A SADDLE POINT LEAST SQUARES APPROACH TO MIXED METHODS

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Abstract. We investigate new PDE discretization approaches for solving variational formulations with different types of trial and test spaces. The general mixed formulation we consider assumes a stability LBB condition and a data compatibility condition at the continuous level. We expand on the Bramble-Pasciak's least square formulation for solving such problems by providing new ways to choose approximation spaces and new iterative processes to solve the discrete formulations. Our proposed method has the advantage that a discrete $\inf - \sup$ condition is automatically satisfied by natural choices of test spaces (first) and corresponding trial spaces (second). In addition, for the proposed iterative solver, a nodal basis for the trial space is not required. Applications of the new approach include discretization of first order systems of PDEs, such as $\text{div} - \text{curl}$ systems and time-hamonic Maxwell equations.

1. Introduction

Over the last two decades, there have been many advances in applying finite element least squares methods to approximate first order systems of PDEs, [6, 7, 8, 14, 15, 16, 17, 18, 26, 27, 28]. However, when compared to the more established field of finite element methods for elliptic problems, a unified theoretical framework for least squares approximation of solutions of first order systems of PDEs is missing. Our proposed framework provides powerful preconditioning techniques, has efficient error estimators, is suitable to multilevel techniques, and leads to robust and easy to implement solvers. We combine known theory and discretization techniques for approximating elliptic problems and for symmetric saddle point problems (see [1, 5, 11, 12, 13, 23, 29, 31, 32, 36]) to obtain a unified framework for discretizing variational formulations with different types of test and trial spaces. In particular, the framework can be applied to least square approximation for a large class of first order systems of PDEs.

For the applications we consider, the solution spaces are L^2 type spaces, and the data can reside in weak negative norm spaces. We require that the test spaces be H^1 type spaces with suitable boundary conditions, and

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the discrete test spaces be conforming finite element spaces built using the action of the continuous differential operator associated with a given problem. Among the advantages of the method are the following: the discretization leads to saddle point variational formulation with automatic discrete $\inf-\sup$ condition, and assembly of stiffness matrices for the trial spaces is avoided.

The general abstract problem that we plan to discretize using a *Saddle Point Least Squares Method* (SPLS) is: Find $p \in Q$ such that

$$b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \text{for all } \mathbf{v} \in \mathbf{V} \quad \text{or} \quad B^* p = \mathbf{f}, \tag{1.1}$$

where $\mathbf{V}$ and Q are infinite dimensional Hilbert spaces and $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, that satisfies a standard $\inf-\sup$ condition, and $\mathbf{f} \in \mathbf{V}^*$ belongs to the range of B^*. In the special case when the operator B associated with the form $b(\cdot,\cdot)$ is injective, our method can be viewed as a conforming Petrov-Galerkin method. From the point of view of choosing the discrete spaces, our method can be characterized as the dual of the Discontinuous Petrov-Galerkin (DPG) method introduced by Demkowicz and Gopalakrishnan in [20, 21]. While both methods have strong connections with the least squares and minimum residual techniques, the proposed discretization process *stands apart* from the DPG approach because of the *opposite order* and *different ways* in which the trial and test spaces are chosen.

We propose the following main steps of our *saddle point least square discretization* method:

- Step 1 Reduce the general problem (1.1) to a *saddle point least square* formulation, using the natural inner product $a_0(\cdot,\cdot)$ on $\mathbf{V} \times \mathbf{V}$ as the $(1,1)$ form of the saddle point system (see problem (2.12)).
- Step 2 Choose a standard *conforming approximation space* $\mathbf{V}_h$ for the *variational space* $\mathbf{V}$.
- Step 3 Construct a discrete *trial space* $\mathcal{M}_h \subset Q$ using the operator B associated with the form $b(\cdot,\cdot)$. For example, take $\mathcal{M}_h := \mathcal{C}^{-1} B \mathbf{V}_h$, or $\mathcal{M}_h := \tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{V}_h$, where $\mathcal{C}^{-1}$ is the Riesz representation operator for the space Q and $\tilde{\mathcal{Q}}_h$ is a *projection* from Q to a subspace $\tilde{\mathcal{M}}_h$. The pair $(\mathbf{V}_h, \mathcal{M}_h)$ will automatically satisfy a discrete $\inf-\sup$ condition.
- Step 4 Write the discrete version of SPLS formulation on $(\mathbf{V}_h, \mathcal{M}_h)$, see (3.3), and replace $a_0(\cdot,\cdot)$ by an equivalent form $a_{prec}(\cdot,\cdot)$ on $\mathbf{V}_h \times \mathbf{V}_h$.
- Step 5 Solve the new discrete SPLS problem using an Uzawa type iterative process that requires only the action of the preconditioner associated with $a_{prec}(\cdot,\cdot)$ and the action of $\mathcal{C}^{-1}B$ or $\tilde{\mathcal{Q}}_h C^{-1} B$.

The paper is organized as follows. In Section 2, we introduce notation and review basic abstract results needed to describe the method. We also include here the first step of reduction of a mixed problem to a Saddle Point Problem (SPP). In Section 3, we present the discretization part of the method (Step 2 and Step 3) and discuss the choice of discrete spaces and their approximability. Uzawa type Iterative solvers without trial space bases are presented in Section 4. The special choice of spaces with discrete $\inf - \sup$ condition and their approximability properties are presented in Section 5. In Section 6 we present an example of SPLS discretization for a $\text{div} - \text{curl}$ system. The Appendix (Section 8) contains some important functional analysis results needed for the proofs of the paper.

2. Notation and Background

In this section, we start with a review of the notation of the classical SPP theory and introduce the spaces, the operators and the norms for the general abstract case. We let $\mathbf{V}$ and Q be two Hilbert spaces with inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, with the corresponding induced norms $|\cdot|_{\mathbf{V}} = |\cdot| = a_0(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_Q = \|\cdot\| = (\cdot,\cdot)^{1/2}$. The dual pairings on $\mathbf{V}^* \times \mathbf{V}$ and $Q^* \times Q$ are denoted by $\langle\cdot,\cdot\rangle$. Here, $\mathbf{V}^*$ and Q^* denote the duals of $\mathbf{V}$ and Q, respectively. With the inner products $a_0(\cdot,\cdot)$ and $(\cdot,\cdot)$, we associate the operators
$\mathcal{A} : V \to V^*$ and $\mathcal{C} : Q \to Q^*$ defined by

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle = a_0(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}$$

and

$$\langle \mathcal{C}p, q\rangle = (p, q) \quad \text{for all } \ p, q \in Q.$$

The operators $\mathcal{A}^{-1} : V^* \to V$ and $\mathcal{C}^{-1} : Q^* \to Q$ are called the Riesz-canonical isometries and satisfy the following propperties

$$a_0(\mathcal{A}^{-1}\mathbf{u}^*, \mathbf{v}) = \langle \mathbf{u}^*, \mathbf{v}\rangle, \ \ |\mathcal{A}^{-1}\mathbf{u}^*|_{\mathbf{V}} = \|\mathbf{u}^*\|_{\mathbf{V}^*}, \mathbf{u}^* \in \mathbf{V}^*, \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

$$(\mathcal{C}^{-1}p^*, q) = \langle p^*, q\rangle, \ \ \|\mathcal{C}^{-1}p^*\| = \|p^*\|_{Q^*}, \ p^* \in Q^*, q \in Q. \tag{2.2}$$

Next, we suppose that $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{V} \times Q$, satisfying the inf-sup condition. More precisely, we assume that

$$\inf_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| \ |\mathbf{v}|} = m > 0 \tag{2.3}$$

and

$$\sup_{p\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)}{\|p\| \ |\mathbf{v}|} = M < \infty. \tag{2.4}$$

Here, and throughout this paper, the "inf" and "sup" are taken over nonzero vectors. With the form b, we associate the linear operators $B : V \to Q^*$ and $B^* : Q \to V^*$ defined by

$$\langle B\mathbf{v}, q\rangle = b(\mathbf{v}, q) = \langle B^*q, \mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{V}, \ q \in Q.$$

Let $\mathbf{V}_0$ be the kernel of B or $\mathcal{C}^{-1}B$, i.e.,

$$\mathbf{V}_0 = Ker(B) = \{\mathbf{v} \in \mathbf{V} |\ B\mathbf{v} = 0\} = \{\mathbf{v} \in \mathbf{V} |\ \mathcal{C}^{-1}B\mathbf{v} = 0\}.$$

Due to (2.4), $\mathbf{V}_0$ is a closed subspace of $\mathbf{V}$.

We notice here that $\mathcal{C}^{-1}B : \mathbf{V} \to Q$ and $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}$ are dual to each other in the Hilbert sense, and consequently, the Schur complement $S_0 := \mathcal{C}^{-1}B\mathcal{A}^{-1}B^*$ is a symmetric operator. In addition, S_0 is a positive definite operator with $\sigma_0(S_0) \subset [m^2, M^2]$, see Lemma 8.1 or [2].

