A NONCONFORMING SADDLE POINT LEAST SQUARES APPROACH FOR ELLIPTIC INTERFACE PROBLEMS

CONSTANTIN BACUTA AND JACOB JACAVAGE

ABSTRACT. We present a non-conforming least squares method for approximating solutions of second order elliptic problems with discontinuous coefficients. The method is based on a general Saddle Point Least Squares (SPLS) method introduced by one of the authors in [9]. The SPLS method has the advantage that a discrete $\inf - \sup$ condition is automatically satisfied for standard choices of test and trial spaces. We explore the SPLS method for non-conforming finite element trial spaces which allow higher order approximation of the fluxes. For the proposed iterative solvers, inversion at each step requires bases only for the test spaces. We focus on using projection trial spaces with local projections that are easy to compute. The choice of the local projections for the trial space can be combined with classical gradient recovery and a posteriori refinement techniques to lead to quasi-optimal approximations of the global flux. Numerical results for 2D an 3D domains are included to support the proposed method.

1. Introduction

Elliptic interface problems have applications in a variety of different fields. In material science, they arise in the study and design of composite materials built from essentially different components, see [3, 20, 23, 11]. In fluid dynamics, they model several layers of fluids with different viscosities or diffusion through heterogeneous porous media [13, 19]. In addition, the elliptic interface problem is used to model stationary heat conduction problems with a conduction coefficient which is discontinuous across a smooth internal interface [21] as well as in biological systems [22].

The Saddle Point Least Squares (SPLS) method introduced in [9], and its version we propose in this paper, can be applied to general first order or second order elliptic PDEs that admit the following mixed variational formulation: Find $p \in Q$ such that

$$b(v,p) = \langle F, v\rangle \quad \text{for all } v \in V, \quad \text{or} \quad B^* p = F, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 74S05, 74B05, 65N22, 65N55.

Key words and phrases. least squares, saddle point systems, mixed methods, multilevel methods, Uzawa type algorithms, conjugate gradient, cascadic algorithm, dual DPG.

The work was supported by NSF, DMS-1522454

Note: The original manuscript was submitted to *De Gruyter, Computational Methods in Applied Mathematics* on December 19, 2017.

where V and Q are infinite dimensional Hilbert spaces, $b(\cdot,\cdot)$ is a continuous bilinear form on $V \times Q$ that satisfies an $\inf - \sup$ condition, and $F \in V^*$ belongs to the range of B^*. The SPLS for (1.1) *bridges* between the field of *least squares methods* and the field of *symmetric saddle point problems.* The discretization approach in this paper can be viewed as a new discontinuous Petrov-Galerkin method. From the point of view of choosing the discrete spaces, it can be characterized as a dual of Demkowicz-Gopalakrishnan's Discontinuous Petrov-Galerkin (DPG) method [17, 18], which is currently undergoing an intensive study. While both methods have strong connections with least squares and minimum residual techniques, our proposed discretization process stands apart from the DPG approach due to the different ways in which the trial and test spaces are chosen. In our approach, we choose a discrete test space first and the trial space is then built in order to satisfy a discrete $\inf - \sup$ condition. For the SPLS method, the trial space is built from the action of the continuous differential operator associated with the problem on the test space. Due to the iterative process we choose to solve the discrete SPLS formulation, assembly of the stiffness matrices for the trial spaces is avoided. The SPLS method can be combined with multilevel adaptivity and preconditioning techniques in order to address particular challenges of the PDE to be solved due to discontinuous coefficients or multidimensional domains.

In contrast with the SPLS work in [9, 10], where both the test and trial spaces were chosen to be conforming finite element spaces, this paper considers trial spaces which are non-conforming finite element spaces. This allows efficient treatment of PDEs with discontinuous coefficients.

The paper is organized as follows. In section 2, we introduce notation for the general non-conforming (n-c) SPLS method and present two types of trial spaces along with stability and approximability properties. In section 3, the general theory will be applied to approximating the solution of second order elliptic problems with discontinuous coefficients. In section 4, numerical results for the SPLS dicretization are presented.

2. The general non-conforming SPLS approach

We first introduce some notation for the spaces and operators for the general abstract setting. Let V and $\tilde{Q}$ be infinite dimensional Hilbert spaces and assume the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)_{\tilde{Q}}$ induce the norms $|\cdot|_V = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_{\tilde{Q}} = \|\cdot\| = (\cdot,\cdot)_{\tilde{Q}}^{1/2}$. We denote the duals of V and $\tilde{Q}$ by V^* and $\tilde{Q}^*$, respectively. The dual pairings on $V^* \times V$ and $\tilde{Q}^* \times \tilde{Q}$ will both be denoted by $\langle \cdot,\cdot \rangle$. With the inner product $(\cdot,\cdot)_{\tilde{Q}}$, we associate the operator $\mathcal{C} : \tilde{Q} \to \tilde{Q}^*$ defined by

$$\langle \mathcal{C}p, q \rangle = (p, q)_{\tilde{Q}} \quad \text{for all} \ \ p, q \in \tilde{Q}.$$

The operator $\mathcal{C}^{-1} : \tilde{Q}^* \to \tilde{Q}$ is the Riesz-canonical isometry. In addition, we let Q be a closed subspace of $\tilde{Q}$ equipped with the induced inner product (from $\tilde{Q}$).

We assume that $b(\cdot,\cdot)$ is a continuous bilinear form on $V \times \tilde{Q}$ satisfying

$$\sup_{p\in\tilde{Q}} \sup_{v\in V} \frac{b(v,p)}{|v| \, \|p\|} = M < \infty, \tag{2.1}$$

and the following $\inf-\sup$ condition on $V \times Q$,

$$\inf_{p\in Q} \sup_{v\in V} \frac{b(v,p)}{|v| \, \|p\|} = m > 0. \tag{2.2}$$

With the form b, we associate the linear operators $B : V \to \tilde{Q}^*$ and $B^* : \tilde{Q} \to V^*$ defined by

$$\langle Bv, q\rangle = b(v,q) = \langle B^* q, v\rangle \quad \text{for all } v \in V,\ q \in \tilde{Q}.$$

Lastly, we define V_0 to be the kernel of B, i.e.,

$$V_0 := \mathrm{Ker}(B) = \{v \in V |\ Bv = 0\}.$$

We consider problems of the form: Given $F \in V^*$, find $p \in Q$ such that (1.1) holds. We note here that for the existence and uniqueness of the solution of the continuous problem (1.1), we use the trial space Q. However, for discretization purposes, we need to consider the form $b(\cdot,\cdot)$ on $V \times \tilde{Q}$. The existence and uniqueness of (1.1) was first studied by Aziz and Babuška in [2]. It is well known that if a bounded form $b : V \times \tilde{Q} \to \mathbb{R}$ satisfies (2.2) and the data $F \in V^*$ satisfies the *compatibility condition*

$$\langle F, v\rangle = 0 \quad \text{for all } v \in V_0, \tag{2.3}$$

then the mixed problem (1.1) has a unique solution, see e.g. [2, 4]. With the mixed problem (1.1), we associate the SPLS formulation: Find $(u,p) \in (V,Q)$ such that

$$\begin{array}{llll} a_0(u,v) & + \quad b(v,p) & = \langle F, v\rangle & \text{for all } v \in V, \\ b(u,q) & & = 0 & \text{for all } q \in Q. \end{array} \tag{2.4}$$

The following statement summarizes the connection between the two variational formulations. The remark was pointed out in [6, 16] and is essential in our approach and (some versions of) the DPG method. It is worth noting that the p component of the solution of (2.4) is in fact the solution of the normal equation that corresponds to our main problem (1.1), see [9].

