - \mathbf{u}| = 9.508e - 07$.

lev	$\|\mathbf{w}_j\|$	rat	$\|q_j\|$	rat	# iter /Ossen problem
k=4	6.12e-08		2.14e-08		50, 45, 47, 48, 47, 48, 47, 47
k=5	2.87e-08	2.13	1.47e-08	1.46	38, 53, 55, 54, 56, 50, 56, 52
k=6	1.38e-08	2.08	5.50e-09	2.67	40, 60, 62, 59, 61, 62, 57, 62
k=7	4.00e-09	3.45	1.40e-09	3.93	38, 62, 63, 62, 62, 64, 56, 62

TABLE 2. A summary of results for the Driven Cavity problem using **RRM-h** and stable pairs . The residual reduction is similar with **NSUM-h** but the number of iterations at each level is reduced by at least a factor of two . For the last Oseen Problem on level $k = 7$ we get $|\mathbf{w} - \mathbf{u}| = 9.46e - 07$.

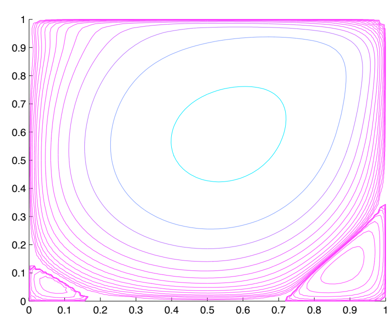

Figure 1: Streamlines for velocity for $\nu = 0.01$, $h = 1/128$.

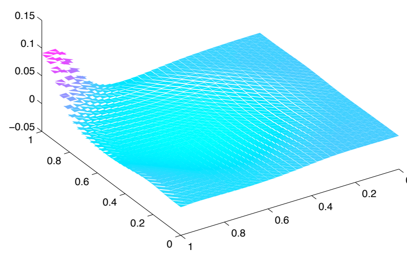

Figure 2: *Discontinuous* approximation for pressure for $\nu = 0.01$, $h = 1/64$.

For the second choice of pressure projector, we take $R_h = R_k$ as the L^2 orthogonal projection onto the space of *scalar continuous* piecewise linear functions with respect to the mesh $\mathcal{T}_k$. The interesting thing about this choice of discrete spaces is that we do not have discrete LBB stability. We do have a discrete LBB condition as shown in Section 5, but the discrete stability constant might be level dependent. In this case, due to lack of stability, for each fixed Oseen problem we stop the iteration process when the residual reduction becomes not significant, more precisely when

$$\sqrt{\frac{\gamma}{\alpha}|\mathbf{w}_k|^2 + \frac{1}{\beta}\|q_k\|^2} > 0.996\ \sqrt{\frac{\gamma}{\alpha}|\mathbf{w}_{k-1}|^2 + \frac{1}{\beta}\|q_{k-1}\|^2}.$$

For the **NSUM-h** algorithm, numerical results for $\beta = 0.04$ are shown in Table 3. For **RRM-h** numerical results are shown in Table 4.

lev	$\|\mathbf{w}_j\|$	rat	$\|q_j\|$	rat	# iter /Ossen problem
k=4	4.83e-02		3.32e-02		168, 109, 36, 8, 11, 7, 4, 8
k=5	1.03e-02	4.69	9.81e-03	3.38	267, 85, 64, 22, 4, 4, 4, 4
k=6	3.35e-03	3.08	3.30e-03	2.97	314, 76, 45, 13, 4, 4, 4, 4
k=7	1.74e-03	1.92	1.14e-03	2.90	360, 61, 40, 15, 10, 6, 5, 6

TABLE 3. A summary of results for the Driven Cavity problem using **NSUM-h** and not-stable pairs. For the last Oseen Problem on level $k = 7$ we have $|\mathbf{w} - \mathbf{u}| = 4.72e - 04$.

lev	$\|\mathbf{w}_j\|$	rat	$\|q_j\|$	rat	# iter /Ossen problem
k=4	1.74e-02		3.59e-02		9, 52, 32, 10, 4, 10, 23, 9
k=5	2.74e-03	6.33	1.14e-02	3.16	69, 27, 10, 11, 11, 12, 10, 5
k=6	8.21e-04	3.34	3.70e-03	3.07	103, 29, 10, 11, 11, 12, 5, 5
k=7	2.71e-04	3.03	1.22e-03	3.02	170, 24, 10, 10, 4, 11, 10, 6

TABLE 4. A summary of results for the Driven Cavity problem using **RRM-h** and not-stable pairs. For the last Oseen Problem on level $k = 7$ we have $|\mathbf{w} - \mathbf{u}| = 3.76e - 04$.

