A New Approach For Solving Stokes Systems Arising from a Distributive Relaxation Method

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The distributed relaxation method for the Stokes problem has been advertised as an adequate change of variables that leads to a lower triangular system with Laplace operators on the main diagonal for which multigrid methods are very efficient. We show that under high regularity of the Laplacian, the transformed system admits almost block-lower triangular form. We analyze the distributed relaxation method and compare it with other iterative methods for solving the Stokes system. We also present numerical experiments illustrating the effectiveness of the transformation which is well established for certain finite difference discretizations of Stokes problems. Numer Methods Partial Differential Eq 000: 000–000, 2010

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I. DESCRIPTION OF THE PROBLEM

We consider the stationary Stokes equations:

$$\begin{array}{lll} -\Delta \mathbf{u} & +\nabla p & = \mathbf{f}, \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} & & = g, \text{ in } \Omega, \end{array} \tag{1.1}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = 0$ on $\partial\Omega$ and g satisfying the constrain:

$$\int_{\Omega} g \, dx = 0.$$

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The "distributive relaxation" as proposed in [1] (see also [2]), reads as follows. Make the following change of variables,

$$\begin{aligned} \mathbf{u} &= \mathbf{w} - \nabla\psi, \\ p &= -\Delta\psi. \end{aligned} \tag{1.2}$$

Substituting (1.2) into (1.1) (assuming enough regularity), leads to

$$\begin{array}{ll} -\Delta\mathbf{w} & = \mathbf{f}, \\ \operatorname{div}\mathbf{w} & -\Delta\psi = g. \end{array} \tag{1.3}$$

The transformed operator has a lower triangular form with Laplace operators on the main diagonal, for which efficient multigrid methods are available. Of course, we did not pay any attention to the boundary conditions. It is easily seen that if the original variable $\mathbf{u}$ vanishes on the boundary Γ of the given domain Ω, then the new variable $\mathbf{w}$ couples with $\nabla\psi$ on Γ. That is, the above system (1.3) does not actually decouple $\mathbf{w}$ from ψ. Our goal in this study is to consider the coupled problem (1.1)–(1.2) in a weak form with proper Sobolev spaces for $\mathbf{u}$, and ψ together with appropriate boundary conditions and then to find an efficient finite element discretization for the coupled (more elliptic) problem.

The remainder of this article is organized as follows. In Section II, we introduce and analyze an Uzawa type algorithm for solving (1.1). We establish sharp rates of convergence in terms of the "inf-sup" condition constant at the continuous level. Even though the convergence results of Uzawa algorithms at the continuous level are known (see, e.g., Refs. [3–7]), the presented analysis is based on fixed point approach and will be used for establishing the convergence of the distributive relaxation iteration.

In Section III, we study the properties of the transformed problem. An analysis of the distributed relaxation method as an iterative process is given, using the results of Section II.

Finally, in Section IV, we report on a number of numerical tests illustrating the performance of the new method based on the transformed system. Our tests are preliminary. Further study on finite element discretizations of the distributed relaxation system would be done elsewhere. Some early results regarding multigrid convergence with distributed (transforming) smoothers are found in Ref. [8].

II. AN UZAWA TYPE ALGORITHM

Let $\mathbf{V} := (H_0^1(\Omega))^d$, where $d = 2$ or $d = 3$, and $P = L_0^2(\Omega)$. We will also denote by $A(\mathbf{u}, \mathbf{v})$ the bilinear form

$$\begin{aligned} A(\mathbf{u}, \mathbf{v}) := (\nabla\mathbf{u}, \nabla\mathbf{v}) &= \sum_{i=1}^{d} (\nabla\mathbf{u}_i, \nabla\mathbf{v}_i) \\ &= (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) + (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}), \ \mathbf{u}, \mathbf{v} \in \mathbf{V}. \end{aligned}$$

We denote the norm induced by A with $|\cdot|_{\mathbf{V}}$. The norm on P is the L^2 standard norm and is simply denoted by $\|\cdot\|$. A variational formulation of (1.1) is

Find $\mathbf{u} \in \mathbf{V}$, $p \in P$ such that

$$\begin{array}{lll} A(\mathbf{u}, \mathbf{v}) & -(p, \operatorname{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \mathbf{v} \in \mathbf{V}. \\ (\operatorname{div}\mathbf{u}, q) & = (g, q), & q \in P. \end{array} \tag{2.1}$$

It is known that the following inf-sup condition holds:

$$\inf_{p\in P}\sup_{\mathbf{v}\in\mathbf{V}}\frac{(p,\operatorname{div}\mathbf{v})}{\|p\|\ |\mathbf{v}|_{\mathbf{V}}}=c_0>0. \tag{2.2}$$

Here, and throughout this article, the "inf" and "sup" are taken over nonzero vectors. Because of the choice of the norm on $\mathbf{V}$, it is clear that we have $c_0\le 1$.

Next, we define $F: P\to P$ by

$$\begin{aligned} F(p)&=p+g-\operatorname{div}\mathbf{u},\ \text{where } \mathbf{u} \text{ solves}\\ A(\mathbf{u},\mathbf{v})&=(\mathbf{f},\mathbf{v})+(p,\operatorname{div}\mathbf{v}),\ \forall\mathbf{v}\in\mathbf{V}. \end{aligned} \tag{2.3}$$

Let $\hat{p}_0=F(0)=g-\operatorname{div}\mathbf{u}_0$, where $\mathbf{u}_0$ is the solution of the problem

$$A(\mathbf{u}_0,\mathbf{v})=(\mathbf{f},\mathbf{v}),\ \forall\mathbf{v}\in\mathbf{V},$$

and let $T: P\to P$ be the linear part of the mapping F, i.e.,

$$\begin{aligned} Tp&=p-\operatorname{div}\mathbf{u},\ \text{where}\\ A(\mathbf{u},\mathbf{v})&=(p,\operatorname{div}\mathbf{v})\quad \forall\mathbf{v}\in\mathbf{V}. \end{aligned} \tag{2.4}$$

We have $F(p)=Tp+\hat{p}_0$. If p is a fixed point of F and $\mathbf{u}$ solves the second equation in (2.3), then $(\mathbf{u}, p)$ is the solution of (2.1). The solution $(\mathbf{u}, p)$ can be found by the functional iteration of F. More precisely, we are led to the following Uzawa type algorithm.