Proposition 2.1. *In the presence of the continuous* $\inf-\sup$ *condition* (2.2) *and the compatibility condition* (2.3)*, we have that* p *is the unique solution of* (1.1) *if and only if* $(u = 0, p)$ *is the unique solution of* (2.4)*.*

2.1. **Non-Conforming SPLS discretization.** The non-conforming (trial space) *SPLS discretization* of (1.1) is defined as a (trial) non-conforming saddle point discretization of (2.4). We consider finite dimensional approximation spaces $V_h \subset V$ and $\mathcal{M}_h \subset \tilde{Q}$ (larger than Q in general) and restrict the forms $a_0(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to the discrete spaces V_h and $\mathcal{M}_h$. Assume that the following discrete $\inf-\sup$ condition holds for the pair $(V_h, \mathcal{M}_h)$:

$$\inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \, \|p_h\|} = m_h > 0. \tag{2.5}$$

We define $V_{h,0}$ to be the kernel of the discrete operator B_h, i.e.,

$$V_{h,0} := \{v_h \in V_h |\ b(v_h, q_h) = 0 \quad \text{for all } q_h \in \mathcal{M}_h\},$$

and let $V_{h,0}^{\perp}$ denote the orthogonal complement of $V_{h,0}$ with respect to the inner product $a_0(\cdot,\cdot)$ on V_h. If $V_{h,0} \subset V_0$, then the compatibility condition (2.3) implies a discrete compatibility condition. Consequently, under the discrete stability assumption (2.5), the problem of finding $p_h \in \mathcal{M}_h$ such that

$$b(v_h, p_h) = \langle F, v_h \rangle \quad \text{for all } v_h \in V_h, \tag{2.6}$$

has a unique solution.

In general, the compatibility condition (2.3) might not hold on $V_{h,0}$. Hence, the discrete problem (2.6) may not be well-posed. In any case, under the assumption (2.5), the standard discrete saddle point problem of finding $(u_h, p_h) \in V_h \times \mathcal{M}_h$ such that

$$\begin{array}{lcll} a_0(u_h, v_h) & + & b(v_h, p_h) & = \langle F, v_h \rangle \qquad \text{for all } v_h \in V_h, \\ b(u_h, q_h) & & & = 0 \qquad\qquad \text{for all } q_h \in \mathcal{M}_h, \end{array} \tag{2.7}$$

does have a unique solution. We call the variational formulation (2.7) the *non-conforming saddle point least squares discretrization* of (1.1). As in the continuous case, it is easy to prove that the p_h part of the solution of (2.7) is the solution of the *normal equation* associated with (2.6).

2.2. **The discrete spaces.** Let V_h be a *finite element subspace* of V and assume that the action of $\mathcal{C}^{-1}$ at the continuous level is easy to obtain.

2.2.1. *No projection trial space.* We first consider the case when $\mathcal{M}_h$ is given by

$$\mathcal{M}_h := \mathcal{C}^{-1} B V_h \subset \tilde{Q}.$$

In this case, we have $V_{h,0} \subset V_0$ and a discrete $\inf-\sup$ condition holds. Indeed, for a generic $p_h = \mathcal{C}^{-1} B w_h \in \mathcal{M}_h$ where $w_h \in V_{h,0}^{\perp}$, we have

$$\begin{aligned} \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \ \|p_h\|} &= \inf_{w_h \in V_{h,0}^{\perp}} \sup_{v_h \in V_h} \frac{(\mathcal{C}^{-1} B v_h, \mathcal{C}^{-1} B w_h)_{\tilde{Q}}}{|v_h| \, \|\mathcal{C}^{-1} B w_h\|} \\ &\geq \inf_{w_h \in V_{h,0}^{\perp}} \frac{\|\mathcal{C}^{-1} B w_h\|^2}{|w_h| \, \|\mathcal{C}^{-1} B w_h\|} = \inf_{w_h \in V_{h,0}^{\perp}} \frac{\|\mathcal{C}^{-1} B w_h\|}{|w_h|} := m_{h,0}. \end{aligned} \tag{2.8}$$

Thus, we have that both variational formulations (2.6) and (2.7) have a unique solution $p_h \in \mathcal{M}_h$. Furthermore, using Proposition 2.1 for the discrete pair $(V_h, \mathcal{M}_h)$, we have that $(u_h = 0, p_h)$ is the solution of (2.7).

2.2.2. *Approximability of no projection trial space.* Note that if p is the solution of (1.1) and p_h is the solution of (2.6), or $(0, p_h)$ is the solution of (2.7), then from (1.1) and (2.6) we obtain

$$0 = b(v_h, p - p_h) = (\mathcal{C}^{-1} B v_h, p - p_h)_{\tilde{Q}} \quad \text{for all } v_h \in V_h.$$

Thus, p_h is the orthogonal projection of p onto $\mathcal{M}_h$ which gives us

$$\|p - p_h\| = \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|.$$

This result is optimal, and in contrast with the standard approximation estimates for saddle point problems, it does not depend on $m_{h,0}$.

2.2.3. *Projection type trial space.* Let $\tilde{\mathcal{M}}_h \subset \tilde{Q}$ be a finite dimensional subspace equipped with the inner product $(\cdot,\cdot)_h$. Define the representation operator $R_h : \tilde{Q} \to \tilde{\mathcal{M}}_h$ by

$$(R_h p, q_h)_h := (p, q_h)_{\tilde{Q}} \quad \text{for all } q_h \in \tilde{\mathcal{M}}_h. \tag{2.9}$$

Here, $R_h p$ is the Riesz representation of $p \to (p, q_h)_{\tilde{Q}}$ as a functional on $(\tilde{\mathcal{M}}_h, (\cdot,\cdot)_h)$.

Remark 2.2. *In the case when $(\cdot,\cdot)_h$ coincides with the inner product on $\tilde{Q}$, we have that R_h is the orthogonal projection onto $\tilde{\mathcal{M}}_h$.*

Since the space $\tilde{\mathcal{M}}_h$ is finite dimensional, there exist constants k_1, k_2 such that

$$k_1 \|q_h\| \le \|q_h\|_h \le k_2 \|q_h\| \quad \text{for all } q_h \in \tilde{\mathcal{M}}_h. \tag{2.10}$$

We further assume that the equivalence is uniform with respect to h, i.e., the constants k_1, k_2 are independent of h. Using the operator R_h, we define $\mathcal{M}_h$ as

$$\mathcal{M}_h := R_h \mathcal{C}^{-1} B V_h \subset \tilde{\mathcal{M}}_h \subset \tilde{Q}.$$

The following proposition gives a sufficient condition on R_h to ensure the discrete $\inf - \sup$ condition is satisfied and relates the stability of the families of spaces $\{(V_h, \mathcal{C}^{-1} B V_h)\}$ and $\{(V_h, R_h \mathcal{C}^{-1} B V_h)\}$.

