

ANISOTROPIC REGULARITY AND OPTIMAL RATES OF CONVERGENCE ON THREE
DIMENSIONAL POLYHEDRAL DOMAINS

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ABSTRACT. We consider the the Poisson problem $-\Delta u = f \in \Omega$, $u = g$ on $\partial\Omega$, where Ω is a bounded domain in \mathbb{R}^n . The objective of the paper is twofold. The first objective is to present the well posedness and the regularity of the problem using appropriate weighted spaces for the data and the solution. The second objective is to illustrate how weighted regularity results for the Laplace operator are used in designing efficient finite element discretizations of elliptic boundary value problems with the focus on discretization of the Poisson problem on polyhedral domains in \mathbb{R}^3 .

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INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. Consider the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g, & \text{on } \Omega, \end{cases} \quad (1)$$

defined on a bounded domain $\Omega \subset \mathbb{R}^n$, where Δ is the *analyst's* Laplacian $\Delta = \sum_{i=1}^d \partial_i^2$. When $\partial\Omega$ is smooth, it is well known that this Poisson problem

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has a unique solution $u \in H^{m+1}(\Omega)$ for any $f \in H^{m-1}(\Omega)$ and $g \in H^{m+1/2}(\partial\Omega)$ [26]. Moreover, u depends continuously on f and g . This result is the *classical well-posedness of the Poisson problem* on smooth domains.

On the other hand, when Ω is not smooth, it is also known [20, 36, 42, 46] that there exists a constant $s = s_\Omega$, such that $u \in H^s(\Omega)$ for any $s < s_\Omega$, but $u \notin H^{s_\Omega}(\Omega)$ in general, even if f and g are smooth functions. For instance, if Ω is a polygonal domain in two dimensions, then $s_\Omega = 1 + \pi/\alpha_{MAX}$, where α_{MAX} is the largest interior angle of Ω . For any polyhedra, we have $s_\Omega < \infty$, and this phenomenon is called *loss of regularity* and is responsible for the loss of accuracy in certain approximation methods for the solutions of Equation (1). It is therefore desirable to establish an alternative well-posedness result on polyhedra.

A thorough analysis of the difficulties that arise for $\partial\Omega$ Lipschitz is contained in the papers of Babuška [4], Baouendi and Sjöstrand [10], Băcuță, Bramble, and Xu [15], Babuška and Guo [32, 31], Brown and Ott [14], Jerison and Kenig [34, 35], Kenig [39], Kenig and Toro [40], Koskela, Koskela and Zhong [44, 45], Mitrea and Taylor [58, 60, 61], Verchota [73], and others (see the references in the aforementioned papers). Other results specific to curvilinear polyhedral domains are contained in the papers of Costabel [18], Dauge [20], Elschner [21, 22], Kondratiev [42, 43], Mazya and Rossmann [54], Rossmann [63] and others. Other relevant references are the monographs of Grisvard [28, 29] as well as the recent books [46, 47, 52, 53, 62]. Another way to obtain a convenient well-posedness result for the Poisson problem on a polyhedron Ω is to use the stratified space geometry of Ω . This leads, by successive conformal changes of the metric, to a metric for which the smooth part of $\bar{\Omega}$ is a smooth manifold with boundary whose double is complete. The resulting Sobolev spaces defined by the new metric will lead to spaces on which the Poisson problem is well-posed [9].

For the discretization on polyhedral domains we build discrete spaces $S_k \subset H_0^1(\Omega)$ and Galerkin finite element projections $u_k \in S_k$ that approximate the solution u of Equation (1) for $f \in H^{m-1}(\Omega)$ arbitrary. We prove that, by using certain spaces of continuous, piecewise polynomials of degree m , the sequence S_k achieves *quasi-optimal rates of convergence*. More precisely we prove the existence of a constant $C > 0$, independent of k and f , such that

$$\|u - u_k\|_{H^1(\Omega)} \leq C \dim(S_k)^{-m/3} \|f\|_{H^{m-1}(\Omega)}, \quad u_k \in S_k. \quad (2)$$

We now describe these results and some extensions in more detail.

1 ISOTROPIC WEIGHTED SOBOLEV SPACES

Using the standard notation for partial derivatives, namely $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, the usual Sobolev spaces on an open set V are

$$H^m(V) = \{u : V \rightarrow \mathbb{C}, \partial^\alpha u \in L^2(V), |\alpha| \leq m\}.$$

To define the weighted analogues of these spaces we would need to introduce the notion of *singular boundary points* for a domain $\Omega \subset \mathbb{R}^n$.