Algorithm 2.1 (Uzawa algorithm)**.** *Let p_0 be any given initial approximation to p, and for $n=1,2,\ldots$ construct $(\mathbf{u}_n, p_n)$ by*

$$\begin{aligned} A(\mathbf{u}_n,\mathbf{v})&=(\mathbf{f},\mathbf{v})+(p_{n-1},\operatorname{div}\mathbf{v}),\ \mathbf{v}\in\mathbf{V},\\ p_n&=F(p_{n-1})=p_{n-1}+g-\operatorname{div}\mathbf{u}_n. \end{aligned}$$

Remark 2.2. The following modification of Uzawa algorithm, referred to as the "iterated penalty method" (cf., e.g., Ref. [9]) is sometimes used in practice. For any parameter $\beta\ge 0$, consider the iteration

$$\begin{aligned} A(\mathbf{u}_n,\ \mathbf{v})+\beta\ (\operatorname{div}\mathbf{u}_n-g,\ \operatorname{div}\mathbf{v})&=(\mathbf{f},\mathbf{v})+(p_{n-1},\ \operatorname{div}\mathbf{v}),\ \forall\mathbf{v}\in\mathbf{V},\\ p_n&=p_{n-1}+g-\operatorname{div}\mathbf{u}_n. \end{aligned}$$

Its convergence can be analyzed similarly to the (unmodified) Uzawa algorithm (see later), with the change $A(\mathbf{u},\ \mathbf{v}):=A(\mathbf{u},\ \mathbf{v})+\beta\ (\operatorname{div}\mathbf{u},\ \operatorname{div}\mathbf{v})$ and $(\mathbf{f},\ \mathbf{v}):=(\mathbf{f},\ \mathbf{v})+\beta\ (g,\ \operatorname{div}\ \mathbf{v})$.

Algorithm 2.1 converges provided $\|T\|<1$.

Theorem 2.3. $\|T\|=1-c_0^2$.

Proof. Let $p,q\in P$, and define $\mathbf{u},\mathbf{w}\in\mathbf{V}$ as the solutions of

$$A(\mathbf{u},\mathbf{v})=(p,\operatorname{div}\mathbf{v})\ \text{ and }\ A(\mathbf{w},\mathbf{v})=(q,\operatorname{div}\mathbf{v}),\ \forall\mathbf{v}\in\mathbf{V},$$

respectively. First, we note that

$$(Tp, q) = (Tp, Tq) + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{w}).$$

Consequently, $T : P \to P$ is a a bounded, symmetric, and non-negative definite operator. By noticing that

$$(Tp, p) = \|p\|^2 - (p, \operatorname{div} \mathbf{u}) = \|p\|^2 - A(\mathbf{u}, \mathbf{u}),$$

we obtain

$$\|T\| = \sup_{p \in P} \frac{(Tp, p)}{\|p\|^2} = \sup_{p \in P} \left(1 - \frac{A(\mathbf{u}, \mathbf{u})}{\|p\|^2}\right) = 1 - \inf_{p \in P} \frac{A(\mathbf{u}, \mathbf{u})}{\|p\|^2}.$$

On the other hand, if G is the functional $v \to (p, \operatorname{div} v)$, then $G \in \mathbf{V}'$ and from Lax-Milgram theorem we get

$$A(\mathbf{u}, \mathbf{u}) = \|G\|^2_{\mathbf{V}'} = \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{|\mathbf{v}|^2_{\mathbf{V}}}.$$

Consequently

$$\|T\| = 1 - \inf_{p \in P} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{\|p\|^2 \, |\mathbf{v}|^2_{\mathbf{V}}} = 1 - c_0^2.$$

■

Corollary 2.4. *The sequence* $\{(\mathbf{u}_n, p_n)\}$ *given by the Algorithm (2.1) converges to the solution* $(\mathbf{u}, p)$ *of* (2.1) *and*

$$\|p_n - p\| \leq (1 - c_0^2) \|p_{n-1} - p\|,$$

and

$$\|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{V}} \leq \|p_{n-1} - p\|.$$

Note that if the second equation in (2.1) is replaced by

$$p_n = p_{n-1} + \alpha(g - \operatorname{div} \mathbf{u}_n),$$

with $\alpha \in (0, 2)$, then we still get convergence of the new algorithm. The operator T_α which gives the convergence factor of the new algorithm is defined by

$$T_\alpha p = p - \alpha \operatorname{div} \mathbf{u}, \text{ where}$$
$$A(\mathbf{u}, \mathbf{v}) = (p, \operatorname{div} \mathbf{v}), \ \mathbf{v} \in \mathbf{V}. \tag{2.5}$$

T_α remains a symmetric operator and as before we get

$$(T_\alpha p, p) = \|p\|^2 - \alpha \, (p, \operatorname{div} \mathbf{u}) = \|p\|^2 - \alpha A(\mathbf{u}, \mathbf{u})$$
$$= \|p\|^2 - \alpha \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{|\mathbf{v}|^2_{\mathbf{V}}}.$$

The following estimate

$$\sup_{p\in P}\sup_{\mathbf{v}\in\mathbf{V}}\frac{(p,\operatorname{div}\mathbf{v})^2}{\|p\|^2|\mathbf{v}|_{\mathbf{V}}^2}=1,$$

together with the inf-sup condition,

$$\inf_{p\in P}\sup_{\mathbf{v}\in\mathbf{V}}\frac{(p,\operatorname{div}\mathbf{v})^2}{\|p\|^2|\mathbf{v}|_{\mathbf{V}}^2}=c_0^2,$$

lead to

Theorem 2.5.

$$\|T_\alpha\|=1-\alpha\ c_0^2,\ \textit{for}\ \ 0<\alpha\le 1,\ \textit{and}$$
$$\|T_\alpha\|=\max\{\alpha-1,|1-\alpha c_0^2|\},\ \textit{for}\ \ 1<\alpha<2.$$

The optimal rate of convergence is obtained when $\alpha=2/(1+c_0^2)$ *and in this case the convergence factor is*

$$\|T_{\text{opt}}\|=\frac{1-c_0^2}{1+c_0^2}.$$

Remark 2.6. The Uzawa algorithm provides a solution to the fixed point problem, $F(p)=p$. In terms of T, it reads $(I-T)p=\hat{p}_0=F(0)$. Thus, we are inverting $I-T$ by a stationary iterative method, which converges, if $\|T\|<1$. Since, $I-T$ is s.p.d., we can use the conjugate gradient (CG) method instead. Its convergence rate is bounded by $\varrho_{T_{\text{CG}}}\le\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, where $\kappa\le\frac{1}{1-\|T\|}$ is an upper bound of the condition number of $I-T$. In the present case, we have

$$\varrho_{T_{\text{CG}}}\le\frac{1-c_0}{1+c_0}.$$

III. DISTRIBUTIVE RELAXATION METHOD AS AN ITERATIVE PROCESS

For this section, we assume that Ω is a convex polygonal domain. We return to the problem (1.1) with the change of variable (1.2). To be more precise, we consider the change of variable $(\mathbf{u},p)\to(\mathbf{w},\psi)$, where

$$-\Delta\psi=p,\ \text{in}\ \Omega,$$
$$\frac{\partial\psi}{\partial n}=0\ \text{on}\ \Gamma, \tag{3.1}$$

and

$$\mathbf{w}=\mathbf{u}+\nabla\psi. \tag{3.2}$$

Then, assuming enough regularity, (1.1) becomes

$$\begin{aligned} -\Delta \mathbf{w} &= \mathbf{f}, \text{ in } \Omega, \\ \mathbf{w} &= \nabla \psi, \text{ on } \Gamma, \\ -\Delta \psi &= g - \operatorname{div} \mathbf{w}, \text{ in } \Omega, \\ \frac{\partial \psi}{\partial n} &= 0, \text{ on } \Gamma. \end{aligned} \tag{3.3}$$