2.1. **The general problem.** Assume that $b : \mathbf{V} \times Q \to \mathbb{R}$ is a bilinear form satisfying (2.3) and (2.4), and let $\mathbf{f} \in \mathbf{V}^*$ be given. Many variational formulations of PDEs, including first order systems, can be written in the mixed form:

Find $p \in Q$ such that

$$b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \qquad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{2.5}$$

The operator form of (2.5) is to solve the following equation for p

$$B^* p = \mathbf{f}. \tag{2.6}$$

The existence and the uniqueness of (2.5) was first studied by Aziz and Babuška in [1] and is known as the Babuška's Lemma.

Lemma 2.1. *(Babuška) Let $b : \mathbf{V} \times Q \to \mathbb{R}$ be a bilinear form satisfying* (2.3) *and* (2.4)*, and let $\mathbf{f} \in \mathbf{V}^*$. The problem: Find $p \in Q$ such that*

$$b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \qquad \textit{for all } \mathbf{v} \in \mathbf{V} \tag{2.7}$$

has a unique solution if and only if

$$\langle \mathbf{f}, \mathbf{v} \rangle = 0, \qquad \textit{for all } \mathbf{v} \in \mathbf{V}_0. \tag{2.8}$$

If (2.8) *holds, and p is the solution of* (2.7)*, then*

$$m\|p\| \le \|p\|_{S_0} := (S_0 p, p)^{1/2} = \|F\|_{\mathbf{V}^*} \le M\|p\|. \tag{2.9}$$

A proof of the Lemma can be found in [1, 2].

To justify our *saddle point least squares* terminology, we write (2.6) in the equivalent form

$$\mathcal{A}^{-1}B^* p = \mathcal{A}^{-1}\mathbf{f}. \tag{2.10}$$

Using that the Hilbert transpose of $\mathcal{A}^{-1}B^* : \mathbf{V} \to Q$ is the operator $\mathcal{C}^{-1}B : \mathbf{V} \to Q$, the normal equation for solving (2.10) in the least square sense is:

$$\mathcal{C}^{-1}BA^{-1}B^* p = \mathcal{C}^{-1}B\mathcal{A}^{-1}\mathbf{f}. \tag{2.11}$$

Since $S_0 := \mathcal{C}^{-1}B\mathcal{A}^{-1}B^*$ is a symmetric positive definite operator, the problem (2.11) has a unique solution p. Then, we consider the well posed saddle point problem:

Find $(\mathbf{u}, p) \in (\mathbf{V}, Q)$ such that

$$\begin{array}{lclcll} a_0(\mathbf{u}, \mathbf{v}) & + & b(\mathbf{v}, p) & = \langle \mathbf{f}, \mathbf{v} \rangle & & \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) & & & = 0 & & \text{for all } q \in Q. \end{array} \tag{2.12}$$

One can immediately check that p is the unique solution of (2.11) if and only if $(\mathbf{u}, p)$ is the unique solution of (2.12), where $\mathbf{u} = \mathcal{A}^{-1}(B^* p - F)$.

Proposition 2.2. *In the presence of the continuous* $\inf-\sup$ *condition* (2.3) *and the compatibility condition* (2.8)*, we have that p is the unique solution of* (2.5) *if and only if* $(\mathbf{u}=0, p)$ *is the unique solution of* (2.12).

Remark 2.3. *Even in the absence of the compatibility condition* (2.8)*, but assuming the continuous* $\inf-\sup$ *condition* (2.3)*, we have that* (2.12) *has unique solution regardless of the fact that* (2.5) *is wellposed or not. The above arguments justify the name of the system* (2.12) *as the SPLS variational formulation of* (1.1).

2.2. **A note on approximating the solution by Uzawa type algorithms.** Given a relaxation parameter $\alpha \in (0, 2/M^2)$, the Uzawa algorithm for approximating the solution $(\mathbf{u}=0, p)$ of (2.12) can be described as follows.

Start with any $p_0 \in Q$. For $j = 1, 2, \ldots$ Compute
$\mathbf{u}_j \in \mathbf{V}$, $q_j, p_j \in Q$ by

$$
\begin{aligned}
a_0(\mathbf{u}_j, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v}\rangle - \mathbf{b}(\mathbf{v}, \mathbf{p_{j-1}}), \ \mathbf{v} \in \mathbf{V}\\
q_j &= \mathcal{C}^{-1} B \mathbf{u}_j,\\
p_j &= p_{j-1} + \alpha\, q_j
\end{aligned}
$$

The convergence of the algorithm is discussed in many publications, see e.g., [5, 13, 22, 23, 30]. It is easy to check that

$$
p - p_j = (I - \alpha \mathcal{C}^{-1} B A^{-1} B^*)(p - p_{j-1}) = (I - \alpha S_0)(p - p_{j-1}). \tag{2.13}
$$

By using the spectral bounds for S_0, we obtain a convergence rate $\gamma := \|I - \alpha S_0\| = \max\{|1-\alpha m^2|, |1-\alpha M^2|\} < 1$, provided that $\alpha \in \left(0, \frac{2}{M^2}\right)$. It is easy to check that for $j \geq 1$, we also have

$$
\mathbf{u}_{j+1} = (I - \alpha \mathcal{A}^{-1} B^* \mathcal{C}^{-1} B)\mathbf{u}_j. \tag{2.14}
$$

Due to the compatibility assumption (2.8), each residual representation $\mathbf{u}_j$ in the Uzawa algorithm, belongs to $\mathbf{V}_0^\perp$. Using the spectral properties of $S_1 := \mathcal{A}^{-1} B^* \mathcal{C}^{-1} B : \mathbf{V}_0^\perp \to \mathbf{V}_0^\perp$ (see Lemma (8.1) (v)) and (2.14), we have that the series $\displaystyle\sum_{j=1}^{\infty} \mathbf{u}_j$ is absolutely convergent. Consequently, if we start the algorithm with $p_0 = 0$, then $p_k = \alpha \mathcal{C}^{-1} B(\displaystyle\sum_{j=1}^{k} \mathbf{u}_j)$ and $p_k \to p = \mathcal{C}^{-1} B\, \mathbf{w}$ with $\mathbf{w} = \alpha \sum_{j=1}^{\infty} \mathbf{u}_j \in \mathbf{V}_0^\perp$. Thus, the Uzawa algorithm not only approximates the solution of (2.7), but also finds a special representation of the solution: $p = \mathcal{C}^{-1} B\, \mathbf{w}$ with $\mathbf{w} \in \mathbf{V}_0^\perp$. Similar remarks hold for Uzawa Gradient (UG) and Uzawa Conjugate Gradient (UCG) algorithms.

3. SPLS discretization

In light of Propsition 2.2, we are led to the corresponding *SPLS discretization* of (1.1). We let $\mathbf{V}_h \subset \mathbf{V}$ and $\mathcal{M}_h \subset Q$ be finite dimensional approximation spaces and consider the restrictions of the forms $a_0(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to the discrete spaces $\mathbf{V}_h$ and M_h. Assume that the following discrete $\inf-\sup$ condition holds for the pair $(\mathbf{V}_h, \mathcal{M}_h)$.

$$\inf_{p_h\in\mathcal{M}_h} \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\|p_h\|\ |\mathbf{v}_h|} = m_h > 0. \tag{3.1}$$

We define the corresponding discrete operators $A_h, \mathcal{C}_h, B_h$, and B_h^*. For example, A_h is the discrete version of A, and is defined by

$$\langle A_h\mathbf{u}_h, \mathbf{v}_h\rangle = a_0(\mathbf{u}_h, \mathbf{v}_h), \quad \text{for all } \mathbf{u}_h \in \mathbf{V}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

We define $\mathbf{V}_{h,0}$ to be the kernel of B_h

$$\mathbf{V}_{h,0} := \{\mathbf{v}_h \in \mathbf{V}_h |\ b(\mathbf{v}_h, q_h) = 0, \quad \text{for all } q_h \in \mathcal{M}_h\},$$

$$\mathbf{V}_{h,1} := \mathbf{V}_{h,0}^{\perp} = A_h^{-1}B_h^*(M_h), \text{ and } S_{h,0} := \mathcal{C}_h^{-1}B_h\mathcal{A}_h^{-1}B_h^*.$$

Remark 3.1. *If $\mathbf{V}_{h,0} \subset \mathbf{V}_0$, then the compatibility condition* (2.8) *implies a discrete compatibility condition. Consequently, under the discrete stability assumption* (3.1)*, the problem: Find $p_h \in \mathcal{M}_h$ such that*

$$b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h\rangle, \quad \textit{for all } \mathbf{v}_h \in \mathbf{V}_h,$$

has unique solution.