Let $\Omega^{(n-2)} \subset \partial\Omega$ be the set of singular (or non-smooth) boundary points of Ω , that is, the set of points $p \in \partial\Omega$ such that $\partial\Omega$ is not smooth in a neighborhood of p . We will denote by $\eta_{n-2}(x)$ the distance from a point $x \in \Omega$ to the set $\Omega^{(n-2)}$ and agree to take $\eta_{n-2} = 1$ if there are no such points. That is, if $\partial\Omega$ is smooth. We define the weighted Sobolev spaces

$$\mathcal{K}_a^\mu(\Omega) = \{u \in L_{\text{loc}}^2(\Omega), \eta_{n-2}^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq \mu\}, \quad \mu \in \mathbb{Z}_+, \quad (3)$$

which we endow with the induced Hilbert space norm. We note that for $n = 3$ for example and Ω a polyhedral domain in \mathbb{R}^3 , we have that $\eta_1(x)$ is the distance to the skeleton made up by the union of the closed edges of $\partial\Omega$.

Similar definitions hold for the spaces on faces of Ω . For example for $n = 3$, we define

$$\mathcal{K}_a^m(\partial\Omega) = \{(u_F), \eta^{|\alpha|-a} \partial^\alpha u_F \in L^2(F)\},$$

where $|\alpha| \leq m$ and F ranges through the set of faces of $\partial\Omega$.

For $s \in \mathbb{R}_+$, we define the space $\mathcal{K}_a^s(\partial\Omega)$ by the standard interpolation.

The following result is proved in [8].

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n = 3$, be a bounded, polyhedral domain and $m \in \mathbb{Z}_+$. Then there exists $\gamma > 0$ such that $\tilde{\Delta}(u) = (\Delta u, u|_{\partial\Omega})$ defines an isomorphism*

$$\tilde{\Delta} : \mathcal{K}_{a+1}^{m+1}(\Omega) \rightarrow \mathcal{K}_{a-1}^{m-1}(\Omega) \oplus \mathcal{K}_{a+1/2}^{m+1/2}(\partial\Omega),$$

for all $|a| < \gamma$. If $m = 0$, the solution u corresponding to a fixed data $(f, g) \in \mathcal{K}_{a-1}^{-1}(\Omega) \oplus \mathcal{K}_{a+1/2}^{1/2}(\partial\Omega)$, is also the solution of the associated variational problem:

$$B(u, v) := \int_{\mathbb{P}} \nabla u \cdot \nabla v dx = \int_{\mathbb{P}} f v dx. \quad (4)$$

For $n = 2$, and Ω a polygonal domain, a similar result was proved by Kondratiev in [42]. In this case, $\gamma = \frac{\pi}{\alpha_{MAX}}$, where α_{MAX} is the measure in radians of the maximum angle of Ω .

Remark 1.2. For general polyhedral domains in \mathbb{R}^n , with n arbitrary, the result is presented in [9]. For elasticity with mixed boundary conditions a similar result is investigated by Mazzucato and Nistor in [56]. For the transmission problems in 2D similar results are done by Li and Nistor. Using countably normed spaces and analytic regularity, we mention the work of Babuška-Guo, and Costabel-Dauge-Nicaise.

2 ANISOTROPIC WEIGHTED SOBOLEV SPACES AND REGULARITY

We start this section by introducing new weighted spaces. To make the presentation easier we assume first that $\Omega = D_\alpha = \{0 < \theta < \alpha\}$, is a *dihedral angle*

with edge along the Oz -axis and that $f \in H^{m-1}(D_\alpha)$. Then $u \in \mathcal{K}_{a+1}^{m+1}(D_\alpha)$ for positive and small enough a . Hence,

$$\partial_z u \in \mathcal{K}_a^m(D_\alpha).$$

However, we also have

$$\Delta \partial_z u = \partial_z \Delta u = \partial_z f \in H^{m-2}(\Omega).$$

Then, using Theorem 1.1 we obtain that

$$\partial_z u \in \mathcal{K}_{a+1}^m(\Omega),$$

which is a better estimate than the previous one.

