

# SHARP STABILITY AND APPROXIMATION ESTIMATES FOR SYMMETRIC SADDLE POINT SYSTEMS

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ABSTRACT. We establish sharp well-posedness and approximation estimates for variational saddle point systems at the continuous level. The main results of this note have been known to be true only in the finite dimensional case. Known spectral results from the discrete case are reformulated and proved using a functional analysis view, making the proofs in both cases, discrete and continuous, less technical than the known discrete approaches. We focus on analyzing the special case when the form  $a(\cdot, \cdot)$  is bounded, symmetric, and coercive, and the mixed form  $b(\cdot, \cdot)$  is bounded and satisfies a standard inf – sup or LBB condition. We characterize the spectrum of the symmetric operators that describe the problem at the continuous level. For a particular choice of the inner product on the product space of  $b(\cdot, \cdot)$ , we prove that the spectrum of the operator representing the system at continuous level is  $\left\{ \frac{1-\sqrt{5}}{2}, 1, \frac{1+\sqrt{5}}{2} \right\}$ . As consequences of the spectral description, we find the minimal length interval that contains the ratio between the norm of the data and the norm of the solution, and prove explicit approximation estimates that depend only on the continuity constant and the continuous and the discrete inf – sup condition constants.

## 1. NOTATION AND STANDARD PROPERTIES

The existing literature on stability and approximation estimates for symmetric Saddle Point (SP) systems is quite rich for both continuous and discrete levels. While at discrete level the estimates can be done using eigenvalue analysis of symmetric matrices and consequently are optimal, at the continuous level, the estimates are presented as inequalities depending on related constants and consequently are not optimal. In this note, we will establish optimal estimates at the continuous level, that can be viewed as generalizations of results at the discrete level. The new spectral estimates provide more insight into the behavior of the general symmetric SP problems. In addition, information about the continuous spectrum and the techniques used to characterize it can lead to efficient analysis of iterative methods for SP systems.

Towards this end, we let  $\mathbf{V}$  and  $Q$  be two Hilbert spaces with inner products given by *symmetric* bilinear forms  $a(\cdot, \cdot)$  and  $(\cdot, \cdot)$  respectively, with

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the corresponding induced norms  $|\cdot|_{\mathbf{V}} = \|\cdot\| = a(\cdot, \cdot)^{1/2}$  and  $\|\cdot\|_Q = \|\cdot\| = (\cdot, \cdot)^{1/2}$ . The dual pairings on  $\mathbf{V}^* \times \mathbf{V}$  and  $Q^* \times Q$  are denoted by  $\langle \cdot, \cdot \rangle$ . Here,  $\mathbf{V}^*$  and  $Q^*$  denote the duals of  $\mathbf{V}$  and  $Q$ , respectively. With the inner products  $a(\cdot, \cdot)$  and  $(\cdot, \cdot)$ , we associate the operators  $\mathcal{A} : V \rightarrow V^*$  and  $\mathcal{C} : Q \rightarrow Q^*$  defined by

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

and

$$\langle \mathcal{C}p, q \rangle = (p, q) \quad \text{for all } p, q \in Q.$$

The operators  $\mathcal{A}^{-1} : V^* \rightarrow V$  and  $\mathcal{C}^{-1} : Q^* \rightarrow Q$  are the Riesz-canonical representation operators and satisfy

$$(1.1) \quad a(\mathcal{A}^{-1}\mathbf{u}^*, \mathbf{v}) = \langle \mathbf{u}^*, \mathbf{v} \rangle, \quad |\mathcal{A}^{-1}\mathbf{u}^*|_{\mathbf{V}} = \|\mathbf{u}^*\|_{\mathbf{V}^*}, \quad \mathbf{u}^* \in \mathbf{V}^*, \mathbf{v} \in \mathbf{V},$$

$$(1.2) \quad (\mathcal{C}^{-1}p^*, q) = \langle p^*, q \rangle, \quad \|\mathcal{C}^{-1}p^*\| = \|p^*\|_{Q^*}, \quad p^* \in Q^*, q \in Q.$$

Next, we suppose that  $b(\cdot, \cdot)$  is a continuous bilinear form on  $\mathbf{V} \times Q$ , satisfying the inf-sup condition. More precisely, we assume that

$$(1.3) \quad \sup_{p \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| |\mathbf{v}|} = M < \infty, \quad \text{and} \quad \inf_{p \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| |\mathbf{v}|} = m > 0.$$

Throughout this paper, the ‘‘inf’’ and ‘‘sup’’ are taken over nonzero vectors. With the form  $b$ , we associate the linear operators  $B : V \rightarrow Q^*$  and  $B^* : Q \rightarrow V^*$  defined by

$$\langle B\mathbf{v}, q \rangle = b(\mathbf{v}, q) = \langle B^*q, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}, q \in Q.$$

Let  $\mathbf{V}_0 = \ker(B) \subset \mathbf{V}$ , be the kernel of  $B$ .