Proposition 2.3. *Assume that*

$$\|R_h q_h\|_h \ge \tilde{c}\, \|q_h\| \quad \textit{for all } q_h \in \mathcal{C}^{-1} B V_h, \tag{2.11}$$

with a constant $\tilde{c}$ independent of h. Then $V_{h,0} \subset V_0$. Furthermore, the stability of the family $\{(V_h, \mathcal{C}^{-1} B V_h)\}$, meaning $m_{h,0}$ defined in (2.8) *satisfies $m_{h,0} > c_0 > 0$ for some constant c_0 independent of h, implies the stability of the family $\{(V_h, R_h \mathcal{C}^{-1} B V_h)\}$.*

Proof. Let $v_h \in V_{h,0}$. Then, for any $p_h \in \mathcal{M}_h$,

$$0 = b(v_h, p_h) = (\mathcal{C}^{-1} B v_h, p_h)_{\tilde{Q}} = (R_h \mathcal{C}^{-1} B v_h, p_h)_h.$$

Taking $p_h = R_h \mathcal{C}^{-1} B v_h$ gives us $\|R_h \mathcal{C}^{-1} B v_h\|_h = 0$ and the inclusion $V_{h,0} \subset V_0$ follows from (2.11). To show the stability, we take a generic function $p_h = R_h \mathcal{C}^{-1} B w_h \in \mathcal{M}_h$ where $w_h \in V_{h,0}^{\perp}$. We have

$$\begin{aligned} m_h = \inf_{p_h \in \mathcal{M}_h} \sup_{v_h \in V_h} \frac{b(v_h, p_h)}{|v_h| \, \|p_h\|_h} &= \inf_{w_h \in V_{h,0}^{\perp}} \sup_{v_h \in V_h} \frac{(\mathcal{C}^{-1} B v_h, R_h \mathcal{C}^{-1} B w_h)_{\tilde{Q}}}{|v_h| \, \|R_h \mathcal{C}^{-1} B w_h\|_h} \\ &= \inf_{w_h \in V_{h,0}^{\perp}} \sup_{v_h \in V_h} \frac{(R_h \mathcal{C}^{-1} B v_h, R_h \mathcal{C}^{-1} B w_h)_h}{|v_h| \, \|R_h \mathcal{C}^{-1} B w_h\|_h} \\ &\geq \inf_{w_h \in V_{h,0}^{\perp}} \frac{\|R_h \mathcal{C}^{-1} B w_h\|_h^2}{|w_h| \, \|R_h \mathcal{C}^{-1} B w_h\|_h} \\ &\geq \tilde{c} \inf_{w_h \in V_{h,0}^{\perp}} \frac{\|\mathcal{C}^{-1} B w_h\|}{|w_h|} = \tilde{c}\, m_{h,0}, \end{aligned}$$

where $m_{h,0}$ is defined in (2.8). □

As a consequence of Proposition 2.3, we have that under the assumption (2.11) both variational formulations (2.6) and (2.7) have unique solution $p_h \in \mathcal{M}_h$. Furthermore, using Proposition 2.1 for the discrete pair $(V_h, \mathcal{M}_h)$, we have that $(u_h = 0, p_h)$ is the solution of (2.7).

2.2.4. *Approximability of projection type trial space.* The following proposition shows that under condition (2.11) we have a quasi-optimal approximability property for the projection type trial space.

Proposition 2.4. *If p is the solution of* (1.1)*, p_h is the solution of* (2.6) *(or the n-c SPLS solution of* (2.7)*), and R_h satisfies* (2.11)*, then*

$$\|p - p_h\| \leq C \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|,$$

where C depends only on $\tilde{c}$ of (2.11) *and the equivalence of norms constants of* (2.10)*.*

Proof. From the assumptions on p and p_h, using (1.1) and (2.6) we obtain

$$0 = b(v_h, p - p_h) = (\mathcal{C}^{-1} B v_h, p - p_h)_{\tilde{Q}} \quad \text{for all} \;\; v_h \in V_h.$$

In turn, this implies

$$(\mathcal{C}^{-1} B v_h, p - Q_h p)_{\tilde{Q}} = (\mathcal{C}^{-1} B v_h, p_h - Q_h p)_{\tilde{Q}} \quad \text{for all } v_h \in V_h, \tag{2.12}$$

where Q_h is the orthogonal projection onto $\mathcal{M}_h$. Note that

$$\|p_h - Q_h p\|_h = \sup_{q_h \in \mathcal{M}_h} \frac{|(p_h - Q_h p, q_h)_h|}{\|q_h\|_h}. \tag{2.13}$$

Using (2.11) and (2.12), we obtain

$$\begin{aligned}
\sup_{q_h\in\mathcal{M}_h}\frac{|(p_h-Q_hp,q_h)_h|}{\|q_h\|_h} &= \sup_{w_h\in V_{h,0}^{\perp}}\frac{|(p_h-Q_hp,R_h\mathcal{C}^{-1}Bw_h)_h|}{\|R_h\mathcal{C}^{-1}Bw_h\|_h}\\
&= \sup_{w_h\in V_{h,0}^{\perp}}\frac{|(p_h-Q_hp,\mathcal{C}^{-1}Bw_h)_{\tilde{Q}}|}{\|R_h\mathcal{C}^{-1}Bw_h\|_h}\\
&= \sup_{w_h\in V_{h,0}^{\perp}}\frac{|(p-Q_hp,\mathcal{C}^{-1}Bw_h)_{\tilde{Q}}|}{\|R_h\mathcal{C}^{-1}Bw_h\|_h}\\
&\le \sup_{w_h\in V_{h,0}^{\perp}}\frac{\|p-Q_hp\|\,\|\mathcal{C}^{-1}Bw_h\|}{\|R_h\mathcal{C}^{-1}Bw_h\|_h}\le\frac{1}{\tilde{c}}\|p-Q_hp\|.
\end{aligned}$$

Hence, from (2.10), (2.13), and the above estimate we have

$$\|Q_hp-p_h\|\le\frac{1}{k_1}\|Q_hp-p_h\|_h\le\frac{1}{\tilde{c}k_1}\|p-Q_hp\|. \tag{2.14}$$

Thus,

$$\begin{aligned}
\|p-p_h\| &\le \|p-Q_hp\|+\|Q_hp-p_h\|\\
&\le\left(1+\frac{1}{\tilde{c}k_1}\right)\|p-Q_hp\| = C\inf_{q_h\in\mathcal{M}_h}\|p-q_h\|.
\end{aligned}$$

□

Remark 2.5. *The no projection trial space described in Section 2.2.1 can be viewed as the special case of the projection type trial space when $R_h = I$.*

2.3. **Iterative solvers.** When solving (2.7) on $(V_h, \mathcal{M}_h = \mathcal{C}^{-1}BV_h)$ or $(V_h, \mathcal{M}_h = R_h\mathcal{C}^{-1}BV_h)$, a global linear system might be difficult to assemble as one may not be able to find simple local bases for the space $\mathcal{M}_h$. Nevertheless, it is possible to solve (2.7) without an explicit basis for $\mathcal{M}_h$ by using the Uzawa (U), Uzawa Gradient (UG), or Uzawa Conjugate Gradient (UCG) algorithm. We will describe each algorithm below. For implementation and convergence analysis for such algorithms, it is essential to use the $(\cdot,\cdot)_h$ inner product on $\mathcal{M}_h$.

Algorithm 2.6. *(U-UG) Algorithms*

Step 1: Set $u_0 = 0 \in V_h$, $p_0 \in \mathcal{M}_h$, **compute** $u_1 \in V_h$, $q_1 \in \mathcal{M}_h$ *by*

$$\begin{aligned}
a_0(u_1,v) &= \langle F,v\rangle - b(v,p_0) && \text{for all } v\in V_h,\\
(q_1,q)_h &= b(u_1,q) && \text{for all } q\in\mathcal{M}_h.
\end{aligned}$$

Step 2: For $j = 1, 2, \ldots,$ **compute** $h_j, \alpha_j, p_j, u_{j+1}, q_{j+1}$ *by*

$$
\begin{array}{lrll}
(\mathbf{U-UG1}) & a_0(h_j, v) = & -b(v, q_j) & \textit{for all } v \in V_h \\
(\mathbf{U}\alpha) & \alpha_j = & \alpha_0 & \textit{for the Uzawa algorithm or} \\
(\mathbf{UG}\alpha) & \alpha_j = & -\dfrac{(q_j, q_j)_h}{b(h_j, q_j)} & \textit{for the UG algorithm} \\
(\mathbf{U-UG2}) & p_j = & p_{j-1} + \alpha_j\ q_j & \\
(\mathbf{U-UG3}) & u_{j+1} = & u_j + \alpha_j\ h_j & \\
(\mathbf{U-UG4}) & (q_{j+1}, q)_h = & b(u_{j+1}, q) & \textit{for all } q \in \mathcal{M}_h.
\end{array}
$$