In general, $\mathbf{V}_{h,0}$ is not contained in $\mathbf{V}_0$. Thus, the compatibility condition (2.8) does not imply a discrete compatibility condition, and the problem: Find $p_h \in \mathcal{M}_h$ such that

$$b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h\rangle, \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h, \text{ or } B_h^* p_h = \mathbf{f_h}, \tag{3.2}$$

might not be well posed. Nevertheless, under the assumption (3.1), the following saddle point discrete variational problem :
Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{M}_h$ such that

$$\begin{array}{llll} a_0(\mathbf{u}_h, \mathbf{v}_h) & + \ b(\mathbf{v}_h, p_h) & = \langle \mathbf{f}, \mathbf{v}_h\rangle & \text{for all } \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) & & = 0 & \text{for all } q_h \in \mathcal{M}_h, \end{array} \tag{3.3}$$

always has a unique solution. Using the corresponding discrete operators, the problem (3.3) is equivalent to: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{M}_h$ such that

$$\begin{array}{lll} \mathbf{u}_h & = & A_h^{-1}(\mathbf{f_h} - B_h^T p_h) \\ \mathcal{C}_h^{-1}B_hA_h^{-1}B_h^* p_h & = & \mathcal{C}_h^{-1}B_hA_h^{-1}\mathbf{f_h}. \end{array} \tag{3.4}$$

Since the operator $S_{h,0} = \mathcal{C}_h^{-1}B_hA_h^{-1}B_h^*$ is a symmetric positive definite on $\mathcal{M}_h$, it is invertible, and the second equation of (3.4) has the unique solution p_h. Thus, one way to solve the system (3.3), would be to solve the second equation of (3.4) for p_h, and then find $\mathbf{u}_h$ from the first equation of (3.4). Since $\mathcal{C}_h^{-1}B_h$ and $A_h^{-1}B_h^*$ are dual to each other in the Hilbert sense and the problem (3.2) is also equivalent with $\mathcal{A}_h^{-1}B_h^* p_h = \mathcal{A}_h^{-1}\mathbf{f_h}$, we have that the solution of the second equation of (3.4) is in fact the *least squares solution* of (3.2). The component $\mathbf{u}_h$ of the solution of (3.3) or (3.4) becomes the representation on $\mathbf{V}_h$ of the residual associated with the least square solution of (3.2).

Another reason for which the *saddle point least squares discretrization* of (1.1) is the variational formulation (3.3) is that (3.3) is the natural discretization of the SPLS formulation (2.12) of (1.1). Using that (3.3) is the discrete variational formulation of (2.12), and based on the classical error analysis for SPP theory, we can find standard estimates for $|\mathbf{0} - \mathbf{u}_h|$ and $\|p - p_h\|$, see [9, 11, 32, 36]. If we assume discrete stability, then the second component p_h of the solution of (3.3) provides a good approximation to the least square solution p of (1.1) even in the absence of the compatibility condition (2.8). In the case when the compatibility condition (2.8) holds, we are dealing with *a special saddle point problem*, and a sharp error estimate for $\|p - p_h\|$ can be proved using the Xu-Zikatanov argument in [36].

Theorem 3.2. *Let* $b : \mathbf{V} \times Q \to \mathbb{R}$ *satisfy* (2.3) *and* (2.4)*, and assume that* $\mathbf{f} \in \mathbf{V}^*$ *is given and satisfies* (2.8)*. Assume that* p *is the solution of* (1.1) *and* $\mathbf{V}_h \subset \mathbf{V}$*,* $M_h \subset Q$ *are chosen such that the discrete* $\inf-\sup$ *condition* (3.1) *holds. Then, if* $(\mathbf{u}_h, p_h)$ *is the solution of* (3.3)*, the following error estimate holds:*

$$\frac{1}{M}|u_h| \le \|p - p_h\| \le \frac{M}{m_h} \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|. \tag{3.5}$$

The proof of the estimate can be found in the Appendix. The right part of the estimate (3.5) is an improvement of the similar estimate presented in [10] that provides a bound that is linear in m_h^{-2}. This improvement is significant if m_h depends on h.

Remark 3.3. *All the considerations made so far in this section make sense if the form* $a_0(\cdot,\cdot)$*, as an inner product on* $\mathbf{V}_h$*, is replaced by another inner product which gives rise to an independent of* h *equivalent norm on* $\mathbf{V}_h$*. Certainly, the definition of* $A_h, S_{h,0}$*, and* m_h*, will change accordingly with the new inner product, but the error estimate* (3.5) *remains valid with different estimating constants that factor in the norm equivalence constants.*

This observation leads to an efficient preconditioning approach for SPLS discretization. More precisely, if the Uzawa algorithm is involved to solve (3.3), then the action of A_h^{-1} can be replaced by the action of any equivalent preconditioner.

From the first equation of (3.3) and (3.5) we get that the solution $(\mathbf{u}_h, p_h)$ satisfies:

$$|\mathbf{u}_h| = \|\mathbf{f_h} - B_h^* p_h\|_{\mathbf{V}_h^*} \le \frac{M^2}{m_h} \inf_{q_h \in \mathcal{M}_h} \|p - q_h\|,$$

where $\mathbf{f_h}$ is the functional $\mathbf{v}_h \to \langle \mathbf{f}, \mathbf{v}_h \rangle$ on $\mathbf{V}_h$. This gives an estimate for the residual associated with the discrete equation (3.2).

Remark 3.4. *The Bramble-Pasciak least squares method in* [10] *for discretizing* (2.7)*, focuses on solving only the variational formulation of the* *second equation of* (3.4)*, i.e., Find* $p_h \in \mathcal{M}_h$ *such that*

$$(\mathcal{C}_h^{-1} B_h A_h^{-1} B_h^* p_h, q_h) = \langle \mathbf{f}, \mathcal{A}_h^{-1} B_h^* q_h \rangle, \qquad \text{for all } \; q_h \in \mathcal{M}_h, \tag{3.6}$$

which can be reformulated as: Find $p_h \in \mathcal{M}_h$ *such that*

$$b(\mathcal{A}_h^{-1} B_h^* q_h, p_h) = \langle \mathbf{f}, \mathcal{A}_h^{-1} B_h^* q_h \rangle, \qquad \text{for all } \; q_h \in \mathcal{M}_h, \tag{3.7}$$

The approach in [10] *to numerically solve* (3.7) *is based on using families of stable pairs* $\{(\mathbf{V}_h, \mathcal{M}_h)\}$*, bases for both* $\mathbf{V}_h$ *and* $\mathcal{M}_h$*, and on replacing the action of* $\mathcal{A}_h^{-1}$ *on* $\mathbf{V}_h$ *by an approximate inverse. Our approach is based on iteratively solving the coupled saddle point system* (3.3) *whose* p_h *component of the solution is the least squares solution of* (3.2) *and on using bases and matrix assembly only for the test space* $\mathbf{V}_h$*.*

Remark 3.5. *The DPG method,* [20, 21]*, for discretizing* (2.7)*, also reduces to a saddle point formulation that is similar to* (3.3)*. The details are presented in Theorem 2.4 of* [20] *and a proof can be found in* [24]*. For the DPG approach, the trial spaces* Q *or* $\mathcal{M}_h$ *are sought as a product between an "interior component" and and an "interface compomonent" at both continuous and discrete levels. Once a finite dimensional subspace* $\mathbf{V}_r$ *of* $\mathbf{V}$ *(with certain properties) is chosen, the discrete test space, using our notation, is* $\mathbf{V}_h = A_r^{-1} B^* \mathcal{M}_h$*, where* A_r^{-1} *is the Riesz representation operator for the Hilbert space* $(\mathbf{V}_r, a_0(\cdot,\cdot))$*. In our approach, we choose the discrete trial space* $\mathbf{V}_h \subset \mathbf{V}$ *first, a finite dimensional subspace* $\tilde{\mathcal{M}}_h \subset Q$*, and then define* $\mathcal{M}_h := \tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{V}_h$*, where* $\mathcal{C}^{-1}$ *is the Riesz representation operator for the space* Q *and* $\tilde{\mathcal{Q}}_h$ *is a projection from* Q *to* $\tilde{\mathcal{M}}_h$*, (see Section 5 for details).*

Given the connection between approximating the solution of the mixed variational problem (1.1) and solving the discrete problem (3.3), one can just rely on the available techniques and solvers for symmetric saddle point problems of type (2.12) on stable families of spaces $\{(\mathbf{V}_h, \mathcal{M}_h)\}_h$ in order to solve (1.1). Nevertheless, if it is difficult to find families of stable pairs or they exist but are difficult to implement, then we present next alternative ways of approximating the solution of (1.1).