Let $H^1_* = H^1_*(\Omega) = \{\varphi \in H^1 \mid (\varphi, 1) = 0\}$ and

$$H^2_* = H^2_*(\Omega) = \left\{ \varphi \in H^2 \mid (\varphi, 1) = 0, \frac{\partial \varphi}{\partial n}|_\Gamma = 0 \right\}.$$

On H^2_* we consider the inner product given by

$$(\varphi, \psi)_2 := (\Delta \varphi, \Delta \psi).$$

It is clear that $(\cdot, \cdot)_2$ is a biliniear form on H^2_*. To prove that $(\cdot, \cdot)_2$ is indeed an inner product, we just have have to justify that $(\varphi, \varphi)_2 = 0$ implies $\varphi = 0$ in H^2_*. From $(\varphi, \varphi)_2 = 0$ and $\varphi \; in H^2_*$, we have

$$\begin{aligned} -\Delta \varphi &= 0, \text{ in } \Omega, \\ \frac{\partial \varphi}{\partial n} &= 0 \text{ on } \Gamma, \\ \int_\Omega \varphi \, dx &= 0. \end{aligned}$$

Using the regularity of the Neumann's problem for the Laplace operator, see, e.g., Proposition 1.2, page 14 in Ref. [10], we can conclude that the above conditions for φ imply $\varphi = 0$ in H^2_*.

The norm on H^2_* is denoted by $\| \cdot \|_2$ and is the norm induced by the $(\cdot, \cdot)_2$-inner product. A variational formulation of (3.3) reads: Find $\mathbf{w} \in H^1(\Omega)$, $\psi \in H^2_*$, such that

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \;\; \text{for all} \;\; \mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \nabla \psi, \text{ on } \Gamma, \end{aligned} \tag{3.4}$$

and

$$(\nabla \psi, \nabla q) = (g - \operatorname{div} \mathbf{w}, q) \;\; \text{for all } q \in H^1_* \, . \tag{3.5}$$

The solution of problem (3.4)–(3.5) can also be written in terms of a fixed point problem on H^2_*. Define $F : H^2_* \to H^2_*$, by $F(\varphi) = \psi$, where we first compute $\mathbf{w}$ by solving

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \;\; \text{for all} \;\; \mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \nabla \varphi \text{ on } \Gamma, \end{aligned} \tag{3.6}$$

and then ψ is computed from

$$(\nabla \psi, \nabla q) = (g - \operatorname{div} \mathbf{w}, q) \;\; \text{for all } q \in H^1_* \, . \tag{3.7}$$

If $\hat{\psi}_0 = F(0)$, then $F(\varphi) = S\varphi + \hat{\psi}_0$ where $S : H^2_* \to H^2_*$ is defined as the linear part of F, i.e., $S(\varphi) = \psi$, where

$$(\nabla \mathbf{w}, \nabla \mathbf{v}) = 0 \ \text{ for all } \ \mathbf{v} \in \mathbf{V},$$
$$\mathbf{w} = \nabla \varphi \quad \text{ on } \Gamma, \tag{3.8}$$

and ψ is given by

$$(\nabla \psi, \nabla q) = -(\operatorname{div} \mathbf{w}, q) \ \text{ for all } q \in H^1_* \,. \tag{3.9}$$

We note that if ψ is a fixed point of F then $(\mathbf{w}, \psi)$, with $\mathbf{w}$ defined by the solution of (3.6) with $\varphi = \psi$, is the solution of (3.4)–(3.5). The existence and the uniqueness of the solution of (3.4)–(3.5) follows from the existence and the uniqueness of the solution of the Stokes system.

Algorithm 3.1 (Distributed relaxation)**.** *Let ψ_0 be any given initial approximation to ψ. For $n = 1, 2, \ldots$, we construct $(\mathbf{w}_n, \psi_n)$ by first solving for $\mathbf{w}_n$*

$$(\nabla \mathbf{w}_n, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \ \textit{for all} \ \mathbf{v} \in \mathbf{V},$$
$$\mathbf{w}_n = \nabla \psi_{n-1}, \ \textit{on} \ \Gamma, \tag{3.10}$$

and then ψ_n is computed from

$$(\nabla \psi_n, \nabla q) = (g - \operatorname{div} \mathbf{w}_n, q) \text{ for all } q \in H^1_*. \tag{3.11}$$

The above algorithm converges provided $\|S\|_2 < 1$, where $\|S\|_2$ is the norm of S as an operator on H^2_*.

Theorem 3.2. $\|S\|_2 = 1 - c_0^2$.

Proof. First we observe that S is a bounded operator from H^2_* to H^2_*. Next, we prove that for $\varphi_1, \varphi_2 \in H^2_* \cap H^3$ we have

$$(S\varphi_1, S\varphi_2)_2 = (\varphi_1, S\varphi_2)_2 - (\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2), \tag{3.12}$$

where $\mathbf{w}_i$ and $\psi_i = S\varphi_i$, $i = 1, 2$, are defined

$$(\nabla \mathbf{w}_i, \nabla \mathbf{v}) = 0 \ \text{ for all } \ \mathbf{v} \in \mathbf{V},$$
$$\mathbf{w}_i = \nabla \varphi_i \text{ on } \Gamma,$$

and

$$(\nabla \psi_i, \nabla q) = -(\operatorname{div} \mathbf{w}_i, q) \ \text{ for all } q \in H^1_* \,.$$

The main ingredient in proving (3.12) is the fact that for sufficiently smooth functions $\mathbf{w}$ with $\Delta \mathbf{w} = 0$, we have

$$\operatorname{curl} \operatorname{curl} \mathbf{w} = \nabla(\operatorname{div} \mathbf{w}). \tag{3.13}$$