Previous argument suggests $\mathcal{D}_a^1(D_\alpha) := \mathcal{K}_1^1(D_\alpha)$ and

$$\mathcal{D}_a^m(D_\alpha) := \{u \in \mathcal{K}_a^m(D_\alpha), \partial_z u \in \mathcal{D}_a^{m-1}(D_\alpha)\}.$$

\mathcal{D}_a^1 are thus independent of a .

If $\Omega = \mathcal{C}$, a cone centered at the origin. $\rho(x) = |x|$ is the distance from x to the origin, then

$$\mathcal{D}_a^1(\mathcal{C}) := \rho^{a-1} \mathcal{K}_1^1(\mathcal{C}) = \{\rho^{a-1} v, v \in \mathcal{K}_1^1(\mathcal{C})\}.$$

For $m \geq 2$, let $\rho \partial_\rho = x \partial_x + y \partial_y + z \partial_z$ be the infinitesimal generator of dilations. Then, for $m \geq 2$, we define by induction

$$\mathcal{D}_a^m(\mathcal{C}) := \{u \in \mathcal{K}_a^m(\mathcal{C}), \rho \partial_\rho(u) \in \mathcal{D}_a^{m-1}(\mathcal{C})\}.$$

For a general bounded polyhedral domain Ω , we define the *anisotropic weighted Sobolev spaces* $\mathcal{D}_a^m(\Omega)$ by localization around vertices, edges, such that in away from the edges these spaces coincide with the usual Sobolev spaces H^m .

THEOREM 2.1. *Let $f \in H^{m-1}(\Omega)$, with $m \geq 1$. Then the Poisson problem (1) with $g = 0$ has a unique solution $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$ for $0 \leq a < \eta = \eta_\Omega$ and*

$$\|u\|_{\mathcal{D}_{a+1}^{m+1}(\Omega)} \leq C_{\Omega,a} \|f\|_{H^{m-1}(\Omega)}.$$

Earlier results of the same type are done by: Arnold-Falk, Apel99, ApelNicaise, Babuška-Guo, Bacuta-Bramble-Xu, Buffo-Costabel-Dauge03, and Kellogg-Osborn.

3 QUASI-OPTIMAL h^m -MESH REFINEMENT

Based on the above regularity result, we will describe in this section a strategy to obtain quasi-optimal h^m -mesh refinement. Given a bounded polyhedral domain Ω and a parameter $\kappa \in (0, 1/2]$, we will provide a sequence \mathcal{T}_n of decompositions of Ω into finitely many tetrahedra, such that if S_n is the finite element space of continuous, piecewise polynomials on \mathcal{T}_n , then $u_{I,n}$ is the Lagrange interpolant of u of order m , has “quasi-optimal” approximability properties. The result can be formulated as follows:

THEOREM 3.1. *Let $a \in (0, 1/2]$ and $0 < \kappa \leq 2^{-m/a}$. Then there exists a sequence of meshes \mathcal{T}_n and a constant $C > 0$ such that for the corresponding sequence of finite element spaces S_n we have*

$$\|u - u_{I,n}\|_{H^1(\Omega)} \leq C 2^{-km} \|u\|_{\mathcal{D}_{a+1}^{m+1}(\Omega)},$$

for any $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$, $u = 0$ on the boundary, and any $k \in \mathbb{Z}_+$.

The main Theorem 2 is now a direct consequence of Theorem 3.1.

3.1 REFINEMENT STRATEGY

Given a point $P \in \bar{\Omega}$, we shall say that P is of *type V* if it is a vertex of Ω ; we shall say that P is of *type E* if it is on an open edge of Ω . Otherwise, we shall say that it is of *type S* (from Smooth!). The type of a point depends only on Ω and not on any partition or meshing. The initial tetrahedralization will consist of edges of type VE, VS, ES, EE:=E², and S². We shall assume that our initial decomposition and initial tetrahedralization was defined, so that no edge of type VV are present. The points of type V will be regarded as more singular than the points of type E, and the points of type E will be regarded as more singular than the points of type S. The triangles will be of one of the types VES, VSS, ESS.

Let AB be a generic edge in the decompositions \mathcal{T}_n . Then, as part of the \mathcal{T}_{n+1} , this edge will be decomposed in two segments, AC and CB , such that $|AC| = \kappa|AB|$ if A is more singular than B (*i.e.*, if AB is of type VE, VS, or ES). Except when $\kappa = 1/2$, C will be closer to the more singular point. This procedure is as in [?, 16, ?]. See Figure 3.1.