4. Iterative solvers without basis for the trial space

The computational challenge we face when solving (3.3) on $(\mathbf{V}_h, \mathcal{M}_h)$ is that one might not be able to find stable pairs $(\mathbf{V}_h, \mathcal{M}_h)$. Even in the case that stable pairs are available, a global linear system corresponding to (3.3) might be difficult to assemble and solve. It is possible to solve (3.3) without even having explicit bases for $\mathcal{M}_h$ by using the Uzawa (U), Uzawa Gradient (UG), or Uzawa Conjugate Gradient (UCG) algorithms. Following [3], we have that the standard U and UG algorithms can be rewritten such that they differ only by the way the parameter α is chosen. For the Uzawa algorithm, we have to choose a fixed number $\alpha = \alpha_0$ in the interval $\left(0, \frac{2}{M_h^2}\right)$. For the UG algorithm, the parameter α is chosen to impose the orthogonality of consecutive residuals associated with the second equation in (3.3). The first step for Uzawa is identical with the first step of UG. We combine the two algorithms in:

Algorithm 4.1. *(U-UG) Algorithms*

Step 1: Set $\mathbf{u}_0 = 0 \in \mathbf{V}_h,\ p_0 \in \mathcal{M}_h,$ **compute** $\mathbf{u}_1 \in \mathbf{V}_h,\ q_1 \in \mathcal{M}_h$ *by*

$$\begin{aligned} a_0(\mathbf{u}_1, \mathbf{v}) &= \langle \mathbf{f}_h, \mathbf{v}\rangle - b(\mathbf{v}, p_0), \quad \text{for all } \mathbf{v} \in \mathbf{V}_h \\ (q_1, q) &= b(\mathbf{u}_1, q), \quad \text{for all } q \in \mathcal{M}_h. \end{aligned}$$

Step 2 : For $j = 1, 2, \ldots,$ **compute** $\mathbf{h}_j, \alpha_j, p_j, \mathbf{u}_{j+1}, q_{j+1}$ *by*

$$\begin{aligned} &(\mathbf{U-UG1}) & a_0(\mathbf{h}_j, \mathbf{v}) &= -\,b(\mathbf{v}, q_j),\ \mathbf{v} \in \mathbf{V}_h \\ &(\mathbf{U}\alpha) & \alpha_j &= \alpha_0 \text{ for the Uzawa algorithm or} \\ &(\mathbf{UG}\alpha) & \alpha_j &= -\frac{(q_j, q_j)}{b(\mathbf{h}_j, q_j)} \text{ for the UG algorithm} \\ &(\mathbf{U-UG2}) & p_j &= p_{j-1} + \alpha_j\ q_j \\ &(\mathbf{U-UG3}) & \mathbf{u}_{j+1} &= \mathbf{u}_j + \alpha_j\ \mathbf{h}_j \\ &(\mathbf{U-UG4}) & (q_{j+1}, q) &= b(\mathbf{u}_{j+1}, q), \quad \text{for all } q \in \mathcal{M}_h. \end{aligned}$$

Here, $\mathbf{f}_h$ is the restriction of $\mathbf{f}$ to $\mathbf{V}_h$. To obtain the UCG algorithm, the UG algorithm is modified as in [9, 35] as follows: First, we define $d_1 := q_1$ in **Step 1**, and then modify **Step 2** by replacing $b(\cdot, q_j)$ with $b(\cdot, d_j)$, where $\{d_j\}$ is a sequence of conjugate directions:

Algorithm 4.2. *(UCG) Algorithm*

Step 1: Set $\mathbf{u}_0 = 0 \in \mathbf{V}_h,\ p_0 \in \mathcal{M}_h.$ **Compute** $\mathbf{u}_1 \in \mathbf{V}_h,\ q_1, d_1 \in \mathcal{M}_h$ *by*

$$\begin{aligned} a_0(\mathbf{u}_1, \mathbf{v}) &= \langle \mathbf{f}_h, \mathbf{v}\rangle - b(\mathbf{v}, p_0),\ \mathbf{v} \in \mathbf{V}_h \\ (q_1, q) &= b(\mathbf{u}_1, q), \quad \text{for all } q \in \mathcal{M}_h, \quad d_1 := q_1. \end{aligned}$$

Step 2 For $j = 1, 2, \ldots,$ **compute** $\mathbf{h}_j, \alpha_j, p_j, \mathbf{u}_{j+1}, q_{j+1}, \beta_j, d_{j+1}$ *by*

$$\begin{aligned} &(\mathbf{UCG1}) & a_0(\mathbf{h}_j, \mathbf{v}) &= -\,b(\mathbf{v}, d_j),\ \mathbf{v} \in \mathbf{V}_h \\ &(\mathbf{UCG}\alpha) & \alpha_j &= -\frac{(q_j, q_j)}{b(\mathbf{h}_j, q_j)} \\ &(\mathbf{UCG2}) & p_j &= p_{j-1} + \alpha_j\ d_j \\ &(\mathbf{UCG3}) & \mathbf{u}_{j+1} &= \mathbf{u}_j + \alpha_j\ \mathbf{h}_j \\ &(\mathbf{UCG4}) & (q_{j+1}, q) &= b(\mathbf{u}_{j+1}, q), \quad \text{for all } q \in \mathcal{M}_h \\ &(\mathbf{UCG}\beta) & \beta_j &= \frac{(q_{j+1}, q_{j+1})}{(q_j, q_j)} \\ &(\mathbf{UCG6}) & d_{j+1} &= q_{j+1} + \beta_j d_j \end{aligned}$$

Each one of the described algorithms converges to the discrete solution $(\mathbf{u}_h, p_h)$ of (3.3). In addition, the following *sharp error estimation* result was proved by one of the authors in a slightly more general context in [3].

Theorem 4.3. *If* $(\mathbf{u}_h, p_h)$ *is the discrete solution of* (3.3)*, and* $(\mathbf{u}_{j+1}, p_j)$ *is the* j^{th} *iteration for U, UG, or UCG, then* $(\mathbf{u}_{j+1}, p_j) \to (\mathbf{u}_h, p_h)$ *and*

$$\begin{aligned} \frac{1}{M^2}\|q_{j+1}\| \le \|p_j - p_h\| \quad &\le \frac{1}{m_h^2}\|q_{j+1}\|, \\ \frac{m_h}{M^2}\|q_{j+1}\| \le |\mathbf{u}_{j+1} - \mathbf{u}_h| &\le \frac{M}{m_h^2}\|q_{j+1}\|. \end{aligned} \tag{4.1}$$

Besides convergence of the iteration processes, the result *entitles* $\|q_{j+1}\|$ as a *computable, robust, efficient, and uniform*-modulo m_h *estimator for all three algorithms*. Under stability presence, we can use the Theorem 3.2 and the estimates (4.1) to build adaptive or multilevel algorithms for SPLS discretization. A cascadic algorithm for solving symmetric SPPs was introduced by one of the authors in [3].

One *major advantage* of solving the system (3.3) as the *SPLS discretization* of (1.1), using one of these three algorithms is that (4.1) remains valid, if the right correction of constant factors is made, when $a_0(\cdot, \cdot)$ is replaced by a uniform equivalent form $a_{prec}(\cdot, \cdot)$.

Similar to the continuous case in Section 2.2, an important observation here is that if the starting initial guess is $p_0 = 0$, then the p_j- iterates remain in the space $\mathcal{C}_h^{-1} B_h \mathbf{V}_h$ and that $\{p_j\}$ approximates and represent the solution p_h in the form $\mathcal{C}_h^{-1} B_h \mathbf{w}_h$, with $\mathbf{w}_h \in \mathbf{V}_{h,0}^{\perp}$. In addition, as presented in the next section, for certain spaces $\mathcal{M}_h$ a basis is not needed for solving the variational problems associated with q_j, and at each step of the U-type iterative processes, only the action of $\mathcal{A}^{-1}$ or a preconditioner requires an inversion process.

5. Construction of special discrete spaces

Next, we will present a method to address the lack of stability of the approximation spaces. Let $\mathbf{V}_h$ be a subset of $\mathbf{V}$, and let $\tilde{M}_h$ be a finite dimensional subspace of Q that has good approximability properties. Typical examples of spaces $\tilde{M}_h$ are the spaces of piecewise polynomials. We equip $\tilde{M}_h$ with an inner product that could differ from the restriction of the Q inner product on $\tilde{M}_h$, but induces an equivalent norm (independent of h). For convenience, we denote the inner product on $\tilde{M}_h$ by $(\cdot, \cdot)$. If $\tilde{\mathcal{Q}}_h : Q \to \tilde{\mathcal{M}}_h$ is the orthogonal projection onto $\tilde{\mathcal{M}}_h$, we simply define the space $\mathcal{M}_h$ by

$$\mathcal{M}_h := \tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{V}_h.$$

We consider the restriction of the form $a_0(\cdot, \cdot)$ to $\mathbf{V}_h \times \mathbf{V}_h$ and the restriction of $b(\cdot, \cdot)$ to $\mathbf{V}_h \times \mathcal{M}_h$, and define the discrete operators $A_h, \mathcal{C}_h, B_h$, and B_h^*

for the pair $(\mathbf{V}_h, \mathcal{M}_h)$. For any $q_h \in \mathcal{M}_h, \mathbf{v}_h \in \mathbf{V}_h$, we have

$$b(\mathbf{v}_h, q_h) = \langle B_h \mathbf{v}_h, q_h \rangle = (\mathcal{C}_h^{-1} B_h \mathbf{v}_h, q_h).$$