For a bounded linear operator  $T : X \rightarrow Y$  between two Hilbert spaces  $X$  and  $Y$ , we denote the Hilbert transpose of  $T$  by  $T^t$ . If  $X = Y$ , we say that  $T$  is symmetric if  $T = T^t$ . For a bounded linear operator  $T : X \rightarrow X$ , we denote the spectrum of the operator  $T$  by  $\sigma(T)$ .

Next, we review the Schur complement operator with the notation introduced in [6]. First, we notice that the operators  $\mathcal{C}^{-1}B : \mathbf{V} \rightarrow Q$  and  $\mathcal{A}^{-1}B^* : Q \rightarrow \mathbf{V}$  are symmetric to each other, i.e.,

$$(1.4) \quad (\mathcal{C}^{-1}B\mathbf{v}, q) = a(\mathbf{v}, \mathcal{A}^{-1}B^*q), \quad \mathbf{v} \in \mathbf{V}, q \in Q.$$

Consequently,  $(\mathcal{C}^{-1}B)^t = \mathcal{A}^{-1}B^*$  and  $(\mathcal{A}^{-1}B^*)^t = \mathcal{C}^{-1}B$ . The Schur complement on  $Q$  is the operator  $S_0 := \mathcal{C}^{-1}B\mathcal{A}^{-1}B^* : Q \rightarrow Q$ . The operator  $S_0$  is symmetric and positive definite on  $Q$ , satisfying

$$(1.5) \quad \sigma(S_0) \subset [m^2, M^2], \quad \text{and} \quad m^2, M^2 \in \sigma(S_0).$$

A proof of (1.5) can be found in [4, 6]. Consequently, for any  $p \in Q$ ,

$$(1.6) \quad M\|p\| \geq \|p\|_{S_0} := (S_0p, p)^{1/2} = |\mathcal{A}^{-1}B^*p|_{\mathbf{V}} \geq m\|p\|.$$

For  $f \in \mathbf{V}^*$ ,  $g \in Q^*$ , we consider the following variational problem:  
Find  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  such that

$$(1.7) \quad \begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle & \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= \langle g, q \rangle & \text{for all } q \in Q, \end{aligned}$$

where the bilinear form  $b : \mathbf{V} \times Q \rightarrow \mathbb{R}$  satisfies (1.3). It is known that the above variational problem has unique solution for any  $f \in \mathbf{V}^*$ ,  $g \in Q^*$  (see some of the original proofs in [2, 3, 18]). Further results on stability and well posedness of the problem, can be found in many publications, e.g., [16, 17, 22, 1, 23, 20, 24, 28, 30, 19, 21, 27, 6].

The operator version of the problem (1.7) is:  
Find  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  such that

$$(1.8) \quad \begin{aligned} \mathcal{A}\mathbf{u} + B^*p &= \mathbf{f}, \\ B\mathbf{u} &= g. \end{aligned}$$

By applying the Riesz representation operators  $\mathcal{A}^{-1}$  and  $\mathcal{C}^{-1}$  to the first and the second equation respectively, we obtain the system

$$(1.9) \quad \begin{pmatrix} I & \mathcal{A}^{-1}B^* \\ \mathcal{C}^{-1}B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{-1}\mathbf{f} \\ \mathcal{C}^{-1}g \end{pmatrix}.$$

Since  $S_0$  is an invertible operator on  $Q$ , (1.9) is also equivalent to

$$(1.10) \quad \begin{pmatrix} I & \mathcal{A}^{-1}B^* \\ S_0^{-1}\mathcal{C}^{-1}B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{-1}\mathbf{f} \\ S_0^{-1}\mathcal{C}^{-1}g \end{pmatrix}.$$

The matrix operators associated with (1.9) and (1.10) will be investigated in Section 2 and in Section 4.

