NEW EXAMPLES OF GRAPHS WITHOUT SMALL CYCLES AND OF LARGE SIZE[1]

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Abstract: For any prime power $q \geq 3$, we consider two infinite series of bipartite q–regular edge–transitive graphs of orders $2q^3$ and $2q^5$ which are induced subgraphs of regular generalized 4–gon and 6–gon, respectively. We compare these two series with two families of graphs, $H_3(p)$ and $H_5(p)$, p is a prime, constructed recently by Wenger ([26]), which are new examples of extremal graphs without 6– and 10–cycles respectively. We prove that the first series contains the family $H_3(p)$ for $q = p \geq 3$. Then we show that no member of the second family $H_5(p)$ is a subgraph of a generalized 6–gon.
Then, for infinitely many values of q, we construct a new infinite series of bipartite q–regular edge–transitive graphs of order $2q^5$ and girth 10.
Finally, for any prime power $q \geq 3$, we construct a new infinite series of bipartite q–regular edge–transitive graphs of order $2q^9$ and girth $g \geq 14$.
Our constructions were motivated by some results on embeddings of Chevalley group geometries in the corresponding Lie algebras and a construction of a blow–up for an incident system and a graph.

INTRODUCTION.

The missing definitions of graph–theoretical concepts which appear in this paper can be found in [3]. All graphs we consider are simple, i.e. undirected without loops and multiple edges. Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. $|V(G)|$ is called the *order* of G, and $|E(G)|$ is called the *size* of G. A path in G is called *simple* if all its vertices are distinct. When it is convenient, we shall identify G with the corresponding antireflexive symmetric binary relation on $V(G)$, i.e. $E(G) \subset V(G) \times V(G)$. The *length* of a path is the number of its edges. The group of all automorphisms of graph G will be denoted by $Aut(G)$. The *girth* of a graph G, denoted by $g = g(G)$, is the length of the shortest cycle in G.

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Examples of graphs with large girth which satisfy certain additional conditions are known to be hard to construct, and they turn out to be useful in various problems in extremal graph theory, in studies of graphs with high degree of symmetry, and in designs of communication networks. There are many references on each of these topics. Here we mention just a few main books and survey papers which also contain extensive bibliographies. On the extremal graph theory: [3,4,11,16]; on graphs with high degree of symmetry: [6,12,13,15, 16,17,19,25]; on communication networks: [2,9].

Let $\mathcal{F}$ be a family of graphs. By $ex(v, \mathcal{F})$ we denote the greatest number of edges in a graph on v vertices which contains no subgraph isomorphic to a graph from $\mathcal{F}$. Let C_m denote the cycle of length $m \geq 3$. According to a well known unpublished result of Erdös (The Even Circuit Theorem), see [16], $ex(v, C_{2k}) = O(v^{1+1/k})$ (for a generalization of this result see [4,11]). This upper bound is known to be sharp for C_4, C_6 and C_{10}. The corresponding construction for C_4 can be found in [7,10,16]. The constructions for C_6 and C_{10} (see [1,16]) are incidence graphs for generalized n-gons, $n = 4, 6$ (geometries of the Chevalley groups $B_2(q)$ and $G_2(q)$). Recently new important examples of graphs with no 6– or 10–cycles were found by Wenger in [26], where they are denoted by $H_3(p)$ and $H_5(p)$ respectively, p is a prime number. These graphs are members of a family $\{H_i(p),\ i \geq 1\}$, of regular bipartite graphs whose vertex sets are disjoint unions of two i–dimensional vector spaces over the prime field $\mathbb{F}_p$, and whose edges are defined by certain systems of equations.

The content of this paper is outlined below.

(i) In Section 1 we present a construction of a blow–up of a graph which is used in subsequent sections.

(ii) In Section 2 we consider a connection between Wenger graphs and generalized n-gons. Let I be the incidence relation and $\{p, l\}$ be a flag of the regular generalized n–gon, $n = 4, 6$, over $\mathbb{F}_q$, $q = p^m$. We consider graphs $S_n(q)$ obtained by restricting I on the set $P_n \cup L_n$, where P_n (L_n) is the set of points (lines) opposite to p (l) in the n-gon. Coordinatizations of the generalized n–gons (see [20], [21]) allow to identify each P_n and L_n with a vector space $\mathbb{F}_q^{n-1}$, and the incidence of vectors from P_n and L_n can be expressed in terms of systems of equations on their coordinates. If $n = 4$, this system coincides with the one for $H_3(p)$. Therefore $S_4(q)$ is a simple generalization of $H_3(p)$. On the other hand, if $n = 6$ and $q = p > 2$, graphs $S_6(p)$ and $H_5(p)$ are not isomorphic: graph $H_5(p)$ contains an 8–cycle, hence it cannot be isomorphic to a subgraph of a generalized 6–gon.

(iii) We construct an infinite series of regular bipartite edge–transitive graphs of girth 10. Having girth 10, they cannot be isomorphic to subgraphs of the generalized 6–gons, but they have asymptotically as many edges as regular generalized 6–gons. (see Section 3).