On the other hand, since $q_h \in \mathcal{M}_h \subset Q$, and $\mathbf{v}_h \in \mathbf{V}_h \subset \mathbf{V}$, we have

$$b(\mathbf{v}_h, q_h) = (\mathcal{C}^{-1} B \mathbf{v}_h, q_h) = (\tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{v}_h, q_h).$$

From the above identities, we get

$$\mathcal{C}_h^{-1} B_h \mathbf{v}_h = \tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{v}_h \qquad \text{for all } \mathbf{v}_h \in \mathbf{V}_h. \tag{5.1}$$

This implies that $\mathcal{C}_h^{-1} B_h$ is onto M_h and, using that $\mathbf{V}_h$ and M_h are finite dimensional spaces, the discrete inf-sup condition (3.1) holds for some $m_h > 0$ that might depend on h. Thus, the problem (3.3) has unique solution $(\mathbf{u}_h, p_h)$, and any of the three Uzawa type algorithms presented in Section 4 can be applied to approximate it. In addition, in light of 5.1, we have that the residual q_j for each Uzawa type algorithm satisfies $q_j = \mathcal{C}_h^{-1} B_h \mathbf{u}_j = \tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{u}_j$. Consequently, the computation of q_j involves the computation $B\mathbf{u}_j$, the action of C^{-1} at the continuous level (often just a multiplication operator) to find $\mathcal{C}^{-1} B \mathbf{u}_j$, and a projection onto the space $\tilde{M}_h$ - that is usually a standard finite element space. Thus, for each of the Uzawa type algorithms, a basis for $\mathcal{M}_h$ is not needed for solving the variational problems associated with q_j.

5.1. **Approximability of the space $\mathcal{M}_h$.** Due to Theorem 3.2, in order to expect small discretization error $\|p - p_h\|$, besides stability, one needs to investigate the minimization problem $\inf_{q_h \in \mathcal{M}_h} \|p - q_h\|$ in the special case when $\mathcal{M}_h$ is a proper subspace of $\tilde{\mathcal{M}}_h$.

The continuous $\inf - \sup$ condition (2.3) that we assume, guaranties that $\mathcal{A}^{-1} B^*$ is injective and has closed range, see Lemma (8.1) (iii). Thus, the dual operator $\mathcal{C}^{-1} B$ is onto Q, and for any $p \in Q$, there exists $\mathbf{u} \in \mathbf{V}$ such that $\mathcal{C}^{-1} B \mathbf{u} = p$. For $p = C^{-1} B \mathbf{u} \in Q$, and any $q_h = \tilde{\mathcal{Q}}_h \ C^{-1} B \mathbf{v}_h \in \mathcal{M}_h$, we have that

$$\begin{aligned} \|p - q_h\| &= \|\mathcal{C}^{-1} B \mathbf{u} - \tilde{\mathcal{Q}}_h \mathcal{C}^{-1} B \mathbf{v}_h\| \\ &\le \|\mathcal{C}^{-1} B \mathbf{u} - \mathcal{C}^{-1} B \mathbf{v}_h\| + \|\tilde{\mathcal{Q}}_h C^{-1} B \mathbf{v}_h - \mathcal{C}^{-1} B \mathbf{v}_h\| \\ &\le M |\mathbf{u} - \mathbf{v}_h| + \|\tilde{\mathcal{Q}}_h C^{-1} B \mathbf{v}_h - \mathcal{C}^{-1} B \mathbf{v}_h\|. \end{aligned} \tag{5.2}$$

In order to get good SPLS approximation properties for the solution p of (1.1), it would be enough to ask for some regularity of a solution $\mathbf{u}$ of $\mathcal{C}^{-1} B \mathbf{u} = p$, for approximation properties of $\mathbf{V}_h$, and for approximation properties of the projection $\tilde{\mathcal{Q}}_h : \mathcal{C}^{-1} B \mathbf{V}_h \to \tilde{\mathcal{M}}_h$. Note that the argument remains valid regardless of the discrete stability presence.

5.2. **The special projection case.** If $\tilde{\mathcal{Q}}_h\mathcal{C}^{-1}B\mathbf{v}_h = 0$ implies $\mathcal{C}^{-1}B\mathbf{v}_h = 0$, for any $\mathbf{v}_h \in \mathbf{V}_h$, (i.e., $\tilde{\mathcal{Q}}_h$ is injective on $\mathcal{C}^{-1}B\mathbf{V}_h$), then $\mathbf{V}_{h,0} \subset \mathbf{V}_0$ and, see Remark 3, the variational formulation (3.2) is well posed, has unique solution $p_h \in \mathcal{M}_h$, and using Proposition 2.2 for the discrete pair $(\mathbf{V}_h, \mathcal{M}_h)$, we have that $(\mathbf{u}_h = 0, p_h)$ is the solution of (3.3). This will always be the case if we choose $\tilde{\mathcal{M}}_h = \mathcal{C}^{-1}B\mathbf{V}_h$, and $\tilde{\mathcal{Q}}_h$ to be the identity operator on $\mathcal{C}^{-1}B\mathbf{V}_h$. In this special case we do have $\mathcal{M}_h = \tilde{\mathcal{M}}_h = \mathcal{C}^{-1}B\mathbf{V}_h$ and, if p is the solution of (1.1) and p_h is the solution of (3.2), from (1.1) and (3.2) we obtain that

$$0 = b(\mathbf{v}_h, p - p_h) = (\mathcal{C}^{-1}B\mathbf{v}_h, p - p_h), \quad \text{for all } \ \mathbf{v}_h \in \mathbf{V}_h.$$

Thus, we simply have that p_h is the orthogonal projection of p onto $\mathcal{M}_h$, and consequently, using the representation $p = C^{-1}B\mathbf{w}$ for some $\mathbf{w} \in \mathbf{V}$, we have

$$\begin{aligned} \|p - p_h\| &= \inf_{q_h \in \mathcal{M}_h} \|p - q_h\| = \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|C^{-1}B\mathbf{w} - \mathcal{C}^{-1}B\mathbf{v}_h\| \\ &\le M \inf_{\mathbf{v}_h \in \mathbf{V}_h} |\mathbf{w} - \mathbf{v}_h|. \end{aligned} \tag{5.3}$$

Compared with (3.5), the discretization error estimate (5.3) has the advantage that is independent of the stability constant m_h and, in addition, it reduces to approximability of functions in $\mathbf{V}$ by discrete functions in $\mathbf{V}_h$. If one of the Uzawa type iterative methods of Section 4 is applied to find the solution $(0, p_h)$ of (3.3) then, according to Theorem (4.3), the iteration error satifies

$$\|p_j - p_h\| \le \frac{1}{m_h^2} \|q_{j+1}\|.$$

If the discretization error order is available, say $\|p - p_h\| = O(h^\alpha)$, and an estimate about m_h is also available, the iteration error can match the discretization error by imposing the stopping criteria

$$\|q_{j+1}\| \le c_0\, m_h^2\, h^\alpha, \tag{5.4}$$

where c_0 is a constant independent of h. Consequently, using one of the Uzawa algorithms, we can approximate the solution p by an iterate $p_j \in \mathcal{C}^{-1}B\mathbf{V}_h$ up to optimal discretization error order, regardless of (discrete B-) stability presence.

From the numerical experiments we performed so far for concrete SPLS discretization problems, projecting on smooth spaces $\tilde{\mathcal{M}}_h$ could lead to better approximation of the continuous solution p, and projecting on coarser spaces $\tilde{\mathcal{M}}_H$ could lead to stability. It is worth noticing here, that if one chooses $\tilde{\mathcal{M}}_h$ such that $\|\tilde{\mathcal{Q}}_h q_h\| \ge c\|q_h\|$, for all $q_h \in \mathcal{C}^{-1}B\mathbf{V}_h$, then stability of $(\mathbf{V}_h, \mathcal{C}^{-1}B\mathbf{V}_h)$ implies stability for the pair $\left(\mathbf{V}_h, \tilde{\mathcal{Q}}_h\mathcal{C}^{-1}B\mathbf{V}_h\right)$.

A typical choice for the space $\tilde{\mathcal{M}}_h$ used to define the projection $\tilde{\mathcal{Q}}_h$ is the space of *continuous* piecewise polynomials of degree m with respect to a mesh $\mathcal{T}_h$. On $\tilde{\mathcal{M}}_h$ we can choose the standard inner product or we can consider the "lumping" inner product introduced in [4]. To define the "lumping" inner product, assume that $\{\phi_1, \phi_2, \dots, \phi_m\}$ is a nodal basis for $\tilde{M}_h$. We can define $(\cdot,\cdot)_l$ by

$$(\phi_i, \phi_j)_l := \delta_i^j \ (1, \phi_i), \quad i, j = 1, 2, \dots, m, \tag{5.5}$$

and extend it to $\tilde{M}_h \times \tilde{M}_h$ by

$$\left(\sum_{i=1}^m \alpha_i \phi_i, \sum_{j=1}^m \beta_j \phi_j \right)_l := \sum_{i=1}^m \alpha_i \beta_i \ (1, \phi_i). \tag{5.6}$$

Using the identity (5.1), see [4], the computation of $\mathcal{C}_h^{-1} B_h \mathbf{v}_h$ becomes

$$\mathcal{C}_h^{-1} B_h \mathbf{v}_h = \sum_{i=1}^m \frac{(\mathcal{C}^{-1} B \mathbf{v}_h, \phi_i)}{(1, \phi_i)} \phi_i = \sum_{i=1}^m \frac{b(\mathbf{v}_h, \phi_i)}{(1, \phi_i)} \phi_i.$$

We note here that by using the lumping inner product, one avoids mass matrix inversion at each iterative step.