## 2. SPECTRAL AND STABILITY ESTIMATES

In this section, we establish sharp stability estimates for the problem (1.7). If  $T$  is a bounded invertible operator on a Hilbert space  $X$ , and  $x \in X$  is the unique solution of  $Tx = y$ , then

$$(2.1) \quad \|T\|^{-1}\|y\|_X \leq \|x\|_X \leq \|T^{-1}\|\|y\|_X,$$

where  $\|\cdot\|_X$  is the Hilbert norm induced by the inner product on  $X$ , and  $\|T\|$  is the standard norm on  $\mathcal{L}(X, X)$  induced by  $\|\cdot\|_X$ . If in addition, we have that  $T$  is symmetric, then  $\sigma(T) \subset \mathbb{R}$  and

$$(2.2) \quad \|T\| = \sup \{|\lambda| : \lambda \in \sigma(T)\}, \text{ and } \|T^{-1}\| = \sup \left\{ \frac{1}{|\lambda|} : \lambda \in \sigma(T) \right\}.$$

We note that the operator  $T : \mathbf{V} \times Q \rightarrow \mathbf{V} \times Q$ ,  $T := \begin{pmatrix} I & \mathcal{A}^{-1}B^* \\ \mathcal{C}^{-1}B & 0 \end{pmatrix}$  is symmetric with respect to the inner product

$$(2.3) \quad \left( \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \right)_{\mathbf{V} \times Q} := a(\mathbf{u}, \mathbf{v}) + (p, q),$$

and the operator  $T_{S_0} : \mathbf{V} \times Q \rightarrow \mathbf{V} \times Q$ ,  $T_{S_0} := \begin{pmatrix} I & \mathcal{A}^{-1}B^* \\ S_0^{-1}\mathcal{C}^{-1}B & 0 \end{pmatrix}$  is symmetric with respect to the  $S_0$ -weighted on  $Q$  inner product

$$\left( \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \right)_{\mathbf{V} \times Q_{S_0}} := a(\mathbf{u}, \mathbf{v}) + (S_0 p, q) = a(\mathbf{u}, \mathbf{v}) + (p, q)_{S_0}.$$

Thus,  $\sigma(T)$  and  $\sigma(T_{S_0})$  are compact subsets of  $\mathbb{R}$ . Due to the close relation of the two operators with  $S_0$  we will establish estimates for  $\sigma(T)$  and find  $\sigma(T_{S_0})$ . First, we introduce the following numerical values

$$\lambda_m^\pm := \frac{1 \pm \sqrt{4m^2 + 1}}{2}, \text{ and } \lambda_M^\pm := \frac{1 \pm \sqrt{4M^2 + 1}}{2}.$$

**Lemma 2.1.** *Assume that  $\mathbf{V}_0 = \ker(B)$  is non-trivial. Then, the spectrum of  $T_{S_0}$  is discrete and*

$$(2.4) \quad \sigma(T_{S_0}) = \left\{ \frac{1 - \sqrt{5}}{2}, 1, \frac{1 + \sqrt{5}}{2} \right\}.$$

*The spectrum of  $T$  satisfies the following inclusion properties*

$$(2.5) \quad \{\lambda_m^\pm, 1, \lambda_M^\pm\} \subset \sigma(T) \subset [\lambda_M^-, \lambda_m^-] \cup \{1\} \cup [\lambda_m^+, \lambda_M^+].$$

To give a more fluid presentation of the main results, we postpone the proof of the Lemma 2.1 for Section 5.

**Remark 2.2.** *In the discrete case, Lemma 2.1 is known, in the context of block diagonal preconditioning of saddle point systems, at least since the work of Kuznetsov in [26], or the works of Silvester and Wathen, in [34] and Murphy, Golub and Wathen in [29]. Other related estimates can be found in in Section 10.1.1 of the review paper of Benzi, Golub, and Liesen [14] and the refereces therein.*

**Theorem 2.3.** *If  $(\mathbf{u}, p) \neq (0, 0)$  is the solution of (1.7) then*

$$(2.6) \quad \frac{\left( \|\mathbf{f}\|_{\mathbf{V}^*}^2 + \|\mathcal{C}^{-1}g\|_{S_0^{-1}}^2 \right)^{1/2}}{\left( |\mathbf{u}|^2 + \|p\|_{S_0}^2 \right)^{1/2}} \in \left[ \frac{\sqrt{5} - 1}{2}, \frac{\sqrt{5} + 1}{2} \right],$$

and

$$(2.7) \quad \frac{\left( \|\mathbf{f}\|_{\mathbf{V}^*}^2 + \|g\|_{Q^*}^2 \right)^{1/2}}{\left( |\mathbf{u}|^2 + \|p\|^2 \right)^{1/2}} \in [|\lambda_m^-|, |\lambda_M^+|].$$