(iv) It is known that $ex(v, \{C_3, C_4, \ldots, C_m\}) \geq c_m v^{1+\frac{1}{m-1}}$ for some positive constant c_m , $m \geq 3$. This result follows from a theorem proved implicitly by Erdös (see [16]) and the proof is nonconstructive. As it was mentioned in [16], it is unlikely that this lower bound is sharp. In Section 4, for any prime power $q \geq 3$, we construct a q–regular bipartite graph $G(q)$ of order $v = 2q^9$, size $e = q^{10}$ and girth ≥ 14, which supports this claim. (See also [24,27].) For these graphs $e \sim 2^{\frac{10}{9}} v^{1+\frac{1}{9}}$, which is better than the best previously known lower bound $c_{13}v^{1+\frac{1}{12}}$. Graph $G(q)$ is also edge–transitive.

(v) To our knowledge, all graphs presented in (ii)–(iv) have greater edge density than known extremal graphs, where the *density* (or *edge density*) of a graph of order v and size e is defined as $e/\binom{v}{2}$. All of them are edge–transitive. We also describe each of these graphs (including $H_i(q)$'s) in group theoretical terms.

Our constructions were motivated by some results on embeddings of Chevalley group geometries in the corresponding Lie algebras [21,22], and a construction of a blow–up for an incidence system and a graph [20,23].

SECTION 1. A blow–up of a graph.

For a positive integer $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ denote the set of all subsets of $[n]$. Let $\mathcal{L}$ be an n–dimensional vector space over some field K with a fixed basis $\{e_i \mid i \in [n]\}$. For an arbitrary subset A of $[n]$, let $\mathcal{L}_A$ denote the subspace of $\mathcal{L}$ spanned by $\{e_i \mid i \in A\}$. By $x|_A$ we denote the canonical projection of a vector $x \in \mathcal{L}$ on $\mathcal{L}_A$. Let G be a graph, and let $\eta : V(G) \to 2^{[n]}$ be a mapping of the set of vertices of G into $2^{[n]}$. Finally, let $*$ denote a skew–symmetric bilinear product on $\mathcal{L}$. Consider a new graph $\widetilde{G}$ with the vertex set $\widetilde{V}$ defined as

$$\widetilde{V} = \{(a, x) | a \in V(G), x \in \mathcal{L}_{\eta(a)}\}.$$

We define two distinct vertices (a, x) and (b, y) of $\widetilde{G}$ to be adjacent if and only if

$$\{a, b\} \in E(G) \quad \text{and} \quad x^{h_b} - x^{h_a}|_{\eta(a)\cap\eta(b)} = x * y|_{\eta(a)\cap\eta(b)} \ ,$$

where, for $a \in V(G)$, $h_a : e_i \mapsto \lambda_a(i)e_i,\ i \in [n]$, is a nonsingular diagonal operator of $\mathcal{L}$ (defined by its action on the vectors from the basis). We call graph $\widetilde{G}$ a *blow–up* of G.

In [20,21], Ustimenko showed that the incidence relation of the geometry $\gamma(G)$ of a Chevalley group G is a blow–up of the geometry $\gamma(W)$ of its Weyl group W. In this case the vector space $(\mathcal{L}, *)$ is the Lie algebra $\mathcal{L}^+ = \sum_{\alpha \in \Phi^+} \mathcal{L}_\alpha$, where Φ^+ is the set of positive roots for W and $\mathcal{L}_\alpha$ is a root subalgebra of $\mathcal{L}$. The basis vectors $e_\alpha, \alpha \in \Phi^+$, are elements of the Chevalley basis. In particular, each regular generalized n–gon , $n = 3, 4, 6$, corresponding to a Chevalley group of rank 2 of normal type is a blow–up of the incidence graph of the ordinary n–gon (geometry of the dihedral group D_{2n}), which is the cycle C_{2n}. This illustrates that by "blowing up" (over $\mathbb{F}_q$) a small bipartite graph one can obtain a graph of high girth and of large size.

All graphs in this paper are blow–ups of $K_{1,1}$ over finite fields, where $K_{1,1}$ is a graph with two vertices joined by an edge. The bilinear product on $\mathcal{L}$ is defined on the basis elements as $e_i * e_j = \lambda_k \cdot e_k$, where k depends on i and j. For every point p and line l, $|\eta(p)| = |\eta(l)| = n - 1$. Let us assume that $\eta(p) = [n] \setminus \{2\}$ and $\eta(l) = [n] \setminus \{1\}$. Then the set of vertices of the bipartite graph $\widetilde{K}_{1,1}$ can be thought a disjoint union of sets P (set of points) and L (set of lines) of the form $P = \{(x_1, x_3, \ldots, x_n) \mid x_i \in \mathbb{F}_q\}$, $L = \{[y_2, y_3, \ldots, y_n)] \mid y_i \in \mathbb{F}_q\}$.