To obtain the UCG algorithm, the UG algorithm is modified as in [14, 26] by the following steps. First, we define $d_1 := q_1$ in **Step 1**. Then, we modify **Step 2** by replacing $b(\cdot, q_j)$ with $b(\cdot, d_j)$, where $\{d_j\}$ is a sequence of conjugate directions. The resulting algorithm is as follows:

Algorithm 2.7. *(UCG) Algorithm*

Step 1: Set $u_0 = 0 \in V_h$, $p_0 \in \mathcal{M}_h$. **Compute** $u_1 \in V_h$, $q_1, d_1 \in \mathcal{M}_h$ *by*

$$
\begin{array}{lll}
a_0(u_1, v) & = \langle F, v\rangle - b(v, p_0) & \textit{for all } v \in V_h, \\
(q_1, q)_h & = b(u_1, q) & \textit{for all } q \in \mathcal{M}_h, \quad d_1 := q_1.
\end{array}
$$

Step 2: For $j = 1, 2, \ldots,$ **compute** $h_j, \alpha_j, p_j, u_{j+1}, q_{j+1}, \beta_j, d_{j+1}$ *by*

$$
\begin{array}{lrll}
(\mathbf{UCG1}) & a_0(h_j, v) = & -\, b(v, d_j) & \textit{for all } v \in V_h \\
(\mathbf{UCG}\alpha) & \alpha_j = & -\dfrac{(q_j, q_j)_h}{b(h_j, q_j)} & \\
(\mathbf{UCG2}) & p_j = & p_{j-1} + \alpha_j\ d_j & \\
(\mathbf{UCG3}) & u_{j+1} = & u_j + \alpha_j\ h_j & \\
(\mathbf{UCG4}) & (q_{j+1}, q)_h = & b(u_{j+1}, q) & \textit{for all } q \in \mathcal{M}_h \\
(\mathbf{UCG}\beta) & \beta_j = & \dfrac{(q_{j+1}, q_{j+1})_h}{(q_j, q_j)_h} & \\
(\mathbf{UCG6}) & d_{j+1} = & q_{j+1} + \beta_j d_j. &
\end{array}
$$

Note that at each iteration step, only one inversion involving the form $a_0(\cdot, \cdot)$ is required. In [5], it was proven that if (u_h, p_h) is the discrete solution of (2.7) and (u_{j+1}, p_j) is the j^{th} iteration for the U, UG, or UCG algorithm, then $(u_{j+1}, p_j) \to (u_h, p_h)$. In addition, there are constants c_1, c_2, independent of h, such that for all $j = 1, 2, \ldots,$ we have

$$
\begin{aligned}
\frac{c_1}{M^2}\|q_{j+1}\| \le \|p_j - p_h\| \le \frac{c_2}{m_h^2}\|q_{j+1}\|, \\
c_1\,\frac{m_h}{M^2}\|q_{j+1}\| \le \|u_{j+1} - u_h\| \le c_2\,\frac{M}{m_h^2}\|q_{j+1}\|.
\end{aligned}
\tag{2.15}
$$

Hence, the first equation in (2.15) entitles $\|q_{j+1}\|$ as a computable, efficient, and uniform iteration error estimator for all three algorithms. The stability and approximability results of Section 2.2 remain valid if $a_0(\cdot,\cdot)$ is replaced by a uniformly equivalent form $a_{prec}(\cdot,\cdot)$.

We note that for the no projection choice of trial space $\mathcal{M}_h$ outlined in Section 2.2.1, the residual q_{j+1} from **Step 1**, **(U-UG4)**, and **UCG4** can be computed using the action of the operator $\mathcal{C}^{-1}B$, i.e.,

$$q_{j+1} = \mathcal{C}^{-1}Bu_{j+1}.$$

Also, for the choice of a projection type trial space for $\mathcal{M}_h$ outlined in Section 2.2.3, the residual q_{j+1} can be computed by applying the operator $\mathcal{C}^{-1}B$ followed by the operator R_h, i.e.,

$$q_{j+1} = R_h(\mathcal{C}^{-1}Bu_{j+1}).$$

3. n-c SPLS for second order elliptic interface problems

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain with $\{\Omega_j\}_{j=1}^N$ a partition of Ω and $\mathbf{n}_j$ be the outward unit normal vector to $\partial\Omega_j$. Define $\Gamma_{km} := \partial\Omega_k \cap \partial\Omega_m$ to be the interface between Ω_k and Ω_m for $1 \le k < m \le N$. Given $f \in L^2(\Omega)$, we consider the problem of finding $u \in H_0^1(\Omega)$ such that

$$-\mathrm{div}(A\nabla u) = f \quad \text{in } \Omega, \tag{3.1}$$

with the continuity of the co-normal derivative condition

$$[\![A\nabla u \cdot \mathbf{n}]\!]_{\Gamma_{km}} = (A_k\nabla u_k \cdot \mathbf{n}_k + A_m\nabla u_m \cdot \mathbf{n}_m)\big|_{\Gamma_{km}} = 0 \quad \text{for all } k < m.$$

We assume the matrix A is symmetric and satisfies

$$a_{min}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le a_{max}|\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^d, \tag{3.2}$$

for positive constants $a_{min} \le a_{max}$. In addition, the entries could by discontinuous, with possibly large jumps, across the subdomain boundaries. In the above equation and for the remainder of this section, we denote the standard Euclidean inner product and norm for vectors in $\mathbb{R}^d$ by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. In addition, $(\cdot,\cdot)$ and $\|\cdot\|$ will denote the standard L^2 inner product and norm for both scalar and vector functions. The primal mixed variational formulation of (3.1) we consider is: Find $\sigma = A\nabla u$, with $u \in H_0^1(\Omega)$, such that

$$(\sigma, \nabla v) = (A\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega). \tag{3.3}$$

To fit (3.3) into the abstract formulation (1.1), we let $V := H_0^1(\Omega)$, $\tilde{Q} := L^2(\Omega)^d, Q := A\nabla V$, and define $b : V \times \tilde{Q} \to \mathbb{R}$ by

$$b(v, q) := (q, \nabla v) \quad \text{for all } v \in V, q \in \tilde{Q}.$$

Also,

$$\langle F, v\rangle := (f, v) \quad \text{for all } v \in V.$$

On V, we consider the standard inner product

$$a_0(u, v) := (\nabla u, \nabla v) \quad \text{for all } u, v \in V,$$

and denote the norm by $|v|_V := ||\nabla v||$. On $\tilde{Q}$, we define the weighted inner product

$$(p, q)_{\tilde{Q}} := (p, A^{-1}q) \quad \text{for all } p, q \in \tilde{Q}.$$

Note that for $\tau_1, \tau_2 \in Q$, we then have

$$(\tau_1, \tau_2)_Q = (\tau_1, \tau_2)_{\tilde{Q}} = (A\nabla u_1, A\nabla u_2)_{\tilde{Q}} = (A\nabla u_1, \nabla u_2).$$

With these inner products on V and $\tilde{Q}$, we have that the operators $B : V \to \tilde{Q}^*$ and $\mathcal{C}^{-1}B : V \to \tilde{Q}$ are given by

$$Bv = \nabla v, \quad \text{and} \quad \mathcal{C}^{-1}Bv = A\nabla v \quad \text{for all } v \in V.$$