*Proof.* Let  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  be the solution of (1.7). Then, using (1.10),  $\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = T_{S_0}^{-1} \begin{pmatrix} \mathcal{A}^{-1}\mathbf{f} \\ S_0^{-1}\mathcal{C}^{-1}g \end{pmatrix}$ . The estimate (2.6) is a direct consequence of (2.1), (2.2), Lemma 2.1, and the fact that

$$\left\| \begin{pmatrix} \mathcal{A}^{-1}\mathbf{f} \\ S_0^{-1}\mathcal{C}^{-1}g \end{pmatrix} \right\|_{\mathbf{V} \times Q_{S_0}}^2 = \|\mathbf{f}\|_{\mathbf{V}^*}^2 + (S_0^{-1}\mathcal{C}^{-1}g, S_0^{-1}\mathcal{C}^{-1}g)_{S_0} = \|\mathbf{f}\|_{\mathbf{V}^*}^2 + \|\mathcal{C}^{-1}g\|_{S_0^{-1}}^2.$$

For the the proof of (2.7) we use that  $\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = T^{-1} \begin{pmatrix} \mathcal{A}^{-1}\mathbf{f} \\ \mathcal{C}^{-1}g \end{pmatrix}$ .  $\square$

As a direct consequence of Theorem 2.3 we have:

**Corollary 2.4.** *If  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  is the solution of (1.7) then*

$$(2.8) \quad (|\mathbf{u}|^2 + \|p\|_{S_0}^2)^{1/2} \leq \frac{2}{\sqrt{5}-1} \left( \|\mathbf{f}\|_{\mathbf{V}^*}^2 + \frac{1}{m^2} \|g\|_{Q^*}^2 \right)^{1/2},$$

$$(2.9) \quad (|\mathbf{u}|^2 + \|p\|_{S_0}^2)^{1/2} \geq \frac{2}{\sqrt{5}+1} \left( \|\mathbf{f}\|_{\mathbf{V}^*}^2 + \frac{1}{M^2} \|g\|_{Q^*}^2 \right)^{1/2},$$

$$(2.10) \quad (|\mathbf{u}|^2 + \|p\|^2)^{1/2} \leq \frac{2}{\sqrt{4m^2+1}-1} (\|\mathbf{f}\|_{\mathbf{V}^*}^2 + \|g\|_{Q^*}^2)^{1/2},$$

$$(2.11) \quad (|\mathbf{u}|^2 + \|p\|^2)^{1/2} \geq \frac{2}{\sqrt{M^2+1}+1} (\|\mathbf{f}\|_{\mathbf{V}^*}^2 + \|g\|_{Q^*}^2)^{1/2}.$$

### 3. THE XU-ZIKATANOV APPROACH FOR THE SYMMETRIC CASE

In this section we present the Xu and Zikatanov estimate for the Galerkin approximation of variational problems in the *symmetric* case and provide a spectral description of the estimating constants.

Let  $X$  be a Hilbert space, let  $\mathcal{B} : X \times X \rightarrow \mathbb{R}$  be a *symmetric* bilinear form satisfying

$$(3.1) \quad M_{\mathcal{B}} := \sup_{x \in X} \sup_{y \in X} \frac{\mathcal{B}(x, y)}{\|x\|_X \|y\|_X} < \infty \quad \text{and} \quad m_{\mathcal{B}} := \inf_{x \in X} \sup_{y \in X} \frac{\mathcal{B}(x, y)}{\|x\|_X \|y\|_X} > 0.$$

For any  $F \in X^*$ , we consider the problem:

Find  $x \in X$  such that

$$(3.2) \quad \mathcal{B}(x, y) = \langle F, y \rangle, \quad \text{for all } y \in X.$$

Let  $T : X \rightarrow X$  be the *symmetric* operator associated to the form  $\mathcal{B}(\cdot, \cdot)$ ,

$$(3.3) \quad \langle Tx, z \rangle_X = \mathcal{B}(x, z), \quad \text{for all } x, z \in X.$$

Next result gives a characterization for the invertibility of a symmetric bounded operator  $T : X \rightarrow X$  with  $X$  a Hilbert space and is a direct consequence of the *bounded inverse theorem*, see e.g. Theorem 3.8 in [33], and of the fact that for an *injective and symmetric* bounded operator  $T : X \rightarrow X$ , we have  $\text{range}(T^t) = \text{range}(T) = X$ .