All graphs in this paper have a group theoretical interpretation as follows. For every $i \in [n]$ and $x \in \mathbb{F}_q$, there exists an automorphism $t_i(x)$ of the graph which acts on the coordinates of vectors of $P \cup L$ by the rule: $x_j \to P_i^j(x, x_1, x_3, \ldots, x_n)$, $y_j \to L_i^j(y, y_2, y_3, \ldots, y_n)$, where P_i^j and L_i^j are polynomials over $\mathbb{F}_q$. The automorphisms

$t_i(x)$ satisfy the following properties:

(a) $t_i(x) \cdot t_i(y) = t_i(x+y)$, and so they are the "generalized exponents", and the group $U_i = \langle t_i(x) \mid x \in \mathbb{F}_q \rangle$ is isomorphic to the additive group of $\mathbb{F}_q$.

(b) Group U generated by all $t_i(x)$ is nilpotent and of order q^{n+1} ("generalized unipotent subgroup")

(c) Graph $\widetilde{K}_{1,1}$ is isomorphic to the incidence graph of the following incident structure: sets P and L are the sets of cosets of U with respect to subgroups U_1 and U_2 respectively, with two cosets (one from P, another from L) being incident if and only if their intersection is nonempty.

SECTION 2. Extremal regular induced subgraphs of generalized 4–and 6–gons.

The *incidence structure* (P, L, I) is a triple where P and L are two disjoint sets (set of *points* and set of *lines*, respectively), and I is a symmetric binary relation on $P \cup L$ (*incidence relation*). As is usually done, we impose the following restrictions on I : two points (lines) are incident if and only if they coincide. Let $B = B((P, L, I))$ be a bipartite graph such that $V(B) = P \cup L$ and $E(B) = \{\{p, l\} \mid pIl,\ p \in P,\ l \in L\}$. We notice that, according to our definition, B is a simple bipartite graph. We call B the *incidence* graph for the incidence structure (P, L, I).

Let P and L be the sets of vertices and sides of an ordinary n–gon, and I be the natural relation of incidence of a vertex and a side. It is easy to see that the incidence graph of this incidence structure is the cycle C_{2n}. Tits [18] introduced the following definition of a *generalized n–gon* as an incidence structure satisfying the following properties:

(i) for any two distinct elements a and b from $P \cup L$ there exists a positive integer s, $s \leq n$, and a sequence $x_0, x_1, \ldots, x_s$ of distinct elements of $P \cup L$ where $x_0 = a$, $x_s = b$, and $x_i I x_{i+1}$ for $i = 0, \ldots, s-1$.

(ii) if $s < n$, then the sequence defined in (i) is unique.

Of course, the ordinary ("geometrical") n–gon is a generalized n–gon, and the girth of the incidence graph of a generalized n–gon is $2n$. It is known ([12]), that apart from the ordinary polygons, finite generalized n–gons exist only for $n = 3, 4, 6, 8, 12$.

Some other examples of generalized n–gons for $n = 3, 4, 6$ are closely connected to Chevalley groups $A_2(q), B_2(q), G_2(q)$ of rank 2 over the finite field $\mathbb{F}_q$ (see [8]).

Let G be a Chevalley group of rank 2 over the field $\mathbb{F}_q$, $q = p^m$, p is prime, $m \geq 1$. Then a Borel subgroup of G is the normalizer in G of a Sylow p–subgroup of G. There are exactly two maximal subgroups P_1 and P_2 of G which contain a fixed Borel subgroup B (see [8]). Let us consider the incidence structure (P, L, I), where P is $(G : P_1)$ – the totality of all left cosets of G by P_1, L is $(G : P_2)$, and elements a and b of $P \cup L$ are incident if and only if the intersection of a and b as cosets of G is nonempty. It can be shown, e.g. see [19], that this incidence structure is a generalized n–gon. The corresponding bipartite incidence graph, which we denote by $B_n(q)$, is $(q+1)$–regular.

Let us consider the orbits of the Borel subgroup B on the sets P and L for our generalized n–gons. The cardinalities of orbits on the set of points and the set of lines are the same and equal $1, q, q^2, \dots q^{n-1}$ (see [8]). Let $S(P)$ and $S(L)$ be the orbits of largest size q^{n-1} on P and L respectively, and $S_n(q)$ be the subgraph of $B_n(q)$ induced on the set $S(P) \cup S(L)$. The importance of graphs $S_n(q)$ in extremal graph theory stems from the fact that they are of girth $2n$ and of size $O(v^{1+\frac{1}{n-1}})$.

Theorem 2.1. *For $n = 4, 6$, graph $S_n(q)$, satisfies the following properties:*

(a) $S_n(q)$ is q–regular of order $2q^{n-1}$ and size q^n

(b) $S_n(q)$ is a graph of girth $2n$.

(c) $S_n(q)$ is edge–transitive

(d) For $q = 2^k, k \geq 1$, $S_4(q)$ is vertex–transitive. For $q = 3^k, k \geq 1$, $S_6(q)$ is vertex–transitive.