Hence,

$$V_0 = Ker(B) = \{v \in V | Bv = 0\} = \{v \in H_0^1(\Omega) | \nabla v = 0\} = \{0\},$$

which implies (2.3) is trivially satisfied. We note that, as presented in [10], the continuity constant satisfies

$$\begin{aligned} M &= \sup_{q\in\tilde{Q}} \sup_{v\in V} \frac{b(v,q)}{|v|_V \, \|q\|_{\tilde{Q}}} = \sup_{q\in\tilde{Q}} \sup_{v\in V} \frac{(q, \nabla v)}{|v|_V \, \|q\|_{\tilde{Q}}} \\ &= \sup_{q\in\tilde{Q}} \sup_{v\in V} \frac{(q, A\nabla v)_{\tilde{Q}}}{|v|_V \, \|q\|_{\tilde{Q}}} \leq \sup_{v\in V} \frac{\|A\nabla v\|_{\tilde{Q}}}{\|\nabla v\|} \leq \sqrt{a_{max}} < \infty. \end{aligned} \tag{3.4}$$

and the $\inf - \sup$ constant satisfies

$$\begin{aligned} m &= \inf_{q=A\nabla u\in Q} \sup_{v\in V} \frac{b(v,q)}{|v|_V \, \|q\|_{\tilde{Q}}} = \inf_{u\in V} \sup_{v\in V} \frac{(A\nabla u, \nabla v)}{(A\nabla u, \nabla u)^{1/2} \, |v|_V} \\ &\geq \inf_{u\in V} \frac{(A\nabla u, \nabla u)}{(A\nabla u, \nabla u)^{1/2} \, \|\nabla u\|} \geq \sqrt{a_{min}} > 0. \end{aligned} \tag{3.5}$$

Consequently, the variational problem (3.3) is well-posed and suitable for n-c SPLS formulation and discretization.

3.1. **n-c SPLS discretization for second order elliptic interface problems.** We take $V_h \subset V = H_0^1(\Omega)$ to be the space of continuous piecewise polynomials of degree k with respect to the *interface-fitted* triangular mesh $\mathcal{T}_h$. We note that while the *no projection trial space* case is similar with the work presented in [10], the *projection trial space* is analyzed using the non-conforming trial space setting and leads to new stability and approximability estimates for the discontinuous coefficients (or interface) case.

3.1.1. *No projection trial space.* Following Section 2.2.1, we define the trial space as

$$\mathcal{M}_h := \mathcal{C}^{-1}BV_h = A\nabla V_h.$$

By similar arguments used to show (3.5), we obtain

$$m_h := \inf_{q_h=A\nabla u_h\in\mathcal{M}_h} \sup_{v_h\in V_h} \frac{b(v_h, q_h)}{|v_h|_V \, \|q_h\|_{\tilde{Q}}} \geq \sqrt{a_{min}} > 0. \tag{3.6}$$

Thus, we do have stability in this case. The discrete mixed variational formulation is: Find $\sigma_h = A\nabla u_h$, with $u_h \in V_h$, such that

$$(\sigma_h, \nabla v_h) = (A\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \tag{3.7}$$

The SPLS discretization (2.7) to be solved is: Find $(w_h, \sigma_h = A\nabla u_h)$ such that

$$\begin{array}{lllll} (\nabla w_h, \nabla v_h) & + & (A\nabla u_h, \nabla v_h) & = (f, v_h) & \text{for all } v_h \in V_h, \\ A\nabla w_h & & & = 0. & \end{array} \tag{3.8}$$

3.1.2. *Projection type trial space.* We define $\tilde{\mathcal{M}}_h \subset \tilde{Q} = L^2(\Omega)^d$ to be

$$\tilde{\mathcal{M}}_h := \bigoplus_{i=1}^{N} AM_{h,0}|_{\Omega_i},$$

where N is the number of subdomains and where each component of $M_{h,0}|_{\Omega_i}$ consists of continuous piecewise polynomials of degree k with respect to the mesh $\mathcal{T}_{h,i} := \mathcal{T}_h|_{\Omega_i}$ with no restrictions on the boundary. We equip $\tilde{\mathcal{M}}_h$ with the inner product

$$(A\tilde{q}_h, A\tilde{p}_h)_h = \sum_{i=1}^{N} (A\tilde{q}_h, A\tilde{p}_h)_{\tilde{Q},\Omega_i} \quad \text{for all } A\tilde{q}_h, A\tilde{p}_h \in \tilde{\mathcal{M}}_h.$$

Here, $(\cdot,\cdot)_{\tilde{Q},\Omega_i}$ is the inner product on $\tilde{Q}$ restricted to the subdomain Ω_i. Using the definition of R_h given in (2.9), we have that for $p \in \tilde{Q}$

$$\begin{aligned} (p, A\tilde{q}_h)_{\tilde{Q}} = (R_h p, A\tilde{q}_h)_h &= \sum_{i=1}^{N} (R_h p, A\tilde{q}_h)_{\tilde{Q},\Omega_i} \\ &= (R_h p, A\tilde{q}_h)_{\tilde{Q}} \quad \text{for all } A\tilde{q}_h \in \tilde{\mathcal{M}}_h. \end{aligned}$$

Thus, $R_h p$ is the orthogonal projection of p onto $\tilde{\mathcal{M}}_h$ in the $(\cdot,\cdot)_{\tilde{Q}}$ inner product. In turn, this implies $R_h p|_{\Omega_j}$ is the orthogonal projection onto $\tilde{\mathcal{M}}_h|_{\Omega_j} = AM_{h,0}|_{\Omega_j}$ in the $(\cdot,\cdot)_{\tilde{Q}}$ inner product. We then define

$$\mathcal{M}_h := R_h A\nabla V_h.$$

The discrete mixed variational formulation in this case is: Find $\sigma_h = R_h A\nabla u_h$, with $u_h \in V_h$, such that

$$(\sigma_h, \nabla v_h) = (R_h A\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \tag{3.9}$$

The n-c SPLS discretization (2.7) to be solved is: Find $(w_h, \sigma_h = R_h A\nabla u_h)$ such that

$$\begin{array}{lllll} (\nabla w_h, \nabla v_h) & + & (R_h A\nabla u_h, \nabla v_h) & = (f, v_h) & \text{for all } v_h \in V_h, \\ R_h A\nabla w_h & & & = 0. & \end{array} \tag{3.10}$$

3.1.3. *Piecewise linear test space.* We make further assumptions to discuss stability for the family $\{(V_h, \mathcal{M}_h)\}$. We assume for simplicity $\Omega \subset \mathbb{R}^2$ is a polygonal domain separated into two subdomains by a smooth interface $\Gamma \subset \Omega$. The results can easily be extended to N subdomains as well as polyhedral domains in $\mathbb{R}^3$. We also assume that the triangular mesh $\mathcal{T}_h$ is locally quasi-uniform. Let $\{z_{1,i}, \dots, z_{N_i,i}\}$ be the set of all nodes of $\mathcal{T}_{h,i}$ and assume all triangles adjacent to $z_{j,i}$ are of regular shape and their area is of order $h_{j,i}^2$. In this notation, the mesh size of $\mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$ is $h := \max\{h_{1,1}, h_{2,1}, \dots, h_{N_1,1}, h_{1,2}, h_{2,2}, \dots, h_{N_2,2}\}$.