All of the above provides a more detailed picture of the *five step SPLS discretization* summarized at the end of introduction. A concrete example of how to apply the method is presented next.

6. SPLS discretization of a div − curl system

Here, we apply the SPLS discretization method for a model div − curl problem on a polyhedral domain $\Omega \subset \mathbb{R}^3$. For a given data, we are looking to find the vector function $\mathbf{h} \in \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} \nabla\times \mathbf{h} \ &= \mathbf{j} \quad \text{in } \Omega \\ \nabla\cdot (\mu \mathbf{h}) \ &= g \quad \text{in } \Omega \\ (\mu \mathbf{h}) \cdot \mathbf{n} \ &= \sigma \quad \text{on } \ \Gamma := \partial\Omega, \end{aligned} \tag{6.1}$$

where μ is a given parametric scalar L^2 function that is strictly positive on Ω. The variational formulation we adopt for (6.1) is similar to the approach of [10]. We multiply the first equation in (6.1) by $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, multiply the second equation by $\varphi \in H^1(\Omega)/\mathbb{R}$ and after we integrate by parts, we obtain

$$\begin{aligned} (\mathbf{h}, \nabla\times \mathbf{w}) \ &= (\mathbf{j}, \mathbf{w}) && \text{for all} \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega) \\ (\mathbf{h}, \mu \nabla\varphi) \ &= \langle G, \varphi \rangle := (-g, \varphi) + (\sigma, \varphi)_\Gamma && \text{for all} \ \ \varphi \in H^1(\Omega)/\mathbb{R}, \end{aligned} \tag{6.2}$$

where $(\cdot,\cdot)$ denotes the standard L^2 type inner product. If we define $\mathbf{V} := \mathbf{H}_0^1(\Omega) \times H^1(\Omega)/\mathbb{R}$, $Q := \mathbf{L}^2(\Omega)$, and the form "$b(\mathbf{v}, p)$" on $(\mathbf{V}, Q)$, by

$$b((\mathbf{w}, \varphi), \mathbf{h}) := (\mathbf{h}, \nabla \times \ \mathbf{w} + \nabla\varphi), \ \text{for all } (\mathbf{w}, \varphi) \in \mathbf{V}, \ \mathbf{h} \in Q,$$

the variational formulation for (6.1) becomes: Find $\mathbf{h} \in Q$ such that

$$b((\mathbf{w}, \varphi), \mathbf{h}) = \langle F, (\mathbf{w}, \varphi) \rangle := (\mathbf{j}, \mathbf{w}) + \langle G, \varphi \rangle, \ \text{for all } (\mathbf{w}, \varphi) \in \mathbf{V}. \tag{6.3}$$

The choice of inner products on the spaces $\mathbf{V}$ and Q is essential in building *robust solvers* with respect to the function μ. By choosing the weighted

$(\cdot,\cdot)_\mu$ inner product on $Q = \mathbf{L}^2(\Omega)$ and the weighted inner product on $\mathbf{V} := \mathbf{H}_0^1(\Omega) \times H^1(\Omega)/\mathbb{R}$ induced by the norm

$$\|(\mathbf{w},\varphi)\|_{\mathbf{V}}^2 := a_{\mu^{-1}}(\mathbf{w},\mathbf{w}) + a_\mu(\varphi,\varphi) := \int_\Omega \mu^{-1}|\nabla \mathbf{w}|^2 + \int_\Omega \mu|\nabla\varphi|^2,$$

it can be proved that, in this case, the form $b(\cdot,\cdot)$ satisfies a continuous $\inf-\sup$ condition with a constant *independent of* μ, see [10]. The $\mathcal{C}^{-1}B$ operator that appears in the SPLS discretization is

$$\mathcal{C}^{-1}B\,(\mathbf{w},\varphi) = \mu^{-1}\operatorname{curl}\mathbf{w} + \nabla\varphi. \tag{6.4}$$

By using a Helmholtz decomposition, it is easy to check that $\mathcal{C}^{-1}B$ is onto $Q = \mathbf{L}^2(\Omega)$. If the data $(\mathbf{j}, g, \sigma)$ is such that the compatibility condition (2.8) is satisfied, then (6.2) is a well-posed problem and our SPLS discretization can be applied.

For *SPLS discretization*, we can chose $\mathbf{V}_h \subset \mathbf{V}$ to be the standard vector space of continuous piecewise functions of degree m with the appropriate boundary conditions for each component of $\mathbf{V}$. For $\mathcal{M}_h$ we consider two choices: Case 1) $\mathcal{M}_h := \mathcal{C}^{-1}B\mathbf{V}_h$, and Case 2) $\mathcal{M}_h := \tilde{\mathcal{Q}}_h(\mathcal{C}^{-1}B\mathbf{V}_h)$, where $\tilde{\mathcal{Q}}_h$ is the orthogonal projection onto $\tilde{\mathcal{M}}_h$- the *continuous* piecewise polynomials of degree m. On $\tilde{\mathcal{M}}_h$ we can consider the standard inner product or the "lumping" inner product defined in (5.6). In both cases, we *automatically* have a positive discrete $\inf-\sup$ constant m_h.

Sufficient conditions that can *establish stability* for this problem *remain to be investigated* for various choices of discrete spaces. For the 2D version of (6.1), it is easy to check that we do have (free) stability for both types of discretization. This is simply due to the fact that the $(\operatorname{curl},\operatorname{curl})$ form is equivalent with the Dirichlet form (∇,∇) on the space $H_0^1(\Omega)$.

In general, establishing stability for the family $\{(\mathbf{V}_h, \mathcal{M}_h = \mathcal{C}^{-1}B\mathbf{V}_h)\}$ might not be an easy problem, even if we focus on $\mathbf{V}_h$ to be a standard conforming space of piecewise polynomial of degree k. For example, if we consider the simple case of the operator B being the $\operatorname{div} : \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$ and $\mathcal{C}^{-1}$ to be just the identity operator, then the problem of discrete B-*stability* is equivalent with establishing the classical Stokes stability for $(\mathbf{V}_h, \operatorname{div}\mathbf{V}_h)$. From the works of Scott and Vogelius on *divergence stability*, [33, 34], for the case of general quasi-uniform meshes, the *divergence stability* remains a 30 year old open problem. For many problems of interest, the B-*stability* can be established based on paricular properties of the operator B. Preliminary numerical results for $\operatorname{div}-\operatorname{curl}$ systems and the Maxwell equations, suggest the presence of B-*stability* for particular choices of test spaces and projection operators.

A more detailed investigation of $B-$stability for the 3D $\operatorname{div}-\operatorname{curl}$ or Maxwell problem remains to be investigated in the near future.

6.1. **Numerical results.** We performed numerical experiments for approximating (6.1) with Uzawa Conjugate Gradient (UCG) and $\mathbf{V}_h$ the conforming P_1 elements. For $\mathcal{M}_h$ we consider both cases described above in this section. In Case 2) we project on $\tilde{\mathcal{M}}_h$ the vector space of continuous P_1 functions, with the componentwise inner product given by (5.6). The domain Ω was chosen the unit cube, and the function μ was chosen to be the restriction to Ω of

$$\mu = \begin{cases} 1, & \text{if } x < \frac{1}{2} \\ \mu_0, & \text{if } x \geq \frac{1}{2}, \end{cases}$$

for various values of the constant μ_0.

First we consider the case when $\mu_0 = 1$ (constant μ) and data chosen such that the exact solution is

$$\mathbf{h} = (x(1-x)yz,\ xy(1-y)z,\ xyz(1-z)).$$

For Case 1), using a power method for the discrete Schur complement $S_{h,0}$ we estimated m_h and imposed the stopping criterion given by (5.4) with $c_0 = 0.4$ and $\alpha = 1$. For Case 2) we just imposed a fixed number of iterations on each level. The numerical results are presented below in Table 1 and Table 2.