Proof. We base our proof on well known properties of the Chevalley groups $B_2(q)$ and $G_2(q)$ (see [8]). The unipotent subgroup U (a fixed Sylow p–subgroup of the group G) acts regularly on the set of edges of $S_n(q)$. Hence, (c) is true. Let $U(p)$ and $U(l)$ be the stabilizers (in U) of point p and line l, respectively. Since their order is q, we have proved part (a). Since $S_n(q)$ is a subgraph of $B_n(q)$ and $g(B_n(q)) = 2n$, then $g(S_n(q)) \geq 2n$. If $g(S_n(q)) > 2n$, then $S_n(q)$ would have more edges than permitted by the Even Circuit Theorem (see Introduction). This proves (b). Under the conditions of (d), there exists an inner automorphism ω of the Chevalley group G, which corresponds to the symmetry of Dynkin diagram. This ω acts on $\gamma(G)$ and interchanges the maximal standard parabolic subgroups. It induces an automorphism $\tilde{\omega}$ of $S_n(q)$ which interchanges the partition sets of this bipartite graph (the set of points and and the set of lines). Therefore the group generated by U and $\tilde{\omega}$ acts transitively on the set of vertices of $S(q)$, and part (d) is proved. ■

Let G be a Chevalley group of normal type corresponding to the Lie algebra $\mathcal{L} = H \oplus \mathcal{L}^+ \oplus \mathcal{L}^-$, where H is the Cartan algebra and $\mathcal{L}^+$ ($\mathcal{L}^-$) is the direct sum of root subalgebras, corresponding to positive (negative) roots. The incidence graph $I(G)$ of the geometry $\gamma(G)$ of group G is a blow–up of the incidence graph $I(W)$ of the geometry of its Weyl group W (see [20,21]). In this case the blow–up $\widetilde{I(W)}$ was constructed by using the Lie algebra $\mathcal{L}^+$ and a fixed Chevalley basis for it.

We restrict our attention to Chevalley groups of rank two of normal type. In this case we obtain a convenient description of graphs $S_n(q)$.

For each $b \in \mathcal{L}$, a linear transformation $ad(b) : x \mapsto [b, x]$ is a nilpotent operator of $\mathcal{L}$. Let $v = ad(te_\alpha)$, where e_α is an element of the Chevalley basis from the root space corresponding to root α, and $t \in I\!F_q$. Let $x_\alpha(t) = 1 + v/1! + v^2/2! + v^3/3! + \ldots$. Then $x_\alpha(t + t') = x_\alpha(t)x_\alpha(t')$, and G is generated by all $x_\alpha(t)$, $\alpha \in \mathcal{L}^+$, $t \in I\!F_q$. For a fixed positive root α, let U_α be a group generated by all $x_\alpha(t)$, $t \in I\!F_q$.

Proposition 2.2. *For $n = 3, 4, 6$, graph $S_n(q)$ is isomorphic to the incidence graph of the group incidence structure $\gamma = \gamma(U, \{U_{\alpha_1}, U_{\alpha_2}\})$.*

Proof. First observe that the orbits of B on sets P and L coincide with those of U. Let cosets aP_1, bP_2 be elements of $S(P)$, $S(L)$, respectively. Then U_{α_1}, U_{α_2} are the respective stabilizers in U of aP_1 and bP_2. Thus the actions of U on $(U : U_{\alpha_1}), (U : U_{\alpha_2})$ are similar to those on $S(P)$, $S(L)$, so that we may identify $S(P) \cup S(L)$ with $(U : U_{\alpha_1}) \cup (U : U_{\alpha_2})$ (as vertices of $S_n(q)$). Since the incidence is preserved under this identification, γ identifies to a spanning subgraph of $S_n(q)$. Since U acts transitively on the edge set of each of γ, $S_n(q)$, it follows that the two graphs are isomorphic. ∎