We take V_h to be the space consisting of piecewise linear polynomials with respect to $\mathcal{T}_h$ vanishing on the boundary of Ω. Also, we take $k = 1$. Hence, each component of $M_{h,0}|_{\Omega_i}$ consists of continuous linear piecewise polynomials with respect to the mesh $\mathcal{T}_{h,i}$. Let $\{\Phi_1^i, ..., \Phi_{2N_i}^i\}$ be a nodal basis for $M_{h,0}|_{\Omega_i}$ and assume that $\Phi_j^i = (\phi_j^i, 0)^T$ and $\Phi_{N_i+j}^i = (0, \phi_j^i)^T$ for $j = 1, \dots, N_i$. Here, $\{\phi_1^i, \dots, \phi_{N_i}^i\}$ is a nodal basis for the space of continuous piecewise linear polynomials with respect to $\mathcal{T}_{h,i}$. With this notation, we note that $\{A\Phi_j^1\}_{j=1}^{N_1} \cup \{A\Phi_j^2\}_{j=1}^{N_2}$ is a basis for $\tilde{\mathcal{M}}_h$. Lastly, we define M_{A_i} to be the Gram matrix of the set $\{A\Phi_j^i\}_{j=1}^{N_i}$ with respect to the $(\cdot, \cdot)_{\tilde{Q}}$ inner product and $D_i := \operatorname{diag}\left(h_{1,i}^2, h_{2,i}^2, \dots, h_{N_i,i}^2, h_{1,i}^2, h_{2,i}^2, \dots, h_{N_i,i}^2\right)$. To prove stability for the family $\{(V_h, \mathcal{M}_h)\}$, we need the following two lemmata. The first lemma follows from a similar result (for no interfaces) proved in [10] and, for completeness, is restated using the notation and assumptions from this section.

Lemma 3.1. *Under the assumptions of Section 3.1.3, we have that for* $i = 1, 2$

$$\langle M_{A_i}\gamma, \gamma\rangle \le c\, a_{max} \langle D_i \gamma, \gamma\rangle \quad \text{for all } \gamma \in \mathbb{R}^{2N_i}. \tag{3.11}$$

Consequently,

$$\langle M_{A_i}^{-1}\gamma, \gamma\rangle \ge \frac{c}{a_{max}} \langle D_i^{-1} \gamma, \gamma\rangle \quad \text{for all } \gamma \in \mathbb{R}^{2N_i}. \tag{3.12}$$

We note that the constant c in the above lemma is generic and does not depend on h. The next result shows that (2.11) is satisfied for the representation operator R_h defined in this section.

Lemma 3.2. *Under the assumptions of Section 3.1.3, there exists a constant c, independent of h, such that*

$$\|R_h A\nabla v_h\|_h \ge c\, \frac{a_{min}}{a_{max}}\, \|A\nabla v_h\|_{\tilde{Q}} \quad \text{for all } v_h \in V_h. \tag{3.13}$$

Proof. First, note that $\{A\Phi_1^1, \dots, A\Phi_{2N_1}^1\}$ and $\{A\Phi_1^2, \dots, A\Phi_{2N_2}^2\}$ are nodal bases for $\tilde{\mathcal{M}}_h|_{\Omega_1}$ and $\tilde{\mathcal{M}}_h|_{\Omega_2}$, respectively. Define $v_h^i := v_h|_{\Omega_i}$ for $v_h \in V_h$. For a fixed $A\nabla v_h$ with $v_h \in V_h$ we define the dual vectors $G_h^1 \in \mathbb{R}^{2N_1}, G_h^2 \in$

$\mathbb{R}^{2N_2}$ by

$$(G_h^1)_i := (A\nabla v_h^1, A\Phi_i^1)_{\tilde{Q}} = (A\nabla v_h^1, \Phi_i^1) \; i = 1, ..., 2N_1,$$
$$(G_h^2)_i := (A\nabla v_h^2, A\Phi_i^2)_{\tilde{Q}} = (A\nabla v_h^2, \Phi_i^2) \; i = 1, ..., 2N_2,$$

and let

$$R_h A\nabla v_h = \begin{cases} \sum_{i=1}^{2N_1} \alpha_i A\Phi_i^1 & \text{in } \Omega_1, \\ \\ \sum_{i=1}^{2N_2} \beta_i A\Phi_i^2 & \text{in } \Omega_2. \end{cases}$$

Thus, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2N_1})^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{2N_2})^T$ are solutions to

$$M_{A_1}\, \alpha = G_h^1, \qquad \text{and} \qquad M_{A_2}\, \beta = G_h^2,$$

respectively. Using (3.12), we obtain

$$\begin{aligned} \|R_h A\nabla v_h\|_h^2 &= \sum_{i,j=1}^{2N_1} \alpha_i\, \alpha_j \left(A\Phi_i^1, \Phi_j^1\right) + \sum_{i,j=1}^{2N_2} \beta_i\, \beta_j \left(A\Phi_i^2, \Phi_j^2\right) \\ &= \left\langle M_{A_1}^{-1} G_h^1, G_h^1 \right\rangle + \left\langle M_{A_2}^{-1} G_h^2, G_h^2 \right\rangle \\ &\geq \frac{1}{a_{max}} \left\langle D_1^{-1} G_h^1, G_h^1 \right\rangle + \frac{1}{a_{max}} \left\langle D_2^{-1} G_h^2, G_h^2 \right\rangle \\ &= \frac{1}{a_{max}} \sum_{i=1}^{2N_1} h_{i,1}^{-2} \left(A\nabla v_h^1, \Phi_i^1\right)^2 + \frac{1}{a_{max}} \sum_{i=1}^{2N_2} h_{i,2}^{-2} \left(A\nabla v_h^2, \Phi_i^2\right)^2. \end{aligned}$$

We recall by definition of D_1, D_2 that we have $h_{i,1} = h_{i+N_1,1}$ for $i = 1, \ldots, N_1$ and $h_{i,2} = h_{i+N_2,2}$ for $i = 1, \ldots, N_2$ in the above. Note that

$$\begin{aligned} \frac{1}{a_{max}} \sum_{i=1}^{2N_1} h_{i,1}^{-2} \left(A\nabla v_h^1, \Phi_i^1\right)^2 &= \frac{1}{a_{max}} \sum_{i=1}^{N_1} h_{i,1}^{-2} \left[\left(a_{11}\frac{\partial v_h^1}{\partial x} + a_{12}\frac{\partial v_h^1}{\partial y}, \phi_i^1 \right)^2 + \left(a_{21}\frac{\partial v_h^1}{\partial x} + a_{22}\frac{\partial v_h^1}{\partial y}, \phi_i^1 \right)^2 \right] \\ &= \frac{1}{a_{max}} \sum_{i=1}^{N_1} h_{i,1}^{-2} \sum_{\tau^1 \subset supp(\phi_i)} \left| \begin{pmatrix} (a_{11}, \phi_i^1)_{\tau^1} & (a_{12}, \phi_i^1)_{\tau^1} \\ (a_{21}, \phi_i^1)_{\tau^1} & (a_{22}, \phi_i^1)_{\tau^1} \end{pmatrix} \begin{pmatrix} \frac{\partial v_h^1|_{\tau^1}}{\partial x} \\ \frac{\partial v_h^1|_{\tau^1}}{\partial y} \end{pmatrix} \right|^2 \\ &\geq c_1 \frac{a_{min}^2}{a_{max}} \sum_{i=1}^{N_1} \sum_{\tau^1 \subset supp(\phi_i)} h_{i,1}^2 |\nabla v_h^1|_{\tau^1}^2 \\ &= c_1 \frac{a_{min}^2}{a_{max}} ||\nabla v_h^1||_{\Omega_1}^2, \end{aligned}$$

where the inequality above follows as the lowest eigenvalue of the matrix

$$\begin{pmatrix} (a_{11}, \phi_i^1)_{\tau^1} & (a_{12}, \phi_i^1)_{\tau^1} \\ (a_{21}, \phi_i^1)_{\tau^1} & (a_{22}, \phi_i^1)_{\tau^1} \end{pmatrix}$$

is bounded below by $c_1\, h_{i,1}^2 a_{min}$ with a constant c_1 independent of τ^1 and h. Similarly, we can show

$$\frac{1}{a_{max}} \sum_{i=1}^{2N_2} h_{i,2}^{-2} \left(A\nabla v_h^2, \Phi_i^2\right)^2 \geq c_2 \frac{a_{min}^2}{a_{max}} ||\nabla v_h^2||_{\Omega_2}^2.$$