**Proposition 3.1.** *A bounded symmetric operator  $T$  on a Hilbert space  $X$  is invertible if and only if there exists  $\delta > 0$  such that  $\|Tx\| \geq \delta\|x\|$  for all  $x \in X$ .*

From the first part of (3.1), we obtain that  $T$  is bounded and  $\|\mathcal{B}\| := \|T\| = M_{\mathcal{B}}$ . The second part of assumption (3.1) implies that  $\|Tx\|_X \geq m_{\mathcal{B}}\|x\|_X$ , for all  $x \in X$ . Since  $T$  is a symmetric operator, by Proposition 3.1, we get that  $T$  is invertible. Consequently, the problem (3.2) has unique solution for any  $F \in X^*$ . We note that the second part of (3.1) is equivalent to  $\|T^{-1}\| = \frac{1}{m_{\mathcal{B}}}$ . Thus, from (2.1), using the Riesz representation theorem, we get that the solution of (3.2) satisfies

$$(3.4) \quad 1/M_{\mathcal{B}} \|F\|_{X^*} \leq \|x\|_X \leq 1/m_{\mathcal{B}} \|F\|_{X^*}.$$

Next, we let  $X_h \subset X$ , be a finite dimensional approximation space, and consider the following discrete variational problem:

Find  $x_h \in X_h$  such that

$$(3.5) \quad \mathcal{B}(x_h, y) = \langle F, y \rangle, \quad \text{for all } y \in X_h.$$

Let  $T_h : X_h \rightarrow X_h$  be the *symmetric* operator defined by

$$\langle T_h x_h, z_h \rangle_X = \mathcal{B}(x_h, z_h), \quad \text{for all } x_h, z_h \in X_h.$$

Assuming the discrete inf – sup condition

$$(3.6) \quad \inf_{x_h \in X_h} \sup_{y_h \in X_h} \frac{\mathcal{B}(x_h, y_h)}{\|x_h\|_X \|y_h\|_X} := m_{\mathcal{B}_h} > 0,$$

the discrete problem (3.5) has unique solution  $x_h$ , called the Galerkin approximation of the continuous solution  $x$  of (3.2). The following result improves on the estimate of Aziz and Babuška in [2], and was proved in a more general case by Xu and Zikatanov in [35].

**Theorem 3.2.** *Let  $x$  the solution of (3.2) and let  $x_h$  be the solution of (3.5). Under the assumptions (3.1) and (3.6), the following error estimate holds*

$$(3.7) \quad \|x - x_h\|_X \leq \frac{M_{\mathcal{B}}}{m_{\mathcal{B}_h}} \inf_{y_h \in X_h} \|x - y_h\|_X.$$

The estimate is based on the fact that the operator  $\Pi : X \rightarrow X_h$  defined by  $\Pi x = x_h$ , where

$$\mathcal{B}(x_h, z_h) = \mathcal{B}(x, z_h), \quad \text{for all } z_h \in X_h,$$

is a projection, and  $\|\Pi\|_{\mathcal{L}(X, X)} = \|I - \Pi\|_{\mathcal{L}(X, X)}$ , see [25, 35]. Since our operators  $T$  and  $T_h$  are symmetric, we can characterize the constants  $M_{\mathcal{B}}$  and  $m_{\mathcal{B}_h}$  using (2.2). Thus, we have

$$M_{\mathcal{B}} = \|T\| = \max\{|\lambda| : \lambda \in \sigma(T)\}, \quad \text{and}$$

$$m_{\mathcal{B}_h} = 1/\|T_h^{-1}\| = \min\{|\lambda| : \lambda \in \sigma(T_h)\}.$$

## 4. APPROXIMATION ESTIMATES FOR THE COERCIVE SYMMETRIC CASE

In this section, we apply the approximation result of Theorem 3.2 to the discrete approximation of problem (1.7). In what follows, we work with the setting and the notation introduced in Section 1. Let  $\mathbf{V}_h$  be a subset of  $\mathbf{V}$  and let  $\mathcal{M}_h$  be a finite dimensional subspace of  $Q$ . We consider the restrictions of the forms  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  to the discrete spaces  $\mathbf{V}_h$  and  $\mathcal{M}_h$  and define the corresponding discrete operators  $A_h, \mathcal{C}_h, B_h$ , and  $B_h^*$ . For example,  $A_h$  is the discrete version of  $A$ , and is defined by

$$\langle A_h \mathbf{u}_h, \mathbf{v}_h \rangle = a(\mathbf{u}_h, \mathbf{v}_h), \quad \text{for all } \mathbf{u}_h \in \mathbf{V}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

We assume that there exists  $m_h > 0$  such that

$$(4.1) \quad \inf_{p_h \in \mathcal{M}_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|p_h\| |\mathbf{v}_h|} = m_h > 0.$$

We define the discrete Schur complement  $S_{0,h} : \mathcal{M}_h \rightarrow \mathcal{M}_h$  by  $S_{0,h} := \mathcal{C}_h^{-1} B_h A_h^{-1} B_h^*$ , and notice that,  $m_h^2$  is the smallest eigenvalues of  $\sigma(S_{0,h})$ .