Let $M_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $M_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, $M_3 = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$. (This is a complete list of the so–called 2×2 Cartan matrices.) In what follows A will represent a matrix from this list. We can consider a lattice $\mathcal{H}$ with basis $\{\alpha_1, \alpha_2\}$, i.e. the set $\{\lambda_1\alpha_1 + \lambda_2\alpha_2 \mid \lambda_1, \lambda_2 \in Z\}$. For an arbitrary 2×2 integer matrix $A = (a_{ij})$, we consider two linear transformations r_1, r_2 of $\mathcal{H}$, where $(\alpha_j)^{r_i} = \alpha_j - a_{ij}\alpha_i, i, j \in \{1, 2\}$. It is easy to check that, if $A = M_k, k = 1, 2, 3$, then $r_i^2 = e, i = 1, 2$ and $(r_1 r_2)^m = e$ for $m = 3$ (if $k = 1$), $m = 4$ (if $k = 2$), $m = 6$ (if $k = 3$), and these conditions are generic relations for a group $W = W(A) = \langle r_1, r_2 \rangle$, i.e., $W(A)$ is isomorphic to the dihedral group D_m. $W(A)$ is usually called the *Weyl group* corresponding to the 2×2 Cartan matrix A. (For more on this, see [5].) The set $\Phi(A) = \{\alpha_i^g \mid g \in W, i = 1, 2\}$ is usually called a *root system*. The set $\Phi(A)$ is a disjoint union of sets $\Phi^+(A)$ and $\Phi^-(A)$, where $\Phi^+(A) = \Phi(A) \cap \{\lambda_1\alpha_1 + \lambda_2\alpha_2 \mid \lambda_i \geq 0, i = 1, 2\}$ (elements of $\Phi^+(A)$ are called *positive roots*) and $\Phi^-(A) = \Phi(A) \cap \{-x \mid x \in \Phi^+(A)\}$ (*negative roots*). We have $\Phi^+(M_1) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \Phi^+(M_2) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}, \Phi^+(M_3) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. Let α_i^*, $i = 1, 2$, be the linear functional on $\mathcal{H}$ such that $\alpha_i^*(\alpha_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. We can consider the dual lattice $\mathcal{H}^* = \{\lambda_1\alpha_1^* + \lambda_2\alpha_2^* \mid \lambda_1, \lambda_2 \in Z\}$. For a given $i, i = 1, 2, 3$, the group $W(A)$ acts on $\mathcal{H}^*$ by the following rule: for a linear functional $l \in \mathcal{H}^*$ and $g \in W$, $l \mapsto l^g$, where $l^g(x) = l(x^{g^{-1}})$ for all $x \in \mathcal{H}$.

Let $H_1(A) = \{(\alpha_1^*)^g \mid g \in W(A)\}$ and $H_2(A) = \{(\alpha_2^*)^g \mid g \in W(A)\}$. We shall say that two functionals $l_1 \in H_1(A)$ and $l_2 \in H_2(A)$ are incident and write $l_1 J l_2$ if and only if for every x from $\Phi(A)$, $l_1(x) \cdot l_2(x) \geq 0$.

Proposition 2.3. *The incidence structure $H(M_k) = (H_1(M_k), H_2(M_k), J)$ is iso-*

morphic to the ordinary m_k–gon, $m_1 = 3, m_2 = 4, m_3 = 6$.

Proof. It is easy to check that

$$H_1(M_1) = \{\alpha_1^*, -\alpha_1^* + \alpha_2^*, -\alpha_1^*\}$$

$$H_2(M_1) = \{\alpha_2^*, -\alpha_2^* + \alpha_1^*, -\alpha_2^*\}$$

$$H_1(M_2) = \{\alpha_1^*, -\alpha_1^* + 2\alpha_2^*, \alpha_1^* - 2\alpha_2^*, -\alpha_1^*\}$$

$$H_2(M_2) = \{\alpha_2^*, \alpha_1^* - \alpha_2^*, -\alpha_1^* + \alpha_2^*, -\alpha_2^*\}$$

$$H_1(M_3) = \{\alpha_1^*, \alpha_1^* - \alpha_2^*, 2\alpha_1^* - \alpha_2^*, -2\alpha_1^* + \alpha_2^*, -\alpha_1^* + \alpha_2^*, -\alpha_1^*\}$$

$$H_2(M_3) = \{\alpha_2^*, 3\alpha_1^* - \alpha_2^*, -3\alpha_1^* + 2\alpha_2^*, 3\alpha_1^* - 2\alpha_2^*, -3\alpha_1^* + \alpha_2^*, -\alpha_2^*\}$$

and the bipartite incidence graph for $H(M_k),\ k = 1, 2, 3$, is the cycle C_{2m_k}. ■

Remark. More general propositions for $n \times n$ Cartan matrices of simple finite dimensional and affine algebras are considered in [21] and [22], respectively.

Let ζ be the vector space of formal linear combinations $t_1\alpha_1^* + t_2\alpha_2^*$, where $t_1, t_2 \in I\!F_q$, let $L = L(A)$ be the set of all linear combinations of the form $\sum_{\alpha\in\Phi^+(A)} t_\alpha e_\alpha$, $t_\alpha \in I\!F_q$, and let $\zeta \oplus L$ be the direct sum of ζ and L.

We define a bilinear product [,] on $\zeta \oplus L$ by its values on elements of the basis in the following way:

$$\begin{aligned} &[\alpha_i^*, \alpha_2^*] = 0, \quad i = 1, 2 \\ &[\alpha_i^*, e_\beta] = \alpha_i^*(\beta) \cdot e_\beta\ , \ \ \beta \in \Phi^+(A), \quad i = 1, 2 \\ &[e_\alpha, e_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi^+(A) \\ (r+1)e_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi^+(A), \end{cases} \end{aligned} \tag{2.1}$$

where r is an integer uniquely determined by the condition $\beta - r\alpha \in \Phi(A)$, $\beta - (r+1)\alpha \notin \Phi(A)$.

It is known (see [8]) that $(\zeta \oplus L, [\ ,\ ])$ is isomorphic to the Borel subalgebra of the Lie algebra for G $(G = A_2(q), B_2(q), G_2(q))$, which is, by definition, the direct sum of the Cartan subalgebra with the sum of root spaces which correspond to positive roots.