Thus,

$$\begin{aligned}\|R_h A\nabla v_h\|_h^2 &\geq \min(c_1, c_2)\frac{a_{min}^2}{a_{max}}(||\nabla v_h^1||_{\Omega_1}^2 + ||\nabla v_h^2||_{\Omega_2}^2)\\ &\geq \min(c_1, c_2)\frac{a_{min}^2}{a_{max}}||\nabla v_h||^2\\ &\geq \min(c_1, c_2)\frac{a_{min}^2}{a_{max}^2}||A\nabla v_h||_{\tilde{Q}}^2.\end{aligned}$$

For the last inequality, we use that

$$\|A\nabla v_h\|_{\tilde{Q}}^2 = (A\nabla v_h, \nabla v_h) \leq a_{max}\|\nabla v_h\|^2.$$

□

As a consequence of Lemma 3.2, equation (3.6), and Proposition 2.3, we have the following result.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\{T_h\}$ be a family of locally quasi-uniform meshes for Ω. For each h, let V_h be the space of continuous linear functions with respect to the mesh $\{\mathcal{T}_h\}$ that vanish on $\partial\Omega$ and $\mathcal{M}_h$ be the corresponding projection type trial space defined in Section 3.1.2. Then the family of spaces $\{(V_h, \mathcal{M}_h)\}$ is stable.*

We note that in the case when $A = I$, we have that **Step 1** of our Uzawa type iterative process (of Section 2.3) coincides with a standard gradient recovery technique with projection operator R_h for solving the Laplace equation. The n-c SPLS iterative process goes beyond the projection of **Step 1**. By computing further p_j iterations in **Step 2**, we approach $R_h\nabla u_h$, which according to Proposition 2.4, is a quasi-optimal approximation of ∇u with functions in $\mathcal{M}_h := R_h\nabla V_h$.

4. Numerical Results

We implemented the n-c SPLS discretization on second order elliptic PDE of the form (3.1). For all of the examples presented, we took Ω to be a bounded polygonal or polyhedral domain and chose the test space $V_h \subset H_0^1(\Omega)$ to be the space of continuous piecewise linear polynomials with respect to the quasi-uniform, or locally quasi-uniform, meshes $\mathcal{T}_h$. We then used the Uzawa conjugate gradient algorithm to approximate the flux $A\nabla u$.

4.1. **Interface problems.** In these examples, the *projection type* trial space was implemented as outlined in Section 3.1.2.

4.1.1. *Intersecting interface example.* For the first example, we took $\Omega = (0,1) \times (0,1)$ with the interface $\Gamma := \Omega \cap \{(x,y) \mid \; x = 1/2 \text{ or } y = 1/2\}$ as considered in [12]. The family of interface-fitted, locally quasi-uniform meshes $\{\mathcal{T}_h\}$ was obtained by a standard uniform refinement strategy starting with a uniform coarse mesh. We computed f such that for

$$A(x,y) = a(x,y)I_2, \text{ where } a(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1/2]^2 \cup [1/2,1]^2, \\ c & \text{if } (x,y) \in \Omega \setminus ([0,1/2]^2 \cup [1/2,1]^2), \end{cases}$$

the exact solution is $u(x,y) = a(x,y)^{-1} \sin(2\pi x)\sin(2\pi y)$. Table 1 shows the results for $c = 1/3, 1/100$, and 50.

error $= \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$

$h = 2^{-k}$	$c = 1/3$			$c = 1/100$			$c = 50$		
k	error	rate	it	error	rate	it	error	rate	it
1	3.122	0.000	4	15.686	0.000	4	1.576	0.000	4
2	0.759	2.039	7	3.947	1.990	12	0.383	2.040	14
3	0.204	1.893	10	1.070	1.882	27	0.103	1.892	37
4	0.057	1.835	12	0.307	1.803	33	0.028	1.868	72
5	0.016	1.848	15	0.086	1.832	44	0.008	1.869	105

Table 1: Interface problem with intersecting interfaces.

4.1.2. *Gradient singularity at the origin.* For the second example, we solved (3.1) where the gradient of the solution is singular at the origin, see [25]. The domain $\Omega = (-1,1)^2$ is decomposed as $\Omega_2 := \{(x,y) \in \Omega \,|\, 0 < \theta(x,y) < \pi/2\}$ and $\Omega_1 := \Omega \setminus \Omega_2$, where $\theta(x,y)$ is the angle in polar coordinates of the point (x,y). The family of interface-fitted, locally quasi-uniform meshes $\{\mathcal{T}_h\}$ was obtained by a graded refinement strategy depending on a refinement parameter κ [7, 8]. The refinement is done by dividing every edge that contains the singular point (the origin) under a fixed ratio κ such that the edge containing the singular point is scaled by κ. We computed f such that for

$$A(x,y) = a(x,y)I_2, \text{ where } a(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \Omega_1, \\ k & \text{if } (x,y) \in \Omega_2, \end{cases}$$

the exact solution, given in polar coordinates, is $u(r,\theta) = r^\lambda (1-r)^2 \mu(\theta)$ where

$$\mu(\theta) = \begin{cases} \cos(\lambda(\theta - \pi/4)) & \text{if } (x,y) \in \Omega_2, \\ b\cos(\lambda(\pi - |\theta - \pi/4|)) & \text{otherwise}, \end{cases}$$

and

$$\lambda = \frac{4}{\pi}\arctan\left(\sqrt{\frac{3+k}{1+3k}}\right), \qquad b = -k\frac{\sin\left(\lambda\frac{\pi}{4}\right)}{\sin\left(\lambda\frac{3\pi}{4}\right)}.$$

Table 2 shows the results for $k = 5$ and 15.

error $= \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$

level k	$k=5$			$k=15$		
	error	rate	it	error	rate	it
1	0.949	0.000	4	2.605	0.000	5
2	0.585	0.699	9	1.504	0.792	15
3	0.151	1.945	16	0.412	1.868	46
4	0.052	1.545	23	0.143	1.529	72
5	0.017	1.602	31	0.047	1.600	94

Table 2: Interface problem with gradient singularity at $(0,0)$.

4.1.3. *3-D example.* For the third example, we took $\Omega \subset \mathbb{R}^3$ to be the unit cube with interface $\Gamma := \Omega \cap \{(x,y,z) \,|\, x = 1/2\}$. We computed f such that for

$$A(x,y,z) = a(x,y,z)I_3, \text{ where } a(x,y,z) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ k & \text{if } x \geq \frac{1}{2}, \end{cases}$$

the exact solution is

$$u(x,y,z) = \begin{cases} k\,x(x-\frac{1}{2})y(y-1)z(z-1) & \text{if } x < \frac{1}{2}, \\ (x-\frac{1}{2})(x-1)y(y-1)z(1-z) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Table 3 shows the results for $k = 5, 25$, and 50.

error $= \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$

$h = 2^{-k}$	$k=5$			$k=25$			$k=50$		
k	error	rate	it	error	rate	it	error	rate	it
1	0.0456	0.000	1	0.2124	0.000	1	0.4208	0.000	1
2	0.0159	1.517	6	0.0744	1.513	14	0.1475	1.512	18
3	0.0042	1.925	9	0.0196	1.922	27	0.0389	1.921	44
4	0.0011	1.879	12	0.0053	1.882	42	0.0106	1.881	67
5	0.0003	1.863	15	0.0014	1.889	65	0.0028	1.889	110

Table 3: 3-*D* interface problem.

We observe for both convex and non-convex domains that the approximation of the flux is *super-linear*, and the method works well no matter the size of the jump discontinuity. We notice also that the number of iterations depends on the size of the jump. This is due to the fact that for the UCG algorithm the number of iterations depends on the condition number of the discrete Schur complement of the problem, which depends on the $\inf-\sup$ condition constant that factors in the size of the jump discontinuity.