We consider the discrete variational form of (1.7):

Find  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{M}_h$  such that

$$(4.2) \quad \begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= \langle \mathbf{f}, \mathbf{v}_h \rangle & \text{for all } \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= \langle g, q_h \rangle & \text{for all } q_h \in \mathcal{M}_h, \end{aligned}$$

It is well known, see e.g., [16, 17, 32, 35, 15], that under the assumption (4.1), the problem (4.2) has unique solution  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{M}_h$ .

**Theorem 4.1.** *If  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  is the solution of (1.7) and  $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{M}_h$  is the solution of (4.2), then the following inequality holds*

$$(4.3) \quad |\mathbf{u} - \mathbf{u}_h|^2 + \|p - p_h\|^2 \leq C_h^2 \left( \inf_{\mathbf{v}_h \in \mathbf{V}_h} |\mathbf{u} - \mathbf{v}_h|^2 + \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|^2 \right),$$

where

$$C_h = \frac{\sqrt{4M^2 + 1} + 1}{\sqrt{4m_h^2 + 1} - 1}.$$

*Proof.* We consider the form  $\mathcal{B}(\cdot, \cdot)$  defined on  $X = \mathbf{V} \times Q$  by

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q).$$

It is easy to check that (1.7) is equivalent to the following problem:

Find  $(\mathbf{u}, p) \in \mathbf{V} \times Q$  such that

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle := \langle \mathbf{f}, \mathbf{v} \rangle + \langle g, q \rangle \quad \text{for all } (\mathbf{v}, q) \in \mathbf{V} \times Q.$$

With the natural inner product defined by (2.3), the operator  $T$  induced by the form  $\mathcal{B}$  is exactly

$$T := \begin{pmatrix} I & \mathcal{A}^{-1} B^* \\ \mathcal{C}^{-1} B & 0 \end{pmatrix},$$

and the corresponding discrete operator is

$$T_h = \begin{pmatrix} I & \mathcal{A}_h^{-1} B_h^* \\ \mathcal{C}_h^{-1} B_h & 0 \end{pmatrix}.$$

Now, we apply Theorem 3.2 for the form  $\mathcal{B}(\cdot, \cdot)$  defined on  $X = \mathbf{V} \times Q$ . Using the description of  $M_{\mathcal{B}}$  and  $m_{\mathcal{B}_h}$  at the end of Section 3 and the spectral estimate (2.5) of Lemma 2.1 for both  $T$  and  $T_h$ , we obtain

$$M_{\mathcal{B}} = \|T\| = \max\{|\lambda| : \lambda \in \sigma(T)\} = \frac{1}{2} \left( \sqrt{4M^2 + 1} + 1 \right),$$

and

$$m_{\mathcal{B}_h} = 1/\|T_h^{-1}\| = \min\{|\lambda| : \lambda \in \sigma(T_h)\} = \frac{1}{2} \left( \sqrt{4m_h^2 + 1} - 1 \right).$$

The estimate (4.3) follows as a direct consequence of (3.7).  $\square$

**Remark 4.2.** *If  $m_h \rightarrow 0$ , then we have that  $C_h = O(m_h^{-2})$ . The order can be improved if we equip  $X = \mathbf{V} \times Q$  or  $X_h = \mathbf{V}_h \times \mathcal{M}_h$  with the weighted on  $Q$  inner product defined by,*

$$\left( \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \right)_{m_h} := a(\mathbf{u}, \mathbf{v}) + m_h^2(p, q).$$

The operators  $T$  and  $T_h$  corresponding to the same form  $\mathcal{B}$  and the new weighted inner product are

$$T = \begin{pmatrix} I & \mathcal{A}^{-1} B^* \\ m_h^{-2} \mathcal{C}^{-1} B & 0 \end{pmatrix}, \quad \text{and} \quad T_h = \begin{pmatrix} I & \mathcal{A}_h^{-1} B_h^* \\ m_h^{-2} \mathcal{C}_h^{-1} B_h & 0 \end{pmatrix}.$$

Using the same arguments as in the proof of Theorem 4.1, we get

$$(4.4) \quad |\mathbf{u} - \mathbf{u}_h|^2 + m_h^2 \|p - p_h\|^2 \leq D_h^2 \left( \inf_{\mathbf{v}_h \in \mathbf{V}_h} |\mathbf{u} - \mathbf{v}_h|^2 + m_h^2 \inf_{q_h \in M_h} \|p - q_h\|^2 \right),$$

where

$$D_h = \|T\| \|T_h^{-1}\| = \frac{\sqrt{4\frac{M^2}{m_h^2} + 1} + 1}{2} \frac{2}{\sqrt{5} - 1} = O(m_h^{-1}).$$