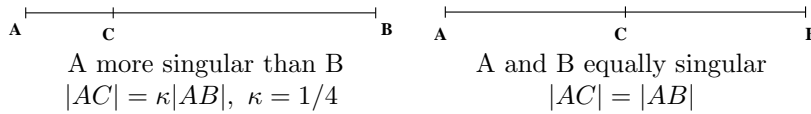
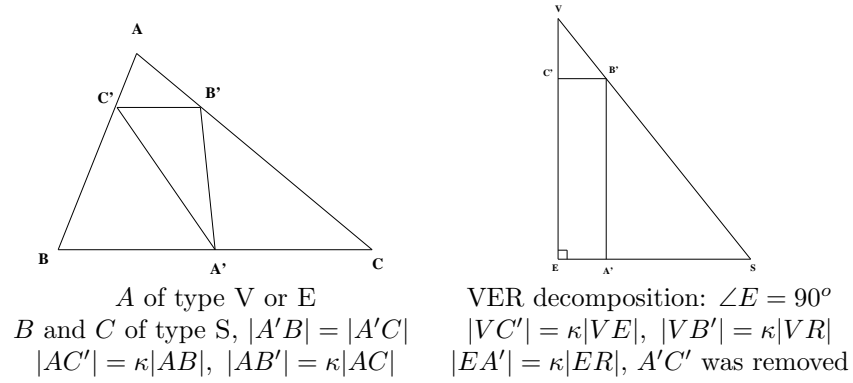
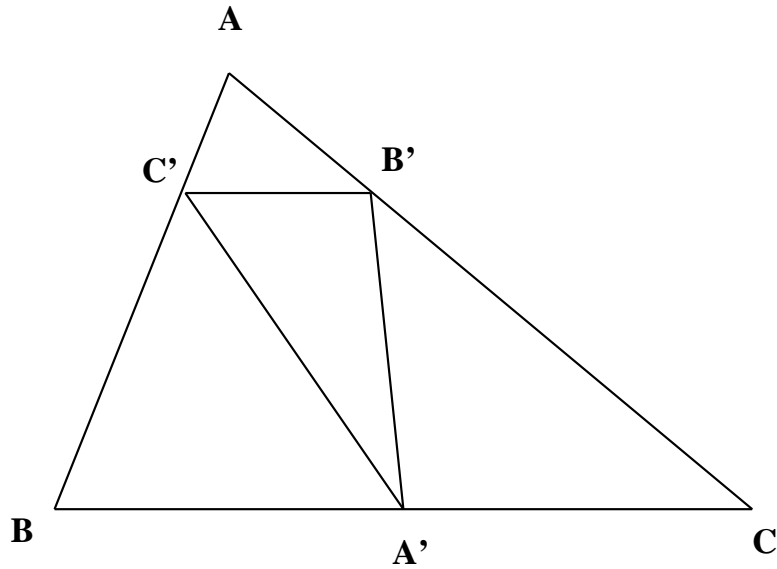


Figure 3.1: Edge decomposition

The above strategy to split edges induces a natural strategy for splitting triangular faces. If ABC is a triangle in the decomposition \mathcal{T}_n , then in \mathcal{T}_{n+1} , the triangle ABC will be divided into four other triangles, according with the edge strategy. The decomposition of triangles of type S³ is obtained for $\kappa = 1/2$. The type VSS triangle decomposition is suggested in Figure 3.3. One exception we have is for the case when ABC is of type VES. In this case we remove the newly introduced segment that is opposite to B , see Figure 3.4, and divide ABC into two triangles and a quadrilateral. The formed quadrilateral will belong to a prism in \mathcal{T}_{n+1} .

Figure 3.2: Triangle decomposition, $\kappa = 1/4$ Figure 3.3: Face decomposition: A of type V or E, B and C of type R, $|AC'| = \kappa|AB|$, $|AB'| = \kappa|AC|$, $|A'B| = |A'C|$, $\kappa = 1/4$