Let us denote by L_α the totality of vectors in L of the form $\lambda\alpha, \lambda \in \mathbb{F}_q$. For an integer a we denote by $\bar{a}$ the residue of $a(\text{mod } p)$. We shall write $\bar{l} = \Sigma\overline{\lambda_i}\alpha_i^*$, where $l = \Sigma\lambda_i\alpha_i^*$ is an element of $\mathcal{H}^*$.

Let $l \in \mathcal{H}^*$ and $\eta(l) = \{\alpha \in \Phi^+ \mid l(\alpha) < 0\}$. We shall consider an incidence structure $\zeta(A,q)$ with the set of points and lines $\zeta_i(A,q) = \{(l,y) \mid l = l(x) \in H_i(A),\ y \in \sum_{\alpha\in\eta(l)} L_\alpha\}$, $i = 1, 2$, and the following incidence relation $\widetilde{J}$:

$$(l(x), y)\widetilde{J}(t(x), z) \Longleftrightarrow lJt \text{ and } [\bar{l} + y, \bar{t} + z] = 0$$

It has been shown in [22,23] that the incidence structure of $\zeta(A,q)$ for $q = p^s,\ p > 3$, and $A \in \{M_1, M_2, M_3\}$ is isomorphic to the generalized n–gon arising from the Chevalley group $A_2(q)$, $B_2(q)$, $G_2(q)$, $n = 3, 4, 6$, respectively. If $A = M_2$, this statement is also valid for $p = 3$.

A mapping $\phi: (P, L, I) \to (P', L', I')$ is called a *morphism* from an incidence system (P, L, I) to an incidence system (P', L', I') if $\phi(P) \subset P'$, $\phi(L) \subset L'$, and pIl implies $\phi(p)I'\phi(l)$.

The following proposition follows immediately from definitions and the above.

Proposition 2.4. *A mapping* $r : \zeta(M_k, q) \to H(M_k)$, $k = 1, 2, 3$, *defined by* $r((h, x)) = h$, *is a morphism of the generalized* m_k*–gon onto the ordinary* m_k*–gon,* $m_k \in \{3, 4, 6\}$. ■

If the characteristic of $\mathbb{F}_q$ is greater than 3, we can identify our graph $S_t(q),\ t = 4, 6$, with the restrictions of the incidence graph for $\zeta(M_k, q), k = 2, 3$, on $r^{-1}(-\alpha_1^*) \cup r^{-1}(-\alpha_2^*)$. It is easy to see that $r^{-1}(-\alpha_i^*) = \{(-\alpha_i^*, x) \mid x \in \sum_{\alpha \in \eta(-\alpha_i^*)} L_\alpha\},\ i = 1, 2$. Therefore $S_t(q)$ is the blow–up of $K_{1,1}$ with vertices $-\alpha_1^*$ and $-\alpha_2^*$.

Let $n = 4, 6$, $P_n = \{(x_1, x_2, \ldots, x_{n-1}) \mid x_i \in \mathbb{F}_q\}$, $L_n = \{[y_1, y_2, \ldots, y_{n-1}] \mid y_i \in \mathbb{F}_q\}$. We define an incidence relation I_n (between P_n and L_n) as: $(a, b, c)I_4[x, y, z]$ if and only if

$$\begin{cases} y - b = xa \\ z - 2c = -2xb, \end{cases}$$

and $(a, b, c, d, f)I_6[x, y, z, u, w]$ if and only if

$$\begin{cases} y - b = xa \\ z - 2c = -2xb \\ u - 3d = -3xc \\ 2w - 3f = 3zb - 3yc + ua. \end{cases}$$

Theorem 2.5. *Let $q = p^m$, p is an odd positive prime. For $p \geq 3$, graph $S_4(q)$, and for $p \geq 5$, graph $S_6(q)$, are isomorphic to the incidence graphs of the incidence structure (P_4, L_4, I_4) and (P_6, L_6, I_6) respectively.* ■

In order to prove Theorem 2.5, it is sufficient to represent elements of P and L by means of their coordinate vectors, and to choose coefficients in (2.1) in a certain way. Our choice is the following:

$$[e_{\alpha_1}, e_{\alpha_2}] = -e_{\alpha_1+\alpha_2},\ \ [e_{\alpha_1}, e_{\alpha_1+\alpha_2}] = -2e_{2\alpha_1+\alpha_2}$$

$$[e_{\alpha_1}, e_{2\alpha_1+\alpha_2}] = 3e_{3\alpha_1+\alpha_2},\ \ [e_{\alpha_2}, e_{3\alpha_1+\alpha_2}] = e_{3\alpha_1+2\alpha_2}$$

$$[e_{\alpha_1+\alpha_2}, e_{2\alpha_1+\alpha_2}] = 3e_{3\alpha_1+2\alpha_2}.$$

Let ϕ be the mapping of $S_6(q)$ to $S_4(q)$ induced by the canonical projections of vector spaces P_6 and L_6 on the first three coordinates. As an immediate corollary of Theorem 2.5, we get

Proposition 2.6. *ϕ is a morphism of the incidence systems.* ■

Remark 1. $S_3(q)$ is the incidence graph of the following incidence system: $P_3 = \{(x_1, x_2)\ |x_i \in I\!\!F_q\}$, $L_3 = \{[y_1, y_2]\ |y_i \in I\!\!F_q\}$, and $(x_1, x_2)I_3[y_1, y_2]$ if and only if $y_2 - x_2 = y_1 x_1$ (affine plane).