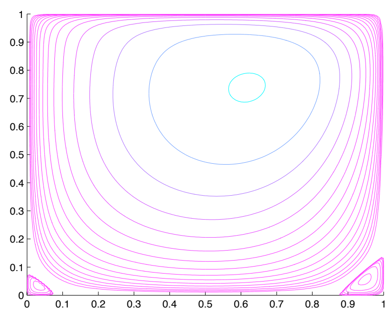

Figure 3: Streamlines for velocity for $\nu = 0.01$, $h = 1/128$.

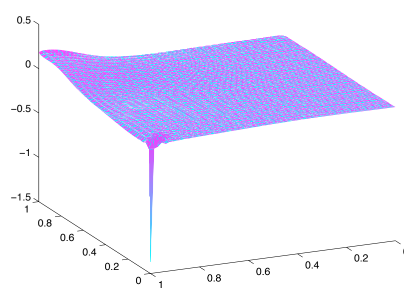

Figure 4: *Continuous* approximation of pressure for $\nu = 0.01$, $h = 1/128$

7. CONCLUSION

We analyzed Uzawa algorithms for non-symmetric saddle point systems using new spectral properties of the Schur complements. We were able to find sharp estimates for one step residual reduction of the non-symmetric Uzawa method. This led to a new Uzawa method, called residual reduction method **(RRM)**, which uses optimal choice of the relaxation parameters at each new iteration. The residual reduction estimate of Theorem 4.1 justifies why in practice the parameter β for the standard Uzawa algorithms needs not to satisfy the strong restriction (3.11). Compared with standard Uzawa methods, **RRM** reduces the number of iterations and the time when dealing with, e.g., the Oseen problem, (see e.g., [18] for a comparison). Using **RRM**, we were able to get good approximation for the solution of the driven cavity model even for the small viscosity $\nu = 0.01$, (see Figures 1-4 and [22] for a comparison). Knowledge on how the residual reduction behaves in terms of the relaxation parameters allowed us to discretize using spaces that are not necessarily stable. The advantage of using smooth but unstable pairs of spaces in the adaptive context will be investigated in a further work.

References

[1] K. Arrow, L. Hurwicz and H. Uzawa. *Studies in Nonlinear Programming*, Stanford University Press, Stanford, CA, 1958.

[2] A. Aziz and I. Babušca, Parti, Survey lectures on mathematical foundations of the finite element method. In A. Aziz, editor, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Academic Press, 1-362, New York, NY, 1972.

[3] C. Bacuta, A Unified Approach for Uzawa Algorithms, SIAM Journal of Numerical Analysis, Volume 44, Issue 6, 2006, pp. 2245-2649.

[4] C. Bacuta, Schur Complements on Hilbert Spaces and Saddle Point Systems, *Journal of Computational and Applied Mathematics*, Volume 225, Issue 2, 2009, pp 581-593.

[5] C. Bacuta and J. H. Bramble. Regularity estimates for solutions of the equations of linear elasticity in convex plane polygonal domains. *Z. angew. Math. Phys.(ZAMP)*, 54(2003), 874-878.

[6] C. Bacuta, J. H. Bramble and J. Pasciak, Using finite element tools in proving shift theorems for elliptic boundary value problems, *Numerical Linear Algebra with Applications*, Volume 10, Issue no 1-2, 2003, pp 33-64.

[7] C. Bacuta, J. H. Bramble and J. Pasciak. New interpolation results and applications to finite element methods for elliptic boundary value problems. *Numerical Linear Algebra with Applications*, vol. 10, Issue no 1-2, 33–64, 2003.

[8] C. Bacuta, J. H. Bramble, J. Pasciak. New interpolation results and applications to finite element methods for elliptic boundary value problems, *East-West J. Numer. Math*, Vol.9, No.3, 2001.