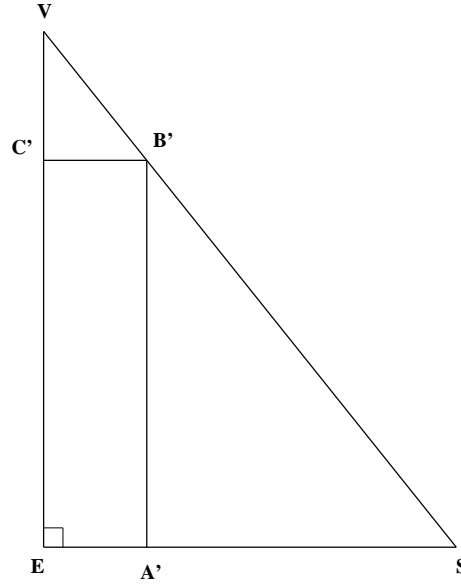


Figure 3.4: VER decomposition: $|VC'| = \kappa|VE|$, $|VB'| = \kappa|VR|$, $|EA'| = \kappa|ER|$, $A'C'$ was removed, $\angle E = 90^\circ$

3.2 DIVISIONS IN TETRAHEDRA AND PRISMS

We will assume, without loss of generality, that our domain Ω is a tetrahedra itself, and describe how to construct the sequence of divisions \mathcal{T}_n for $n \geq 0$. For the first level of semi-uniform refinement of a prism, more details are presented in [8].

We start with an initial division \mathcal{T}'_0 of Ω in straight triangular prisms and tetrahedra of types VESS and VS^3 , having a vertex in common with Ω , and an interior region Λ_0 , see Figure 3.5 and define.

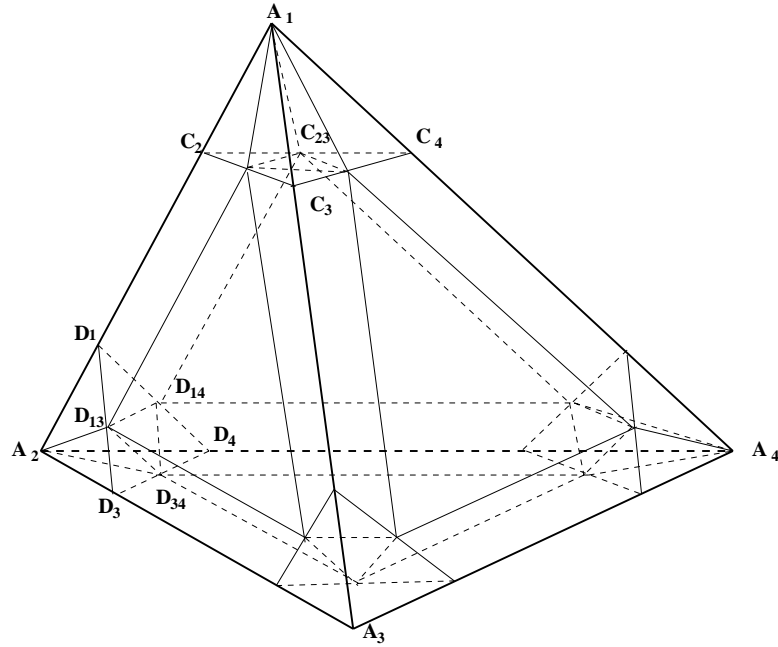


Figure 3.5: Initial decomposition.

We can further assume that some of the edge points (as in Figure 3.5) are moved along the edges so that the prisms become straight triangular prisms i.e., the edges are perpendicular to the bases. For $n \geq 1$, the mesh \mathcal{T}_n is obtained from $\mathcal{T}_{n'}$ (with prisms and tetrahedra) by a canonical procedure, that can be described as follows: **THIS NEEDS ATENTION!**

1) Split each prism into 3 tetrahedra. See for example, Figure 3.6, where the straight prism $ABCA'B'C'$ is divided into three tetrahedra.

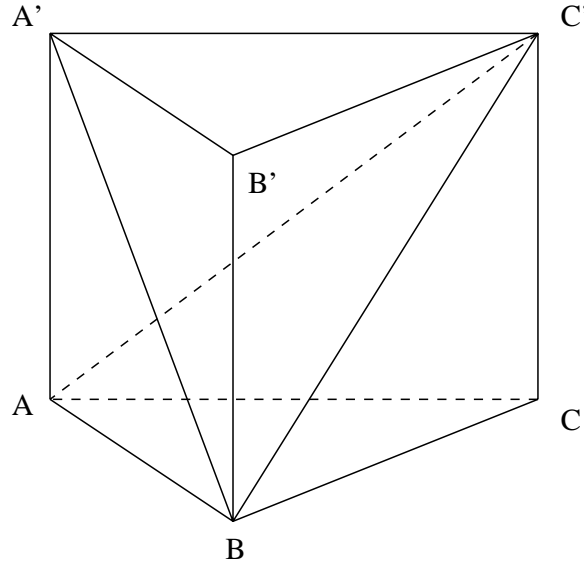


Figure 3.6: Marking a prism: $BC' = \text{mark}$, $AA' \parallel BB' \parallel CC' \perp ABC$ and $A'B'C'$