For the 2D case for example, the proof of (3.12) is seen as follows:

$$\begin{aligned}(S\varphi_1, S\varphi_2)_2 &= (\psi_1, \psi_2)_2 = (\Delta\psi_1, \Delta\psi_2) = (\operatorname{div}\mathbf{w}_1, \operatorname{div}\mathbf{w}_2)\\ &= -(\mathbf{w}_1, \nabla(\operatorname{div}\mathbf{w}_2)) = -(\mathbf{w}_1, \operatorname{curl}\operatorname{curl}\mathbf{w}_2)\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) + (\mathbf{w}_1\cdot\tau, \operatorname{curl}\mathbf{w}_2)_\Gamma\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) + (\nabla\varphi_1\cdot\tau, \operatorname{curl}\mathbf{w}_2)_\Gamma\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) - (\nabla\varphi_1, \operatorname{curl}\operatorname{curl}\mathbf{w}_2)\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) - (\nabla\varphi_1, \nabla(\operatorname{div}\mathbf{w}_2))\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) + (\Delta\varphi_1, \operatorname{div}\mathbf{w}_2)\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) + (\Delta\varphi_1, \Delta\psi_2)\\ &= -(\operatorname{curl}\mathbf{w}_1, \operatorname{curl}\mathbf{w}_2) + (\varphi_1, S\varphi_2)_2.\end{aligned}$$

Here, τ represents the tangential vector to Γ. For the 3D case, "$\cdot\tau$" is replaced by "$\times\tau$". From (3.12) and the density of $H^2_* \cap H^3$ in H^2_*, we conclude that S is a symmetric operator and in particular,

$$(S\varphi, \varphi)_2 = (S\varphi, S\varphi)_2 + (\operatorname{curl}\mathbf{w}, \operatorname{curl}\mathbf{w}),$$

which proves that S is a non-negative operator. Thus,

$$\|S\|_2 = \sup_{\varphi\in H^2_*} \frac{(S\varphi,\varphi)_2}{\|\varphi\|_2^2} = \sup_{\varphi\in H^2_*} \frac{(\operatorname{div}\mathbf{w}, \Delta\varphi)}{\|\Delta\varphi\|^2},$$

where $\mathbf{w}$ is defined by (3.8). Assuming that $\varphi \in H^2_* \cap H^3$, we let $\mathbf{u} = \mathbf{w} - \nabla\varphi$. It follows that

$$\operatorname{div}\mathbf{w} = \operatorname{div}\mathbf{u} + \Delta\varphi,$$

and

$$-\Delta\mathbf{u} = \nabla(\Delta\varphi).$$

The variational form of the above identity is

$$A(\mathbf{u}, \mathbf{v}) = (-\Delta\varphi, \operatorname{div}\mathbf{v}) \text{ for all } \mathbf{v} \in \mathbf{V}.$$

In particular, taking $\mathbf{v} = \mathbf{u}$, we get

$$A(\mathbf{u}, \mathbf{u}) = (-\Delta\varphi, \operatorname{div}\mathbf{u}).$$

It follows that

$$\begin{aligned}\|S\|_2 &= \sup_{\varphi\in H^2_*\cap H^3} \frac{(S\varphi,\varphi)_2}{\|\varphi\|_2^2} = \sup_{\varphi\in H^2_*\cap H^3} \frac{(\operatorname{div}\mathbf{u} + \Delta\varphi, \Delta\varphi)}{\|\Delta\varphi\|^2}\\ &= \sup_{\varphi\in H^2_*} \left(1 - \frac{A(\mathbf{u},\mathbf{u})}{\|\Delta\varphi\|^2}\right).\end{aligned}$$

Denoting $-\Delta\varphi = q$, we have $q \in L_0^2 = P$ and

$$A(\mathbf{u}, \mathbf{v}) = (q, \operatorname{div} \mathbf{v}),\ \mathbf{v} \in \mathbf{V}.$$

Therefore (see the proof of Theorem 2.3),

$$\|S\|_2 = \sup_{q \in P} \left(1 - \frac{A(\mathbf{u}, \mathbf{u})}{\|q\|^2}\right) = 1 - \inf_{q \in P} \frac{A(\mathbf{u}, \mathbf{u})}{\|q\|^2} = 1 - c_0^2.$$

■

This result proves that the distributive relaxation method as an iterative method has similar convergence properties with the Uzawa method. We can also modify the Algorithm (3.14)–(3.15) to obtain a distributive relaxation algorithm with a faster rate of convergence. The new algorithm reads as follows: Let $\alpha \in (0, 2)$ and let ψ_0 be any approximation of ψ. For $n = 1, 2, \ldots$ construct $(\mathbf{w}_n, \psi_n)$ by

$$\begin{aligned} (\nabla \mathbf{w}_n, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \ \text{ for all } \ \mathbf{v} \in \mathbf{V}, \\ \mathbf{w}_n &= \alpha\ \nabla \psi_{n-1}, \ \text{ on } \Gamma, \end{aligned} \tag{3.14}$$

and ψ_n

$$\alpha\ (\nabla \psi_n, \nabla q) = (g - \operatorname{div} \mathbf{w}_n, q) \ \text{ for all } q \in H_*^1\ . \tag{3.15}$$

The operator S_α involved in the convergence analysis is $S_\alpha : H_*^2 \to H_*^2$ defined by $S_\alpha \varphi = \psi$, where

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= 0 \ \text{ for all } \ \mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \alpha\ \nabla \varphi, \ \text{ on } \Gamma, \end{aligned} \tag{3.16}$$

and

$$\alpha\ (\nabla \psi, \nabla q) = -(\operatorname{div} \mathbf{w}, q) \ \text{ for all } q \in H_*^1\ . \tag{3.17}$$

Following the proof of Theorem 3.2, we can see that S_α satisfies

$$(S_\alpha \varphi, \varphi)_2 = \|\Delta \varphi\|^2 - \alpha^{-2} A(\mathbf{u}, \mathbf{u}).$$

Thus, a similar version of Theorem 2.5 holds:

Theorem 3.3. $\|S_\alpha\|_2 < 1$ *for* $\alpha \in (0, 2)$. *The optimal* α *is*

$$\alpha = \sqrt{\frac{1 + c_0^2}{2}},$$

and the optimal convergence factor for the modified distributive relaxation algorithm is

$$\|S_\alpha\|_2 = \frac{1 - c_0^2}{1 + c_0^2}.$$

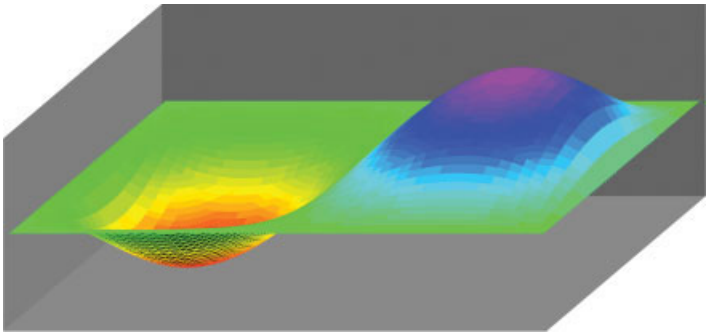

FIG. 1. The first component of $\mathbf{u}$ in numerical test (4.1). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

Remark 3.4. Similarly to Remark 2.6, we can use CG method instead of stationary iteration process to solve the fixed point problem $F(\varphi) = \varphi$, which in terms of S reads $(I - S)\varphi = F(0) = \hat{\psi}_0$. The operator $I - S$ is s.p.d. in the inner product $(\cdot,\ \cdot)_2 = (\Delta\cdot,\ \Delta\cdot)$. The use of CG will require applying Laplace operator to compute the inner products involved in the CG method. With this comment in mind, the same converge rate as in Remark 2.6 holds here as well.

IV. NUMERICAL EXPERIMENTS

In this section, we present numerical tests for our new algorithm. We present the test in two parts. We first apply the traditional P_4-P_3 mixed finite element for the Stokes equations, where the velocity is approximated by C^0- P_4 polynomials and the pressure by C^{-1}-P_3 polynomials. Such a finite element pair satisfies the uniform (in grid size h) inf-sup condition and produces pointwise divergence-free solutions for the velocity. The linear system of finite element equations is solved by the iterated penalty method (mentioned in Remark 2.2, see also later), with an inner multigrid iteration. In the second part of our test, we derive the same P_4-P_3 mixed finite element solutions by the distributive relaxation method [i.e., using the weak variational formulation (3.4)–(3.5)]. More specifically, we represent the finite element solutions $(\mathbf{u}_h, p_h)$ of the original problem with the (C^0-P_4, C^1-P_5) solutions $(\mathbf{w}_h, \psi_h)$ of the weak formulation of the transformed problem. Here, C^1-P_5 stands for the well known Argyris element of continuously differentiable, piecewise P_5 polynomials.

We solve numerically the stationary Stokes equations (1.1). In the numerical test, the domain Ω is a unit square $(0, 1)^2$. We choose the exact solution (Figs. 1 and 2):

$$\mathbf{u} = \operatorname{curl} s(x, y) = \begin{pmatrix} s_y \\ -s_x \end{pmatrix}, \quad p = -s_{xx}(x, y), \tag{4.1}$$

where

$$s(x, y) = 2^8(x - x^2)^2(y - y^2)^2.$$

Therefore $\int_\Omega p d\mathbf{x} = -\int_\Omega s_{xx} d\mathbf{x} = 0$ and in (1.1), $g = 0$ and

$$f = \begin{pmatrix} -s_{yxx} - s_{yyy} - s_{xxx} \\ s_{xxx} + s_{xyy} - s_{xxy}. \end{pmatrix}$$

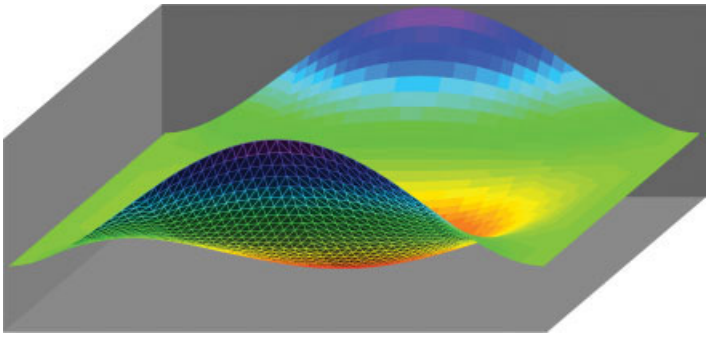

FIG. 2. The solution p in numerical test (4.1). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

We apply several steps of successive refinement to an initial triangulation of Ω, shown in Fig. 3. We denote the grids by $\{\mathcal{T}_h\}$. We use the standard P_4-P_3 mixed element, cf. Ref. [11], in the computation, i.e., the Scott-Vogelius P_4 divergence free element. Here, the finite element spaces approximating the velocity and the pressure are

$$\mathbf{V}_h = \left\{\mathbf{v}_h \in (H_0^1(\Omega))^2 \mid \mathbf{v}|_K \in (P_4)^2 \ \forall K \in \mathcal{T}_h\right\} \subset \mathbf{V}, \tag{4.2}$$

$$P_h = \left\{q_h \in L_0^2(\Omega) \mid q_h = \operatorname{div} \mathbf{v}_h \ \text{ for } \ \mathbf{v}_h \in \mathbf{V}_h\right\} \subset P. \tag{4.3}$$

We note that, as proved by Scott and Vogelius [11], when the underlying grid $\mathcal{T}_h$ is free of singular vertex, P_h is precisely the full space of discontinuous piecewise P_3 polynomials with zero mean value. However, for our grids shown in Fig. 3, we do have two singular vertices, $(0,0)$ and $(1,1)$, where a P_h function has value 0. Alternatively, for the grids in Fig. 3, P_h is also defined by, cf. Scott and Vogelius [11],

$$P_h = \left\{q_h \in L_0^2(\Omega) \mid q_h|_K \in P_3 \ \forall K \in \mathcal{T}_h, \text{ and } \ q_h(0,0) = q_h(1,1) = 0\right\}.$$

The finite element approximation problems for (1.1) read: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$A(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.4}$$

$$(q_h, \operatorname{div} \mathbf{u}_h) = 0 \qquad \forall q_h \in P_h. \tag{4.5}$$

The system of linear equations (4.4)–(4.5) are solved by the following iterated penalty method.

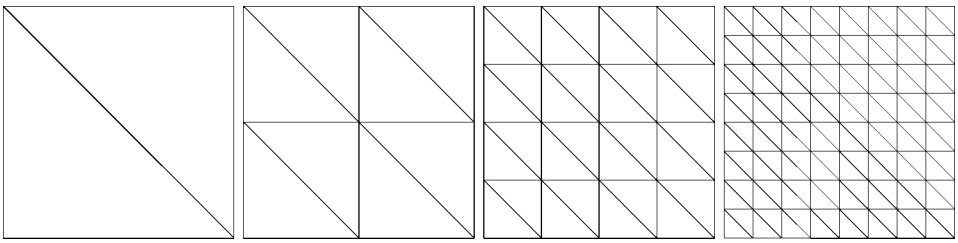

FIG. 3. The multilevel grids in numerical computation.