[9] C.Bacuta, J. H. Bramble and J. Pasciak. Using finite element tools in proving shift theorems for elliptic boundary value problems. *Numerical Linear Algebra with Application*, Volume 10, Issue no 1-2, 33-64, 2003.

[10] C.Bacuta, J. H. Bramble and J. Pasciak. Shift theorems for the biharmonic Dirichlet problem, with J. H. Bramble and J. Pasciak, *Recent Progress in Computational and Applied PDEs*, Kluwer Academic/Plenum Publishers, volume of proceedings for the CAPDE conference held in Zhangjiajie, China, July 2001.

[11] C. Bacuta, V. Nistor and L. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polyhedra I: apriori estimates: Mesh Refinements and Interpolation, with V. Nistor and L. Zikatanov, *Numerical Functional Analysis and Optimization*, Vol. 26, No 6, 2005, pp. 613-639.

[12] C. Bacuta, V. Nistor and L. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polyhedra II: Mesh Refinements and Interpolation, Numerical Functional Analysis and Optimization, Vol. 28, No 7-8, 2007, pp. 775-824.

[13] E. Bansch, P. Morin and R.H. Nochetto. An adaptive Uzawa fem for the Stokes problem:Convergence without the inf-sup condition. *Siam J. Numer. Anal.*, 40:1207-1229, 2002.

[14] M. Benzi, G.H. Golub and J. Liesen, Numerical Solutions of Saddle Point Problems. *Acta Numerica*, 14 (2005), pp. 1-137.

[15] J.H Bramble and J.E. Pasciak, A new approximation technique for div-curl systems, *Math. Comp.*, 73(2004), 1739-1762.
[16] J.H Bramble, J.E. Pasciak and P.S. Vassilevski, Computational scales of Sobolev norms with application to preconditioning, *Math. Comp.*, 69 (2000) 463-480.
[17] J. H. Bramble, J. Pasciak and A.T. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problems. *Siam J. Numer. Anal.*, 34:1072-1092, 1997.
[18] J. H. Bramble, J. Pasciak and A.T. Vassilev. Uzawa type algorithm for nonsymetric saddle point problems. *Math. Comp.*, 69 (1999) 667-689.
[19] J. H. Bramble and X. Zhang. The analysis of multigrid methods, in: *Handbook for Numerical Analysis*, Vol. VII, 173-415, P. Ciarlet and J.L. Lions, eds., North Holland, Amsterdam, 2000.
[20] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York. 1991.
[21] S. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.
[22] H.C. Elman, D. Silvester and A. Wathen. *Finite Elements and Fast Iterative Solvers*, Oxford Science Publications, Oxford, 2005.
[23] H.C. Elman and G.H. Golub. Inexact and preconditioned Uzawa Algorithms for saddle point problems. *Siam J. Numer. Anal.*, 31:1645-1661, 1994.
[24] M. Fortin and R. Glowinski *Augmented Lagrangian Methods: Applications to the numerical solutions of boundary value problems.* Studies in Mathematics and Applications, Vol 15, North-Holland (1983).
[25] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, Berlin, 1986.
[26] Q. Hu and J. Zou. Nonlinear Inexact Uzawa Algorithms for Linear and Nonlinear Saddle-Point Problems, *SIAM J. OPTIM.*, Vol. 16, No.3, pp. 798-825,2006
[27] Q. Hu and J. Zou. Two new variants of nonlinear inexact Uzawa algorithms for saddle-point problems. *Numer. Math.*, 93:333-359, 2002.
[28] Y. Kondratyuk and R. Stevenson. An Optimal Adaptive Finite Element Method for the Stokes Problem. *Siam J. Numer. Anal.*, 46:746-775, 2008.
[29] R.H. Nochetto and J. Pyo. Optimal relaxation parameter for the Uzawa Method. *Numer. Math.*, 98:695-702, 2004.
[30] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*, Springer, Berlin, 1994.
[31] P. Monk. *Finite element methods for Maxwell's Equations*, Clarendon Press, Oxford, 2003.
[32] R. Temam. *Navier-Stokes Equations*, North-Holland, 1984.
[33] R. Verfurth. *A Review of A Posterirori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1996.
[34] J. Xu and L. Zikatanov. Some observations on Babuška and Brezzi theories. *Numer. Math.*, 94(1):195-202, 2003.

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