$\mu = 1$ / Level	$P_1 - \mathcal{C}^{-1}BP_1$			
	$\lVert h - h_{comp} \rVert$	$Order$	# of it.	m_h
1	0.0196		1	0.5632
2	0.0125	0.65	1	0.3493
3	0.0071	0.82	1	0.1739
4	0.0038	0.91	2	0.1366
5	0.0020	0.93	2	0.1106

Table 1: UCG for SPLS-P_1-discretization Case 1)

$\mu = 1$ / Level	$P_1 - \tilde{Q}_h\mathcal{C}^{-1}B(P_1)$		
	$\lVert h - h_{comp} \rVert$	$Order$	# of it.
1	0.01632		15
2	0.00505	1.67	15
3	0.00148	1.76	15
4	0.00047	1.66	15
5	0.00017	1.48	15

Table 2: UCG for SPLS-P_1-discretization Case 2)

Next, we consider the case of discontinuous μ ($\mu_0 = 50$) and data chosen such that the exact solution is

$$\mathbf{h} = (x(1-x)(\frac{1}{2}-x)yz,\ x(\frac{1}{2}-x)y(1-y)z,\ x(\frac{1}{2}-x)yz(1-z)).$$

The numerical results are presented below in Table 3 and Table 4.

For Case 1), with μ having $2D$ *checkerboard* jump discontinuities (1 on "white squares", μ_0 on the "black squares" and constant 1 in the z-direction), we obtained similar results. The estimates for the discrete $\inf - \sup$

Level \ $\mu_0 = 50$	$P_1 - \mathcal{C}^{-1}BP_1$			
	$\|h - h_{comp}\|$	*Order*	# of it.	m_h
2	0.0401		1	0.2981
3	0.0265	0.60	3	0.1517
4	0.0159	0.73	5	0.1467
5	0.0091	0.81	6	0.1208

Table 3: UCG for SPLS-P_1-discretization Case 1)

Level \ $\mu_0 = 50$	$P_1 - \tilde{\mathcal{Q}}_h\mathcal{C}^{-1}B(P_1)$		
	$\|h - h_{comp}\|$	*Order*	# of it.
2	0.0201		15
3	0.0071	1.51	15
4	0.0028	1.32	15
5	0.0013	1.15	15

Table 4: UCG for SPLS-P_1-discretization Case 2)

constant m_h are presented in Table 5. While we can notice instability with respect to h (or Level) we can also notice numerical stability with respect to μ_0.

Level \ μ_0	1	5	10	20	100
1	0.5632	0.5819	0.5823	0.582	0.5816
2	0.3096	0.3195	0.319	0.3183	0.3168
3	0.3096	0.3195	0.319	0.3183	0.3168
4	0.1361	0.1475	0.1437	0.1384	0.1500
5	0.1083	0.1182	0.1223	0.1254	0.1193

Table 5: Estimates for m_h for different values, μ_0

Remark 6.1. *We note that for Case 1), the discretization error* $\|\mathbf{h} - \mathbf{h}_h\|$ *satisfies* (5.3) *and for the conforming choice* P_1 *for* $\mathbf{V}_h$ *we have* $\|\mathbf{h} - \mathbf{h}_h\| = O(h)$. *Then, since* $\|h - h_{comp}\| \leq \|\mathbf{h} - \mathbf{h}_h\| + \|\mathbf{h}_h - \mathbf{h}_{comp}\|$, *and the iteration error* $\|\mathbf{h}_h - \mathbf{h}_{comp}\|$ *matches the discretization error (see Theorem 4.3) we also expect* $O(h)$ *for* $\|h - h_{comp}\|$. *This is numerically observed in Table 1 and Table 3.*

For Case 2), the discretization error $\|\mathbf{h} - \mathbf{h}_h\|$ *satisfies an estimate of type* (5.2). *In this case, our estimate for the discretization error is not optimal, because the functions* $\mathcal{C}^{-1}B\mathbf{v}_h$ *for* $\mathbf{v}_h \in \mathbf{V}_h$ *are piece wise constants, and the most we can expect from* $\|\tilde{\mathcal{Q}}_h C^{-1}B\mathbf{v}_h - \mathcal{C}^{-1}B\mathbf{v}_h\|$ *is* $O(h^{1/2})$. *Nevertheless, the computation we performed in this case, (see Table 2 and Table 4), show* *that the $\|h - h_{comp}\|$ is at least $O(h)$. The supper convergence behavior in this case remains to be theoretically and numerically investigated.*

7. Conclusions

We presented a *saddle point least squares* method to discretize variational formulations with different types of trial and test spaces. To the authors' knowledge, the proposed SPLS approach is different from the DPG formulations as presented in [20, 21, 19], where a trial space is chosen firstly, and a (close to optimal) test space that provides stability of the pairs is chosen secondly. In the first order systems case, the essential differences between the proposed SPLS discretization and the classical least squares finite element method approach, as described in [6, 7, 8, 16, 17] or the FOSL & FOSLL* approaches in [14, 15, 18, 26, 27, 28], are that for SPLS discretization *the test spaces and the trial spaces* are of *a different nature*. This makes *SPLS discretization* more general, but requires special attention in finding stable spaces for discretization or in finding efficient iterative solvers.

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8. Appendix

8.1. **Functional analysis results.** For a bounded linear operator $T : X \to Y$ between two Hilbert spaces X and Y, we denote by T^t the Hilbert transpose of T. If $X = Y$, we say that T is symmetric if $T = T^t$. For a bounded linear operator $T : X \to X$, we denote the spectrum of the operator T by $\sigma_0(T)$. The following lemma provides important properties of norms and operators to be used in this paper. A proof of it can be found in [2].

Lemma 8.1. *Let $\mathcal{A}, \mathcal{C}, B$, and B^* be the operators associated with the spaces $\mathbf{V}, Q$ and the connecting form $b(\cdot,\cdot)$. Assume that* (2.3) *and* (2.4) *are satisfied.*

i) **The operators $\mathcal{C}^{-1}B : \mathbf{V} \to Q$ and $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}$ are symmetric to each other**, *i.e.,*

$$(\mathcal{C}^{-1}B\mathbf{v}, q) = a_0(\mathbf{v}, \mathcal{A}^{-1}B^*q), \ \mathbf{v} \in \mathbf{V}, q \in Q, \tag{8.1}$$

consequently,

$$(\mathcal{C}^{-1}B)^t = \mathcal{A}^{-1}B^* \textit{and} \ \ (\mathcal{A}^{-1}B^*)^t = \mathcal{C}^{-1}B.$$

ii) **The Schur complement on** Q *is the operator* $S_0 := \mathcal{C}^{-1}B\mathcal{A}^{-1}B^* : Q \to Q$. *The operator S_0 is symmetric and positive definite on Q, satisfying*

$$(S_0 p, p) = \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v}, p)^2}{|\mathbf{v}|^2}. \tag{8.2}$$

Consequently, $m^2, M^2 \in \sigma_0(S_0)$ *and*

$$\sigma_0(S_0) \subset [m^2, M^2]. \tag{8.3}$$

iii) *The following estimate holds*

$$\|p\|_{S_0} := (S_0 p, p)^{1/2} = |\mathcal{A}^{-1} B^* p|_{\mathbf{V}} \geq m\|p\| \quad \text{for all} \ \ p \in Q. \tag{8.4}$$

Consequently, $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}$ *has closed range,* $\mathbf{V}_1 := \mathcal{A}^{-1}B^*(Q)$ *is a closed subspace of* $\mathbf{V}$ *and* $\mathcal{A}^{-1}B^* : Q \to \mathbf{V}_1$ *is an isomorphism.*

iv) **The Schur complement on $\mathbf{V}$** *is defined as the operator* $S := \mathcal{A}^{-1}B^*\mathcal{C}^{-1}B : \mathbf{V} \to \mathbf{V}$. *The operator* S *is symmetric and non-negative definite on* $\mathbf{V}$, *with* $Ker(S) = \mathbf{V}_0$, $S(\mathbf{V}) = \mathbf{V}_1$, *and satisfies*

$$a_0(S\mathbf{u}, \mathbf{v}) = (\mathcal{C}^{-1}B\mathbf{u}, \mathcal{C}^{-1}B\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}. \tag{8.5}$$

v) **The Schur complement on $\mathbf{V}_1 = \mathbf{V}_0^{\perp}$** *is the restriction of* S *to* $\mathbf{V}_1$, *i.e.,* $S_1 := \mathcal{A}^{-1}B^*\mathcal{C}^{-1}B : \mathbf{V}_1 \to \mathbf{V}_1$. *The operator* S_1 *is symmetric and positive definite on* $\mathbf{V}_1$, *satisfying*

$$\sigma_0(S_1) = \sigma_0(S_0) \subset [m^2, M^2]. \tag{8.6}$$

8.2. **Proof of Theorem 3.2.**

Proof. Let p be the solution of (1.1) and assume that$(\mathbf{u}_h, p_h)$ is the solution of (3.3). First, we notice that the operator $T_h : Q \to Q$ defined by $T_h p = p_h$ is linear and idempotent. To prove that T_h is idempotent we consider the problem:

Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{V}_h \times \mathcal{M}_h$ such that

$$\begin{array}{lllll} a_0(\tilde{\mathbf{u}}_h, \mathbf{v}_h) & + & b(\mathbf{v}_h, \tilde{p}_h) & = b(\mathbf{v}_h, p_h) & \text{for all } \mathbf{v}_h \in \mathbf{V}_h, \\ b(\tilde{\mathbf{u}}_h, q_h) & & & = 0 & \text{for all } q_h \in \mathcal{M}_h. \end{array} \tag{8.7}$$

Due to the assumption (3.1), we get that (8.7) has *unique* solution and by noticing that $(\tilde{\mathbf{u}}_h, \tilde{p}_h) = (\mathbf{0}, p_h)$ solves the problem, we can conclude that $T_h p_h = p_h$ and consequently, $T_h^2 = T_h$.