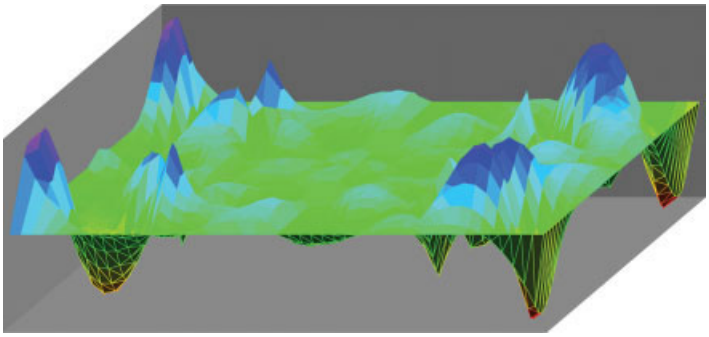

FIG. 4. The velocity error on Level 3 (Fig. 3), $(\mathbf{u}-\mathbf{u}_h)_1$, by (4.6). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

The iterated penalty method (mentioned in Remark 2.2) in the present setting takes the following particular form.

Definition 4.1 (The iterated penalty method)**.** *Let the initial iterate be* $\mathbf{u}_h^0 = \mathbf{0}$*. The rest iterates* $\mathbf{u}_h^n$ *are defined sequentially to be the unique solution of*

$$a\big(\mathbf{u}_h^n, \mathbf{v}_h\big) + \beta\big(\operatorname{div}\mathbf{u}_h^n, \operatorname{div}\mathbf{v}_h\big) = (\mathbf{f}, \mathbf{v}_h) - \left(\operatorname{div}\sum_{j=0}^{n-1}\beta\mathbf{u}_h^j, \operatorname{div}\mathbf{v}_h\right) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.6}$$

$n = 1, 2, \ldots$. *Here,* β *is positive constant. At the end of iteration, we let*

$$p_h^n = -\operatorname{div}\sum_{j=0}^{n}\beta\mathbf{u}_h^j.$$

Remark 4.1. In the iterated penalty method, we only implement the C^0-P_4 finite element for the vector Laplacian-like equations. The solution p_h is then obtained as a byproduct.

In our numerical test, we chose in (4.6)

$$\beta = 500, \text{ and } n = 4,$$

i.e., only four steps of iterated penalty method are applied. In addition, we apply the multigrid method to the Laplace-like equation (4.6). It is shown (e.g., Brenner and Scott [9], or by our analysis based on Remark 2.2) that the iterated penalty method converges with a constant rate (depending on β and the uniform inf-sup condition). Combined with the optimal order multigrid solver, the finite element systems of equations (4.4) are solved in optimal order of arithmetic operations, i.e., in $O(N)$ steps, where N is the number of unknown values of $\mathbf{u}_h$. In Fig. 4, the error $(\mathbf{u}-\mathbf{u}_h)_1$ for the first component of finite element solution $\mathbf{u}_h$ on the third level grid, shown in Fig. 3, is depicted.

In Fig. 5, the error for the pressure, $p - p_h$ on the third level grid is depicted. We note that big nodal error occurs at boundary, especially the four corners. It is common for discontinuous pressure finite elements to have bigger L^∞ error when compared with continuous pressure finite

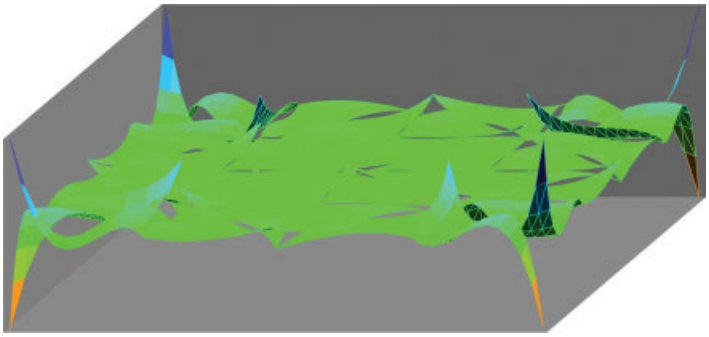

FIG. 5. The pressure error on Level 3 (Fig. 3), $p - p_h$, by (4.6). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

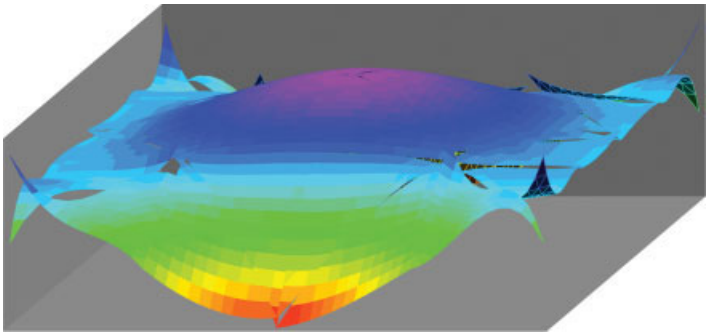

FIG. 6. The Level 3 solution p_h by (4.6). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

elements. However, the former would provide better solutions in the case of less regularity. In addition, we also plot the third level solution p_h in Fig. 6.

In Table I, we list the errors and the the orders of convergence for the finite element solutions to (4.4), obtained by the iterated penalty method (4.6). All the orders of convergence match those

TABLE I. The errors and orders of convergence, by (4.4) and (4.6).

Level	$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty}$	h^m	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2}$	h^m	$\|\mathbf{u} - \mathbf{u}_h\|_{(H^1)^2}$	h^m
1	3.1071203		2.1609024		16.85397	
2	0.1935545	4.00	0.1420790	3.93	2.39238	2.82
3	0.0158297	3.61	0.0055444	4.68	0.21127	3.50
4	0.0006994	4.50	0.0001507	5.20	0.01196	4.14
5	0.0000254	4.78	0.0000040	5.21	0.00067	4.14
6	0.0000008	4.90	0.0000001	5.15	0.00003	4.09
	$\|p - p_h\|_{L^\infty}$	h^m	$\|p - p_h\|_{L^2}$	h^m		
1	341.23126		94.63412			
2	38.09799	3.16	8.32775	3.51		
3	5.86842	2.70	0.66941	3.64		
4	0.52093	3.49	0.03419	4.29		
5	0.03809	3.77	0.00165	4.36		
6	0.00256	3.89	0.00008	4.30		