2) Quasi-uniformly tetrahedralize the interior region of type Λ_0 by using uniform refinement. Perform uniform refinement of a tetrahedra of type S^4 , by dividing along the planes given (in affine coordinates) by : $x_i + x_j = k/2^n, 1 \leq k \leq 2^n$, where x_j are affine barycentric coordinates. The division is compatible with adjacent faces.

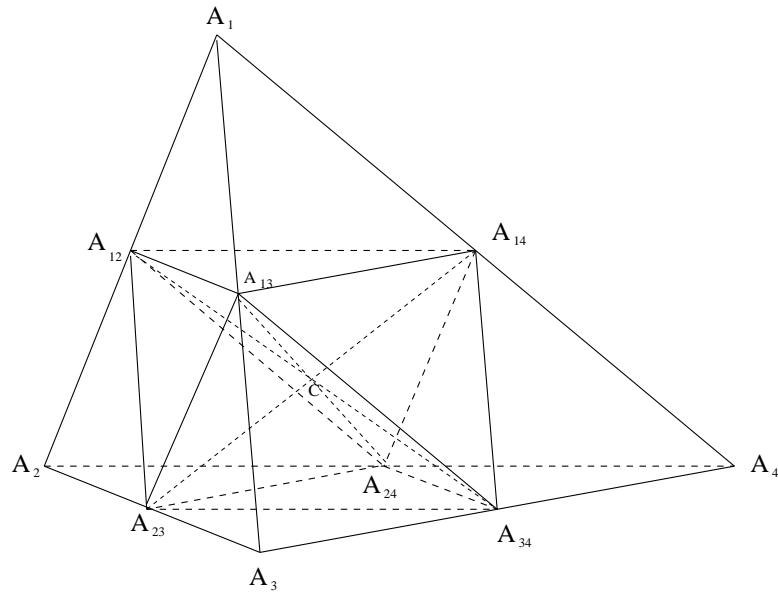


Figure 3.7: First level of uniform refinement

3) Perform semi-uniform refinement for prism with (only) one edge as part of an edge of Ω . The idea is suggested in Figure 3.8.

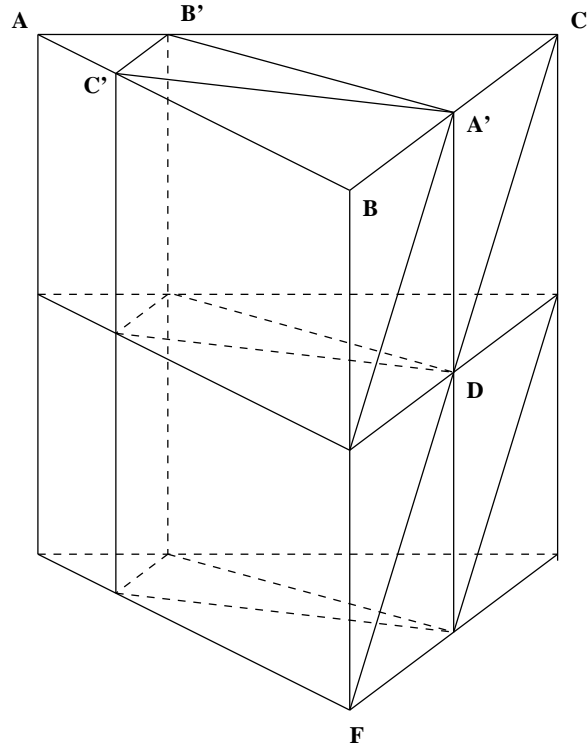


Figure 3.8: First level of semi-uniform refinement of a prism, $CD = \text{mark}$

4) Perform non-uniform refinement for tetrahedron of type VS^3 and VES^3 . We divide a tetrahedron of type VS^3 into 12 tetrahedra as in the uniform strategy, with the edges through the vertex of type V divided in the ratio κ . One tetrahedron of type VS^3 and 11 tetrahedra of type S^4 . We iterate for the small tetrahedron of type VS^3 , the tetrahedra of type S^4 are divided uniformly. See Figure 3.9.