According to Kato [25], and Xu and Zikatanov [36], we have

$$\|I - T_h\|_{\mathcal{L}(Q,Q)} = \|T_h\|_{\mathcal{L}(Q,Q)}.$$

Next, for any $q_h \in \mathcal{M}_h$ we have

$$\begin{aligned} \|p - p_h\| = \|(I - T_h)p\| = \|(I - T_h)(p - q_h)\| &\leq \|I - T_h\|\|p - q_h\| \\ &= \|T_h\|\|p - q_h\|. \end{aligned} \tag{8.8}$$

To estimate $\|T_h\|$ we use (3.1):

$$\|T_h p\| \leq \frac{1}{m_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, T_h p)}{|\mathbf{v}_h|} = \frac{1}{m_h} \sup_{\mathbf{v}_h \in \mathbf{V}_{h,1}} \frac{b(\mathbf{v}_h, p_h)}{|\mathbf{v}_h|}.$$

Solving for $b(\mathbf{v}_h, p_h)$ from the first equation of (3.3), and using the fact that $\langle \mathbf{f}, \mathbf{v}_h \rangle = b(\mathbf{v}_h, p)$ for all $\mathbf{v}_h \in \mathbf{V}_h$, we further get

$$
\begin{aligned}
\|T_h p\| \leq & \frac{1}{m_h} \sup_{\mathbf{v}_h \in \mathbf{V}_{h,1}} \frac{b(\mathbf{v}_h, p) - a_0(\mathbf{u}_h, \mathbf{v}_h)}{|\mathbf{v}_h|} = \frac{1}{m_h} \sup_{\mathbf{v}_h \in \mathbf{V}_{h,1}} \frac{b(\mathbf{v}_h, p)}{|\mathbf{v}_h|} \\
\leq & \frac{M}{m_h} \|p\|.
\end{aligned} \tag{8.9}
$$

Thus, $\|T_h\| \leq \frac{M}{m_h}$, and from (8.8) we obtain the right side of (3.5). To prove the left side of (3.5), from (1.1) and the first equation of (3.3) we get

$$
b(\mathbf{v}_h, p - p_h) = a_0(\mathbf{u}_h, \mathbf{v}_h), \qquad \text{for all } \mathbf{v}_h \in \mathbf{V}_h.
$$

Then,

$$
|\mathbf{u}_h| = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a_0(\mathbf{u}_h, \mathbf{v}_h)}{|\mathbf{v}_h|} = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, p - p_h)}{|\mathbf{v}_h|} \leq M\|p - p_h\|,
$$

which concludes the validity of the estimate (3.5) . □

References

[1] A. Aziz and I. Babuška. Survey lectures on mathematical foundations of the finite element method. *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. Aziz, editor*, 1972.

[2] C. Bacuta. Schur complements on Hilbert spaces and saddle point systems. *J. Comput. Appl. Math.*, 225(2):581–593, 2009.

[3] C Bacuta. Cascadic multilevel algorithms for symmetric saddle point systems. *Comput. Math. Appl.*, 67(10):1905–1913, 2014.

[4] C. Bacuta, B. McCracken, and L. Shu. Residual reduction algorithms for nonsymmetric saddle point problems. *J. Comput. Appl. Math.*, 235(6):1614–1628, 2011.

[5] M. Benzi, G. Golub, and J. Liesen. Numerical solutions of saddle point problems. *Acta Numerica*, pages 1–137, 2005.

[6] P. Bochev, Z. Cai, T. A. Manteuffel, and S. F. McCormick. Analysis of velocity-flux first-order system least-squares principles for the Navier-Stokes equations. I. *SIAM J. Numer. Anal.*, 35(3):990–1009, 1998.

[7] P. Bochev and M.D. Gunzburger. Least-squares finite element methods. In *International Congress of Mathematicians. Vol. III*, pages 1137–1162. Eur. Math. Soc., Zürich, 2006.

[8] P. Bochev and M.D. Gunzburger. *Least-squares finite element methods*, volume 166 of *Applied Mathematical Sciences*. Springer, New York, 2009.

[9] D. Braess. *Finite Elements. Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, Cambridge, 1997.

[10] J. H. Bramble and J. Pasciak. A new approximation technique for div-curl systems. *Math. Comp.*, 73:1739–1762, 2004.

[11] S. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.

[12] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 8(R-2):129–151, 1974.

[13] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.

[14] Z. Cai, C.-O. Lee, T. A. Manteuffel, and S. F. McCormick. First-order system least squares for the Stokes and linear elasticity equations: further results. Iterative methods for solving systems of algebraic equations.

[15] Z. Cai, C.-O. Lee, T. A. Manteuffel, and S. F. McCormick. First-order system least squares for linear elasticity: numerical results. *SIAM J. Sci. Comput.*, 21(5):1706–1727, 2000. Iterative methods for solving systems of algebraic equations (Copper Mountain, CO, 1998).

[16] Z. Cai, T. A. Manteuffel, and S. F. McCormick. First-order system least squares for the Stokes equations, with application to linear elasticity. *SIAM J. Numer. Anal.*, 34(5):1727–1741, 1997.

[17] Z. Cai, T. A. Manteuffel, S. F. McCormick, and S. V. Parter. First-order system least squares (FOSLS) for planar linear elasticity: pure traction problem. *SIAM J. Numer. Anal.*, 35(1):320–335 (electronic), 1998.

[18] Z. Cai, T. A. Manteuffel, S. F. McCormick, and J. Ruge. First-order system LL^* (FOSLL*): scalar elliptic partial differential equations. *SIAM J. Numer. Anal.*, 39(4):1418–1445, 2001.

[19] W. Dahmen, C. Huang, C. Schwab, and G. Welper. Adaptive Petrov-Galerkin methods for first order transport equations. *SIAM J. Numer. Anal.*, 50(5):2420–2445, 2012.

[20] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. Part I: the transport equation. *Comput. Methods Appl. Mech. Engrg.*, 199(23-24):1558–1572, 2010.

[21] L. Demkowicz and J. Gopalakrishnan. A primal DPG method without a first-order reformulation. *Comput. Math. Appl.*, 66(6):1058–1064, 2013.

[22] M. Fortin and R. Glowinski. *Augmented Lagrangian Methods Applications to the numerical solutions of boundary value problems*, volume 15. North-Holland, 1983.

[23] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations*, volume 15. Springer-Verlag, Berlin, 1986.

[24] Jay Gopalakrishnan. Five lectures on dpg methods. *arXiv 1306.0557*, 2013.

[25] T. Kato. Estimation of iterated matrices, with application to the Von Neumann condition. *Numer. Math.*, 2:22–29, 1960.

[26] B. Lee, T. A. Manteuffel, S. F. McCormick, and J. Ruge. First-order system least-squares for the Helmholtz equation. *SIAM J. Sci. Comput.*, 21(5):1927–1949, 2000. Iterative methods for solving systems of algebraic equations (Copper Mountain, CO, 1998).

[27] E. Lee and T.A. Manteuffel. FOSLL* method for the eddy current problem with three-dimensional edge singularities. *SIAM J. Numer. Anal.*, 45(2):787–809, 2007.

[28] T. A. Manteuffel, S. F. McCormick, J. Ruge, and J. G. Schmidt. First-order system.

[29] P Monk. *Finite element methods for Maxwell's equations.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[30] R.H. Nochetto and J. Pyo. Optimal relaxation parameter for the Uzawa method. *Numer. Math*, 98:695–702, 2004.

[31] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Clarendon Press, Oxford, 1999.

[32] F.J. Sayas. Infimum-supremum. *Bol. Soc. Esp. Mat. Apl. S$\vec{e}$MA*, (41):19–40, 2007.

[33] L. R. Scott and M. Vogelius. Conforming finite element methods for incompressible and nearly incompressible continua. In *Large-scale computations in fluid mechanics, Part 2 (La Jolla, Calif., 1983)*, volume 22 of *Lectures in Appl. Math.*, pages 221–244. Amer. Math. Soc., Providence, RI, 1985.

[34] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO Modél. Math. Anal. Numér.*, 19(1):111–143, 1985.

[35] R. Verfürth. A combined conjugate gradient-multigrid algorithm for the numerical solution of the Stokes problem. *IMA J. Numer. Anal.*, 4(4):441–455, 1984.

[36] J. Xu and L. Zikatanov. Some observations on Babuška and Brezzi theories. *Numer. Math.*, 94(1):195–202, 2003.

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