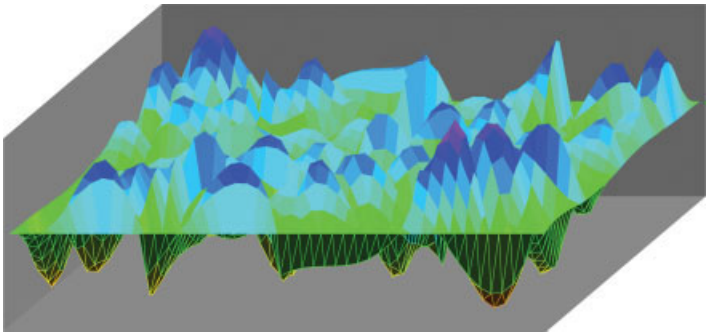

FIG. 7. The Level 3 error $(\mathbf{u}-\mathbf{u}_h)_1$, by (4.8)–(4.10). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

provided by the theory, cf. Ref. [11]. They are all optimal, i.e., $O(h^4)$ for the $\mathbf{u}_h$ error in H^1, and $O(h^5)$ in L^2 and L^∞, and for the pressure p_h error, $O(h^4)$ in L^2 and L^∞ norms. We would comment that the above method of the Scott-Vogelius divergence free elements [11] could be one of the most efficient methods.

In the second part of our numerical test, we use the distributive relaxation method defined in (3.1)–(3.3) to solve the test problem (1.1) with (4.1). In addition to the C^0-P_4 space $\mathbf{V}_h$ and the C^{-1}-P_3 space P_h defined in (4.2) and (4.3), we need the following Argyris C^1 finite element space:

$$S_h = \left\{ \psi_h \in H^2(\Omega) \mid \psi_h|_K \in P_5,\ \forall K \in \mathcal{T}_h,\ \text{and}\ \int_\Omega \psi_h d\mathbf{x} = 0 \right\}. \tag{4.7}$$

In the distributive relaxation method, we solve the following (C^0-P_4, C^1-P_5) coupling problems: Find $(\mathbf{w}_h, \psi_h)$ such that $(\mathbf{w}_h - \nabla\psi_h) \in \mathbf{V}_h$, $\psi_h \in S_h$, and that

$$(\nabla(\mathbf{w}_h - \nabla\psi_h), \nabla\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.8}$$

$$(\nabla\psi_h, \nabla\phi_h) = (-\operatorname{div}\mathbf{w}_h, \phi_h) \quad \forall \phi_h \in S_h. \tag{4.9}$$

The finite element solutions for the original Stokes equations are defined by

$$\mathbf{u}_h = \mathbf{w}_h - \nabla\psi_h \in \mathbf{V}_h, \ \text{ and } \ p_h = \operatorname{div}\mathbf{w}_h \in P_h. \tag{4.10}$$

When solving the coupled system (4.8)–(4.9), we apply the iterative algorithm in (3.10)–(3.11). The algorithm is shown to have a constant rate of convergence, depending only on the inf-sup constant. That is, in our numerical test, we do 10 steps of following block Gauss-Seidel iterations, with $\psi_h^n = 0$: for $n = 1, 2, \ldots, 10$,

$$\left(\nabla\left(\mathbf{w}_h^n - \nabla\psi_h^{n-1}\right), \nabla\mathbf{v}_h\right) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.11}$$

$$\left(\nabla\psi_h^n, \nabla\phi_h\right) = \left(-\operatorname{div}\mathbf{w}_h^n, \phi_h\right) \quad \forall \phi_h \in S_h. \tag{4.12}$$

In Fig. 7, we plot the error of the first component of $\mathbf{u}_h$ on the third level, obtained by (4.8)–(4.10). Comparing the two errors in Figs. 4 and 7, by the two methods, see also Table II, we can see that the new method has much smaller nodal errors for $\mathbf{u}_h$ at the boundary of the domain Ω.

TABLE II. The errors and orders of convergence, by (4.10) and (4.11)–(4.12).

Level	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{L^\infty}$	h^m	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{L^2}$	h^m	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{(H^1)^2}$	h^m
1	0.85606886		0.43641566		3.63567362	
2	0.08656817	3.31	0.03997762	3.45	0.79585096	2.19
3	0.00544815	3.99	0.00179199	4.48	0.08349327	3.25
4	0.00028759	4.24	0.00009114	4.30	0.00682338	3.61
	$\lVert p-p_h\rVert_{L^\infty}$	h^m	$\lVert p-p_h\rVert_{L^2}$	h^m		
1	25.32609624		6.91154459			
2	8.02465563	1.66	1.14672539	2.59		
3	1.34588347	2.58	0.09082632	3.66		
4	0.11132024	3.60	0.00543629	4.06		

The same phenomenon happens to the pressure solution p_h. In Fig. 8, the third level error for the pressure p_h from (4.8)–(4.10) is depicted. Comparing Figs. 5 and 8, the nodal error for the new method (4.8)–(4.9) is also smaller than that obtained by (4.4)–(4.5). This is probably due to the stronger constraint, pointwise divergence free, for the Scott-Vogelius element. In the distributive relaxation method, the solution $\mathbf{u}_h$ is not divergence free, which may make it flexible. In Fig. 9, the third level solution for p_h in (4.8)–(4.10) is depicted. We may compare it with that in Fig. 6.

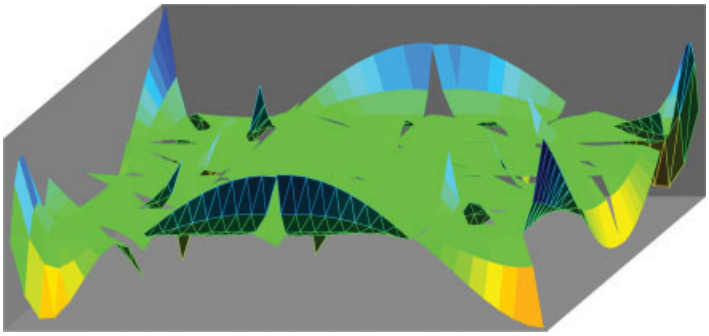

FIG. 8. The Level 3 error $p - p_h$, by (4.8)–(4.10). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

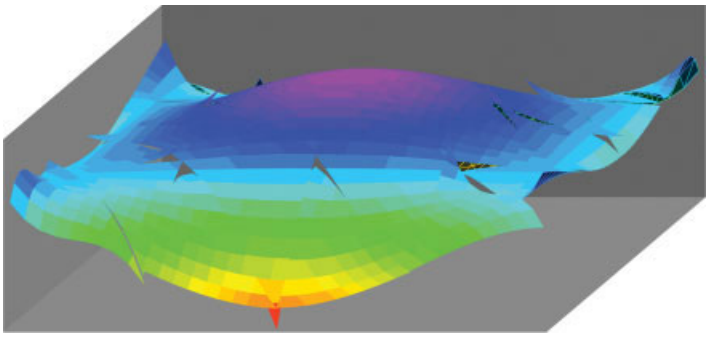

FIG. 9. The Level 3 solution p_h, by (4.8)–(4.10). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

TABLE III. The errors and orders of convergence, by (4.10), for (4.13).