4.2. **Flux recovery for highly oscillatory coefficients.** We note that the stability and approximation results of Section 2.2 can be applied to the case when the coefficients of the PDE (the entries of A) are smooth functions. We would like to illustrate the advantages of our n-c SPLS discretization with projection on an example where the matrix A has highly oscillatory coefficients. We solved (3.1) on $\Omega = (0,1) \times (0,1)$ with $A = a(x,y)I_2$, where

$$a(x,y) = \frac{1}{4 + P(\sin(2\pi x/\varepsilon) + \sin(2\pi y/\varepsilon))}.$$

We computed f such that the exact solution is given by

$$u(x,y) = \frac{\sqrt{4-P^2}}{2}(x^2+y^2)\exp\left(\frac{1}{x^3-x} + \frac{1}{y^3-y}\right).$$

This is a small modification of a similar example presented in [24]. Table 4 shows the results for various values of ε. In all computations, we chose $P = 1.8$.

error $= \|A\nabla u - R_h A\nabla u_h\|_{\tilde{Q}}$

$h = 2^{-k}$	$\varepsilon = 0.4$			$\varepsilon = 0.2$			$\varepsilon = 0.1$			$\varepsilon = 0.05$		
k	error	rate	it	error	rate	it	error	rate	it	error	rate	it
5	2.13e-04	1.42	4	1.34e-04	2.96	4	1.71e-04	1.88	3	3.23e-04	0.27	2
6	4.70e-05	2.18	7	5.65e-05	1.24	6	5.46e-05	1.65	5	6.44e-05	2.33	4
7	1.28e-05	1.88	10	1.42e-05	1.99	9	1.34e-05	2.03	8	1.22e-05	2.40	7
8	3.51e-06	1.87	13	4.07e-06	1.80	12	2.57e-06	2.39	12	2.38e-06	2.36	11

Table 4: Highly oscillatory coefficients example.

The numerical results show almost $O(h^2)$ order of approximation for the flux for meshes that are small enough to capture the high frequency of the coefficients due to the size of ε.

5. Conclusion

We presented a saddle point least squares method with non-conforming trial spaces for discretization of second order PDEs with discontinuous coefficients. The proposed method is easy to implement using Uzawa type algorithm and leads to higher order approximation of the flux if compared with standard finite element (non-mixed) techniques based on linear element approximation. In addition, the method works well when solving second order problems with variable coefficients, including highly oscillatory coefficients, and can be combined with known gradient recovery techniques and adaptive refinement in order to construct optimal or quasi-optimal discrete approximation spaces for the flux.

We plan to further combine the n-c SPLS method with known multilevel and adaptive techniques, [15, 1] for designing robust iterative solvers for more general second order elliptic PDE that are parameter dependent and exhibit singular solutions due to non-convex domains or discontinuous coefficients.

References

[1] M. Ainsworth and J.T. Oden. *A Posteriori Error Estimation in Finite Element Analysis.* Wiley-Interscience, New York, 2000.

[2] A. Aziz and I. Babuška. Survey lectures on mathematical foundations of the finite element method. *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. Aziz, editor*, 1972.

[3] I. Babuška, G. Caloz, and J. E. Osborn. Special finite element methods for a class of second order elliptic problems with rough coefficients. *SIAM Journal on Numerical Analysis*, 31(4):945–981, 1994.

[4] C. Bacuta. Schur complements on Hilbert spaces and saddle point systems. *J. Comput. Appl. Math.*, 225(2):581–593, 2009.

[5] C. Bacuta. Cascadic multilevel algorithms for symmetric saddle point systems. *Comput. Math. Appl.*, 67(10):1905–1913, 2014.

[6] C. Bacuta and P. Monk. Multilevel discretization of symmetric saddle point systems without the discrete LBB condition. *Appl. Numer. Math.*, 62(6):667–681, 2012.

[7] C. Bacuta, V. Nistor, and L. Zikatanov. Improving the rate of convergence of 'high order finite elements' on polygons and domains with cups. *Numerische Mathematik*, 100(2):165 –184, 2005.

[8] C. Bacuta, V. Nistor, and L. Zikatanov. Improving the rate of convergence of high-order finite elements on polyhedra. I. A priori estimates. *Numer. Funct. Anal. Optim.*, 26(6):613–639, 2005.

[9] C. Bacuta and K. Qirko. A saddle point least squares approach to mixed methods. *Comput. Math. Appl.*, 70(12):2920–2932, 2015.

[10] C. Bacuta and K. Qirko. A saddle point least squares approach for primal mixed formulations of second order PDEs. *Comput. Math. Appl.*, 73(2):173–186, 2017.

[11] N. Bakhvalov and G. Panasenko. *Homogenisation: averaging processes in periodic media*, volume 36 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1989. Mathematical problems in the mechanics of composite materials, Translated from the Russian by D. Leĭtes.

[12] R. E. Bank and J. Xu. Asymptotically exact a posteriori error estimators II: General unstructured grids. *SIAM J. Numer. Anal.*, 41(6):2313–2332, 2003.

[13] C. Bernardi and R. Verfürth. Adaptive finite element methods for elliptic equations with non-smooth coefficients. *Numerische Mathematik*, 85(4):579–608, Jun 2000.

[14] D. Braess. *Finite Elements. Theory, Fast Solvers, and Applications in Solid Mechanics.* Cambridge University Press, Cambridge, 1997.

[15] C. Carstensen, M. Eigel, R. H. W. Hoppe, and C. Löbhard. A review of unified a posteriori finite element error control. *Numer. Math. Theory Methods Appl.*, 5(4):509–558, 2012.

[16] A. Cohen, W. Dahmen, and G. Welper. Adaptivity and variational stabilization for convection-diffusion equations. *ESAIM Math. Model. Numer. Anal.*, 46(5):1247–1273, 2012.

[17] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. Part I: the transport equation. *Comput. Methods Appl. Mech. Engrg.*, 199(23-24):1558–1572, 2010.

[18] L. Demkowicz and J. Gopalakrishnan. A primal DPG method without a first-order reformulation. *Comput. Math. Appl.*, 66(6):1058–1064, 2013.

[19] Y. Efendiev and T. Hou. Multiscale finite element methods for porous media flows and their applications. *Applied Numerical Mathematics*, 57(5):577 – 596, 2007.

[20] D. Gueribiz, F. Jacquemin, and S. Fréour. A moisture diffusion coupled model for composite materials. *European Journal of Mechanics - A/Solids*, 42:81 – 89, 2013.

[21] A. Hansbo and P. Hansbo. An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. *Computer Methods in Applied Mechanics and Engineering*, 191(47-48):5537–5552, November 2002.

[22] B. C. Khoo, Z. Li, and P. Lin, editors. *Interface problems and methods in biological and physical flows*, volume 17 of *Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009. Selected papers from the Workshop on Moving Interface Problems and Applications held in Singapore, January 8–March 31, 2007.

[23] A. Knyazev and O. Widlund. Lavrentiev regularization + ritz approximation = uniform finite element error estimates for differential equations with rough coefficients. *Mathematics of Computation*, 72(241):17–40, 2003.

[24] L. Mu, J. Wang, and X. Ye. A weak galerkin generalized multiscale finite element method. *Journal of Computational and Applied Mathematics*, 305(Supplement C):68 – 81, 2016.

[25] M. Petzoldt. *Regularity and Error Estimators for Elliptic Problems with Discontinuous Coefficients*. PhD thesis, 2001.

[26] R. Verfürth. A combined conjugate gradient-multigrid algorithm for the numerical solution of the Stokes problem. *IMA J. Numer. Anal.*, 4(4):441–455, 1984.

University of Delaware, Department of Mathematics, 501 Ewing Hall 19716
E-mail address: bacuta@udel.edu

University of Delaware, Department of Mathematics, 501 Ewing Hall 19716
E-mail address: jjacav@udel.edu