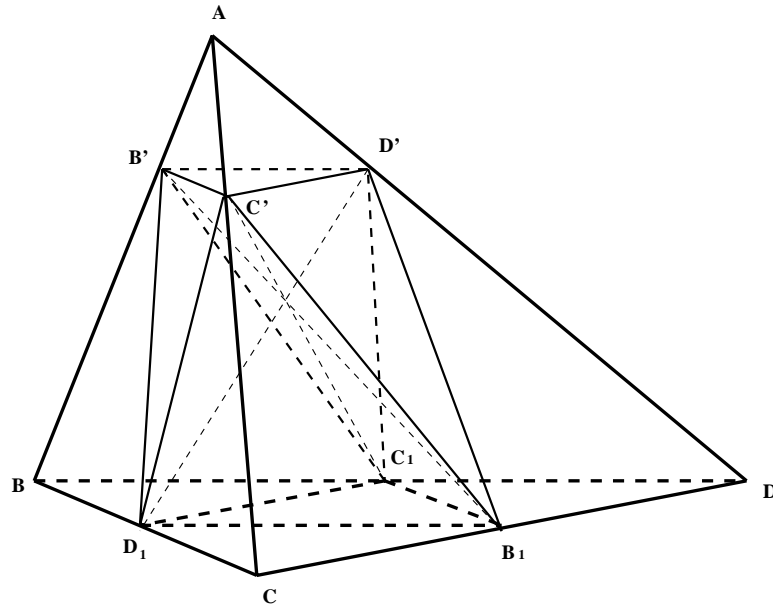


Figure 3.9: A of type V, B, C, D of type R

For tetrahedron of type VESS we divide it into 6 tetrahedra of type S^4 , one tetrahedron of type VS^3 , and a prism. The vertex of type E of will belong only to the prism. See Figure 3.10.

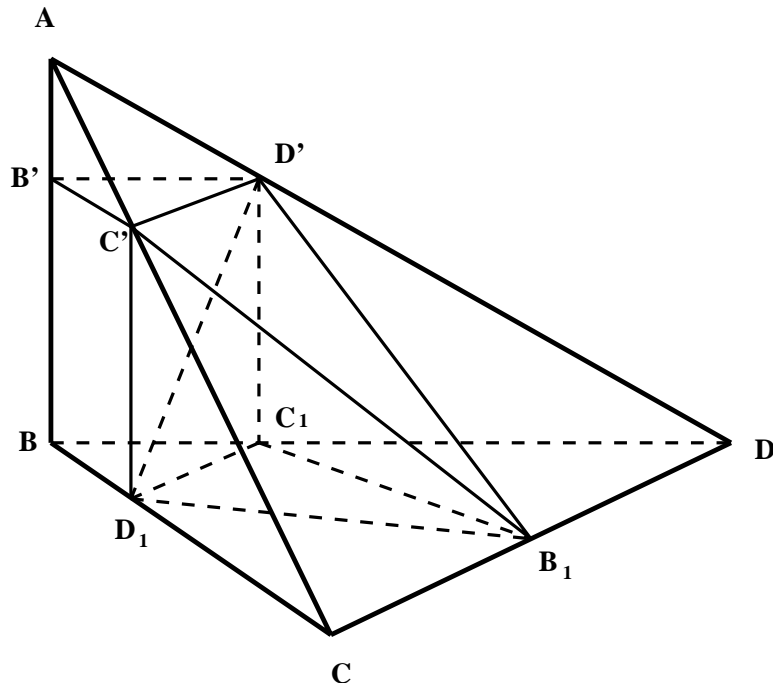


Figure 3.10: A of type V, B of type E, C, D of type R and D_1D' = mark for the prism $BD_1C_1D'C_1B'$

The main ideas of refinement can be formulated as follows: Each edge, triangle, or quadrilateral that appears in a tetrahedron or prism in the decomposition \mathcal{T}_n is divided in the decomposition \mathcal{T}_{n+1} in an intrinsic way, which depends only on the type of the vertices of that edge, triangle, or quadrilateral. In particular, the way that a face in \mathcal{T}_n is divided to yield \mathcal{T}_{n+1} does not depend on the type of the other vertices of the tetrahedron or prism to which it belongs. This ensures that the tetrahedralization \mathcal{T}'_{n+1} , which is obtained from \mathcal{T}_{n+1} by dividing each prism in three tetrahedra, is a conforming mesh.

4 CONCLUSION

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