Level	$\|\mathbf{w}-\mathbf{w}_h\|_{L^2}$	h^m	$\|p-p_h\|_{L^\infty}$	h^m	$\|p-p_h\|_{L^2}$	h^m
1	0.03601346		3.09074074		0.96264473	
2	0.00707924	2.35	1.41327380	1.13	0.17544289	2.46
3	0.00035085	4.33	0.20789334	2.77	0.01504262	3.54
4	0.00001233	4.83	0.01954351	3.41	0.00103766	3.86
5	0.00000040	4.96	0.00148678	3.72	0.00006670	3.96
6	0.00000001	5.00	0.00010228	3.86	0.00000419	3.99
	$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty}$	h^m	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	h^m	$\|\mathbf{u}-\mathbf{u}_h\|_{(H^1)^2}$	h^m
1	0.58271118		0.38940050		1.86070462	
2	0.06898693	3.08	0.04703416	3.05	0.45931549	2.02
3	0.00284940	4.60	0.00091451	5.68	0.03437422	3.74
4	0.00013477	4.40	0.00002492	5.20	0.00235581	3.87
5	0.00006270	1.10	0.00000614	2.02	0.00161727	0.54
6	–		–		–	

In Table II, we list the errors and the the orders of convergence for the finite element solutions to (4.4), obtained by (4.10) with the iteration (4.11)–(4.12). All the orders of convergence are optimal, i.e., $O(h^4)$ for the $\mathbf{u}_h$ error in H^1, and $O(h^5)$ in L^2 and L^∞, and for the pressure p_h error, $O(h^4)$ in L^2 and L^∞ norms. When compared with the data in Table I, the new method (4.8)–(4.9) is slightly better than the original mixed finite element method (4.4)–(4.5). However, the numerical error would prevent us to go to higher levels in Table II. We used the Linux Fortran 77 in the computation, where the nodal basis functions for the C^1-P_5 Argyris element are obtained numerically by inverting a 21×21 ill-conditioned linear system. To see the round-off error, we compute another solution of (1.1) on the unit square, where

$$\mathbf{u} = \operatorname{curl} s_1, \quad p = -\Delta s_1, \text{ and } s_1 = 2^8(x-x^2)^3(y-y^2)^3. \tag{4.13}$$

In this case, the solutions $\mathbf{w}_h$ and ϕ_h in (4.8) and (4.9) are decoupled. So, we computed $\mathbf{w}_h$ in Table III. Apparently, the solutions $\mathbf{w}_h$ and p_h converge in the optimal order. After we obtained $\mathbf{w}_h$, we generate the right-hand side of (4.9) and compute ψ_h and $\mathbf{u}_h$. We only need one Uzawa iteration. However, the round-off error is already too big on level 5, shown by the second portion of data in Table III.

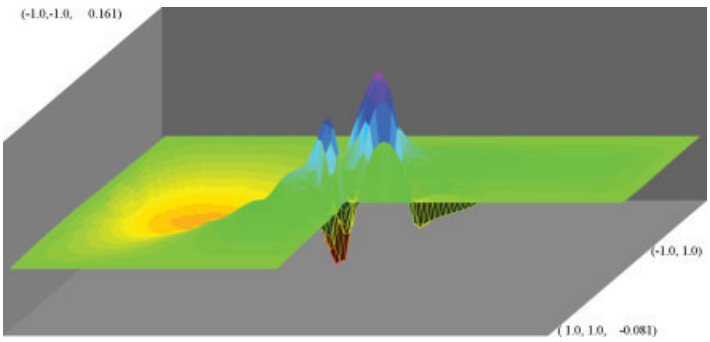

FIG. 10. The first component of a numerical solution $\mathbf{u}_h$ in (4.14). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

Another brief numerical test is on the regularity requirement of the distributive relaxation method. Here, we consider the Stokes equations (1.1) on an L-shaped domain shown in Fig. 10. Here, in (1.1) we have

$$\mathbf{f} = \begin{pmatrix} y \\ -x \end{pmatrix}, \quad \text{and} \quad g = 0. \tag{4.14}$$

In this case, we do not know the exact solution $\mathbf{u}$ and p of (1.1). However, the solution pair should be less smooth because of the reentrant corner. In particular, the continuous solution $\mathbf{w}$ for the continuous problem (4.8) would have a singularity of $O(r^{2/3})$ at the reentrant corner $r = 0$. This can be seen from the first component of a numerical solution $\mathbf{u}_h$ on Level 3 (Fig. 10). This would greatly reduce the convergence order of finite element solutions. Further, as grad ψ_h on the boundary couples the iterative solutions $\mathbf{w}_h$ and ψ_h, we observed a slow convergence of the Uzawa iteration in the finite element spaces (4.11) and (4.12). We will further analyze the finite element method for the distributive relaxation method (3.3).

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References

1. A. Brandt and N. Dinar, Multigrid solutions to flow problems, S. Parter, editor, Numer methods partial differential equations, Academic Press, New York, 1979, pp. 53–147.
2. U. Trottenberg, C. Oosterlee, and A. Schüller, Multigrid, Academic Press, San Diego, 2001.
3. C. Bacuta, Schur Complements on Hilbert Spaces and Saddle Point Systems, J Comput Appl Math 225 (2009), 581–593.
4. C. Bacuta, A unified approach for Uzawa algorithms, SIAM J Numer Anal 44 (2006), 2245–2649.
5. M. Fortin and R. Glowinski, Augmented Lagrangian methods: Applications to the numerical solutions of boundary value problems, Vol. 15: Studies in Mathematics and Applications, North-Holland, New York, 1983.
6. R. H. Nochetto and J. Pyo, Optimal relaxation parameter for the Uzawa Method, Numer Math 98 (2004), 695–702.
7. R. Temam, Navier-Stokes equations, North-Holland, New York, 1984.
8. G. Wittum, On the convergence of multi–grid methods with transformed smoothers, Numer Math 57 (1990), 15–38.
9. S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York, 1994.
10. V. Girault and P. A. Raviart, Finite element methods for Navier-Stokes equations, Springer-Verlag, Berlin, 1986.
11. L. R. Scott and M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, RAIRO, Modelisation Math Anal Numer 19 (1985), 111–143.