Then, from (4.4), we obtain

$$(4.5) \quad |\mathbf{u} - \mathbf{u}_h| \lesssim \frac{1}{m_h} \inf_{\mathbf{v}_h \in \mathbf{V}_h} |\mathbf{u} - \mathbf{v}_h| + \inf_{q_h \in M_h} \|p - q_h\|,$$

and

$$(4.6) \quad \|p - p_h\| \lesssim \frac{1}{m_h^2} \inf_{\mathbf{v}_h \in \mathbf{V}_h} |\mathbf{u} - \mathbf{v}_h| + \frac{1}{m_h} \inf_{q_h \in M_h} \|p - q_h\|.$$

By  $A(h) \lesssim B(h)$ , we understand that  $A(h) \leq cB(h)$ , for a constant  $c$  independent of  $h$ . The estimates (4.5) and (4.6) can be useful when the solution  $(\mathbf{u}, p)$  exhibits extra regularity, [5, 8, 9, 10], and the discretization is done on pairs  $(\mathbf{V}_h, \mathcal{M}_h)$  that are not necessarily stable, but the inf – sup constant  $m_h$  can be theoretically or numerically estimated.



## 5. THE PROOF OF LEMMA 2.1

We start this section by reviewing a known functional analysis result that describes the spectrum of a bounded symmetric operator on a Hilbert space. According to one of the referees, “the result has actually been known for over a century, and is known as *Weyl’s criterion*”. It can be found in [31], Theorem VII.12, p.237, and it also follows from Proposition 3.1.

**Proposition 5.1.** *The spectrum  $\sigma(T)$  of a bounded symmetric operator  $T$  on a Hilbert space  $X$  satisfies*

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T),$$

where  $\sigma_p(T)$  is the point spectrum of  $T$  and consists of all eigenvalues of  $T$ , and  $\sigma_c(T)$  is the continuous spectrum of  $T$  and consists of all  $\lambda \in \sigma(T)$  such that  $T - \lambda I$  is an one-to-one mapping of  $X$  onto a dense proper subspace of  $X$ . Consequently, (using Proposition 3.1) for any  $T$  is symmetric, we have that  $\lambda \in \sigma(T)$  if and only if there exists a sequence  $\{x_n\} \subset X$ , such that

$$\|x_n\| = 1, \quad \text{for all } n, \quad \text{and} \quad \|(T - \lambda)x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Next, we are ready to present a proof for Lemma 2.1.

*Proof.* (Lemma 2.1) First, we will justify (2.4). Let  $\lambda \in \sigma_p(T_{S_0})$  be an eigenvalue and let  $\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \in \mathbf{V} \times Q$  be a corresponding eigenvector. Then,

$$T_{S_0} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix},$$

which leads to

$$(5.1) \quad \begin{aligned} \mathcal{A}^{-1}B^*p &= (\lambda - 1)\mathbf{u}, \\ S_0^{-1}C^{-1}B\mathbf{u} &= \lambda p. \end{aligned}$$

From (5.1) it is easy to see that for any non-zero  $\mathbf{u}_0 \in \mathbf{V}_0 = \ker(B)$  we have that  $\begin{pmatrix} \mathbf{u}_0 \\ 0 \end{pmatrix}$  is an eigenvector for  $T_{S_0}$  corresponding to  $\lambda = 1$ . Thus  $1 \in \sigma_p(T_{S_0})$ . If  $\lambda \neq 1$ , we can assume  $p \neq 0$ , and by substituting  $\mathbf{u}$  from the first equation of (5.1) into the second equation of (5.1), we get

$$S_0^{-1}S_0p = \lambda(\lambda - 1)p, \text{ or } \lambda(\lambda - 1) = 1,$$

which gives  $\lambda = \lambda_{\pm} := \frac{1 \pm \sqrt{5}}{2}$ . Note that for any  $p \neq 0$  we have that  $\begin{pmatrix} \lambda_{\pm} \mathcal{A}^{-1}B^*p \\ p \end{pmatrix}$  is an eigenvector for  $T_{S_0}$  corresponding to  $\lambda = \lambda_{\pm}$ . Thus,

$$\sigma_p(T_{S_0}) = \left\{ \frac{1 - \sqrt{5}}{2}, 1, \frac{1 + \sqrt{5}}{2} \right\}.$$

Next, we prove that the continuous spectrum of  $T_{S_0}$  is empty. If we let  $\lambda \in \sigma_c(T_{S_0})$ , then  $\lambda \neq 1, \lambda \neq \frac{1 \pm \sqrt{5}}{2}$  and, according to Proposition 5.1, there exists a sequence  $\begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \in \mathbf{V} \times Q$ , such that

$$\left\| \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \right\| = 1 \quad \text{and} \quad \left\| T_{S_0} \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} - \lambda \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This leads to

$$(5.2) \quad \begin{aligned} \mathcal{A}^{-1}B^*p_n + (1 - \lambda)\mathbf{u}_n &\rightarrow 0, \\ S_0^{-1}\mathcal{C}^{-1}B\mathbf{u}_n - \lambda p_n &\rightarrow 0. \end{aligned}$$