Remark 2. Under the assumptions of Theorem 2.5, the operator $x_\alpha(t)$ from U preserves the set of vertices of $S_n(q)$, $n = 3, 4, 6$, and its restriction on this set is an automorphism of $S_n(q)$.

Let $H_n(q)$ be a blow–up of $K_{1,1}$ in the case when

(a) $(\mathcal{L}, \ast\ )$ is an n–dimensional algebra over $I\!\!F_q$ with a basis $\{e_1, e_2, \ldots, e_n\}$ and a multiplciation $\ast$ satisfying

$$\begin{aligned} & e_i \ \ast \ e_1 = e_{i+1}, \quad e_1 \ast e_i = -e_{i+1}, \ i = 2, \ldots, n-1 \\ & (1 \notin \{i, j\}) \Rightarrow e_i \ast e_j = 0 \end{aligned} \tag{2.2}$$

(b) $\eta(p) = \{2, 3, \ldots, n\}$ and $\eta(l) = \{1, 3, \ldots, n\}$ for every point (p) and every line $[l]$.

Let $P_h(n) = P_h$ and $L_h(n) = L_h$ be the sets of points and lines of $H_n(q)$. It is easy to see, that point $(p) = (p_2, p_3, \ldots, p_n)$ and line $[l] = [l_1, l_3, \ldots, l_n]$ are incident if and only if the following conditions are satisfied

$$\begin{aligned} & l_3 - p_3 = l_1 \cdot p_2 \\ & l_4 - p_4 = l_1 \cdot p_3 \\ & \ldots\ldots\ldots\ldots\ldots \\ & l_n - p_n = l_1 \cdot p_{n-1} \end{aligned} \tag{2.3}$$

Let us consider nondegenerate polynomial (biregular in terms of algebraic geometry) transformation $t_i(x)$ of disjoint union of vector spaces P_h and L_h which is defined by the rules:

$(a_1, a_3, \ldots, a_n)^{t_2(x)} = (a_1, a_3 + a_1 \cdot x, a_4, \ldots, a_n)$

$[b_2, b_3, \ldots, b_n]^{t_2(x)} = [b_2 + x, b_3, \ldots, b_n]$

$(a_1, \ldots, a_k, \ldots, a_n)^{t_k(x)} = (a_1, \ldots, a_{k-1}, a_k + x, a_{k+1} + a_1 \cdot x, \ldots, a_n),\ k > 2$

$[b_2, \ldots, b_k, \ldots, b_n]^{t_k(x)} = [b_1, \ldots, b_{k-1}, b_k + x, b_{k+1}, \ldots, b_n],\ k > 2$

$(a_1, \ldots, a_n)^{t_1(x)} = (a_1 + x, a_3, \ldots, a_k - c^1_{k-3} \cdot a_{k-1} x + \ldots + c^i_{k-3} \cdot a_{k-i} \cdot x^i \cdot (-1)^i + \ldots)$

$[b_2, b_3, \ldots, b_n]^{t_1(x)} = [b_2, b_3 - b_2 \cdot x, \ldots, b_k - c^1_{k-2} \cdot b_{k-1} \cdot x + \ldots c^i_{k-2} \cdot b_{k-i} \cdot x^i \cdot (-1)^i + \ldots]$

A straightforward computation gives us the following

Proposition 2.7. *For all $i \in [n]$, and all $x, y \in I\!F_q$, $t_i(x) \in Aut(H_n(q))$, and $t_i(x + y) = t_i(x) \cdot t_i(y)$.* ■

Let $UH^n(q)$ be a group, generated by all $t_i(x)$, $i \in [n]$, $x \in I\!F_q$. Let UH_i denote the subgroup $\langle t_i(x) \mid x \in I\!F_q \rangle$.

Analogously to the case of $S_4(q)$ we obtain

Proposition 2.8. *Incidence structure $H_n(q)$ is isomorphic to group incidence structure $\gamma(UH^n(q), \{UH_1, UH_2\})$.* ■

Corollary. *Graph $H_n(q)$ is edge–transitive.*

Theorem 2.9. *Graph $S_4(q)$ is isomorphic to $H_3(q)$.*

Proof. There exists an isomorphism φ of the unipotent subgroup U of $B_2(q)$ into $UH^3(q)$ such that $\varphi(x_{\alpha_1}(t)) = t_1(x)$ and $\varphi(x_{\alpha_2}(t)) = t_2(x)$. This isomorphism induces the bijective morphism of $\gamma(U, \{U_1, U_2\})$, on $\gamma(UH^n(q), \{UH_1, UH_2\})$. ■