From (5.2), using that  $\mathcal{C}^{-1}B$  and  $S_0^{-1}$  are continuous operators, we get

$$(5.3) \quad \begin{aligned} (1 - \lambda)\mathcal{C}^{-1}B\mathbf{u}_n + S_0 p_n &\rightarrow 0 \\ \mathcal{C}^{-1}B\mathbf{u}_n - \lambda S_0 p_n &\rightarrow 0. \end{aligned}$$

This implies that  $S_0 p_n \rightarrow 0$ , and consequently  $p_n \rightarrow 0$ . From the first part of (5.2) we can also conclude that  $\mathbf{u}_n \rightarrow 0$ . The convergence  $(\mathbf{u}_n, p_n) \rightarrow (0, 0)$  contradicts  $\left\| \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \right\| = 1$ . Thus,  $\sigma_c(T_{S_0}) = \emptyset$  and the proof of (2.4) is complete.

To prove (2.5), we start by observing that, as in the previous case,  $\lambda = 1 \in \sigma_p(T)$ . If  $\lambda \neq 1$  is any other spectral value of  $\sigma(T)$ , then there exists a sequence  $\begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \in \mathbf{V} \times Q$ , such that

$$\left\| \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \right\| = 1 \quad \text{and} \quad \left\| T \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} - \lambda \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The convergence part of the above statement implies

$$(5.4) \quad \begin{aligned} \mathcal{A}^{-1}B^*p_n + (1 - \lambda)\mathbf{u}_n &\rightarrow 0, \\ \mathcal{C}^{-1}B\mathbf{u}_n - \lambda p_n &\rightarrow 0. \end{aligned}$$

From the first equation of (5.4) and  $|\mathbf{u}_n|^2 + \|p_n\|^2 = 1$  we get that

$$1 = \frac{1}{(\lambda - 1)^2} \|p_n\|_{S_0}^2 + \|p_n\|^2 \leq \left( \frac{M^2}{(\lambda - 1)^2} + 1 \right) \|p_n\|^2.$$

From (5.4) we obtain that

$$S_0 p_n - \lambda(\lambda - 1)p_n \rightarrow 0.$$

Thus, using the last two statements and Proposition 5.1 for characterizing the spectral values of  $S_0$ , we obtain that  $\lambda(\lambda - 1) \in \sigma(S_0) \subset [m^2, M^2]$ , which proves the right inclusion of (2.5). To complete the proof of (2.5), we have to show that  $\lambda_m^\pm$  and  $\lambda_M^\pm \in \sigma(T)$ . From (1.5), we have that  $m^2 \in \sigma(S_0)$ . In light of Proposition 5.1, we can find a sequence  $(p_n) \subset Q$  such that

$$\|p_n\| = 1 \quad \text{for all } n, \quad \text{and} \quad S_0 p_n - m^2 p_n \rightarrow 0.$$

Then, if we define  $\mathbf{u}_n := \frac{1}{\lambda_m^\pm} \mathcal{A}^{-1} B^* p_n$ , it is easy to check that

$$|\mathbf{u}_n|^2 + \|p_n\|^2 \geq \frac{m^2}{(\lambda_m^\pm - 1)^2} + 1, \quad \text{for all } n,$$

and

$$\left\| T \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} - \lambda \begin{pmatrix} \mathbf{u}_n \\ p_n \end{pmatrix} \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This proves that  $\lambda_m^\pm \in \sigma(T)$ . The proof of  $\lambda_M^\pm \in \sigma(T)$  is similar.  $\square$

## 6. CONCLUSION

We presented sharp stability and approximation estimates for a general class of symmetric saddle point variational systems. The estimates are based on spectral description of the continuous and discrete symmetric operators that represent the systems. The spectrum characterization we provided is an useful tool for analysis at continuous or discrete levels, and can be applied sucesfully to the convergence analysis of iterative methods that are aiming directly to the solution of a continuous saddle point problem. Example of such iterative methods include the Uzawa type algorithms of [6, 7, 11, 12, 13]. Using a Schur complement norm for the second variable of a saddle point system with a coercive symmetric form  $a$  and a mixed form  $b$  satisfying an inf – sup condition, we established that the ratio between the norm of the data and the norm of the solution lies in  $[\frac{1}{\varphi}, \varphi]$ , where  $\varphi$  is the *golden ratio*.

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