Theorem 2.10. *For each $n \geq 3$, $g(H_n(q)) = 8$.*

Proof. Graph $H_n(q)$ contains no odd cycles, since it is bipartite. Incidence structure $H_n(q)$ is a semiplane. So $H_n(q)$ does not contain C_4. Let φ be a morphism of $H_n(q)$ on $H_3(q)$ corresponding to the projection of $P_h(n) \cup L_h(n)$ on first 3 coordinates. The existence of a cycle of length 6 means the existence of point p and line l, connected by two interior vertex disjoint paths of the form $pIxIyIl$. Let us point out that vectors x and y are defined uniquely by their first coordinates. So morphism φ induces an injection between simple paths of length 3 with endpoints at p and l, and paths of the same length between $\varphi(p)$ and $\varphi(l)$. But according to Theorem 2.1, there is no more than one such a path between $\varphi(p)$ and $\varphi(l)$. The obtained contradiction proves that $H_n(q)$ has no C_6. Finally we consider the path $\delta = ((0, \ldots, 0)I[0, \ldots, 0]I(1, 0, \ldots, 0)I[1, -1, 0, \ldots, 0]I(0, -1, 0, \ldots, 0))$. Let us consider the transformation $t = t_2(1)$, which fixes the first and last vertices of δ and moves all others. Then paths δ and δ^t define a C_8. ■

Wenger [26] proved that $H_5(p)$ contains no C_{10}. His proof can be easily modified to obtain that $H_n(q)$, $q = p^m$, contains no C_{10}. This result, together with Theorem 2.10, implies

Proposition 2.11. *Graph $H_5(q)$ of order $2q^5$ and size q^6 contains no C_{10} and is not isomorphic to $S_6(q)$.* ■

In fact, graph $H_5(q)$ does not contain C_{10} and, having girth 8, cannot be embedded into a generalized 6–gon. Other examples of "magnitude extremal" bipartite graphs of girth at most $2k-2$, but containing no C_{2k}, can be found in [14].

SECTION 3. Construction of graphs of order $2q^5$, size q^6, and girth 10 which is not based on a classical root system.

As we have mentioned, it was shown in [12] that there are no generalized 5–gons whose vertices have degree ≥ 3. This makes the construction of this section different from the one in Section 2.

Let P and L be two 5–dimensional vector spaces over the finite field $\mathbb{F}_q$. We assume that a basis in each of these spaces is chosen. Then the vectors of P and L can be thought as ordered 5–tuples of elements from $\mathbb{F}_q$. We define an incidence structure with point set P and line set L. It will be convenient for us to denote vectors from P as $x = (x) = (x_1, x_2, x_3, x_4, x_5)$ and vectors from L as $y = [y] = [y_1, y_2, y_3, y_4, y_5]$. The parentheses and brackets will allow us to distinguish vectors of different types (points and lines). We say that point $p = (p_1, \ldots, p_5)$ is incident with line $l = [l_1, \ldots, l_5]$, and we write it pIl or $(p)I[l]$, if and only if the following conditions are satisfied:

$$\begin{cases} l_2 - p_2 = l_1 p_1 \\ l_3 - p_3 = l_1 p_2 \\ l_4 - p_4 = p_1 l_2 \\ l_5 - p_5 = 2 l_1 p_4 - p_1 l_3 \end{cases} \tag{3.1}$$

This incidence defines a bipartite graph $B = B(q)$ whose vertex partition sets are P and L, and a point (p) and a line $[l]$ are connected by an edge if and only if $(p)I[l]$. The following theorem is the main result of this section.

Theorem 3.1. *The bipartite graph $B(q)$ satisfies the following properties:*
(a) $B(q)$ is q–regular of order $2q^5$ and size q^6

(b) For infinitely many values of q, $g(B(q)) = 10$

(c) For infinitely many values of q, $B(q)$ is not isomorphic to a subgraph of a generalized 6–gon.

Proof. (a) Obviously, $|V(B)| = |P| + |L| = q^5 + q^5 = 2q^5$. It is immediate from (3.1) that for a fixed $(p) \in V(B)$, the components of a line $[l] \in V(B)$ incident to (p) are determined uniquely by the value of l_1, which can be any element of the field. Therefore, the degree of (p) in B is q. In the same way we obtain that the degree of a line $[l]$ in B is also q. Therefore B is q–regular and $|E(B)| = q^6$.

Our proof of part (b) will be facilitated by the following two observations. First we notice that a graph G contains no $C_{2k}, k \geq 2$, if there is at most one simple path of length k between any two of its vertices. We will show that any pair of vertices of B is connected by at most one simple path of length $k, k = 2, 3, 4$. This will imply that $g(B) \geq 10$ since, being a bipartite graph, B contains no odd cycles.

Another observation is the existence of certain automorphisms of B. Let $x \in$

$I\!\!F_q$, and $t_i(x), i = 0, \ldots, 5$, be the mappings $V(B) \to V(B)$ defined as

$$
\begin{aligned}
(p)^{t_0(x)} &= (p_1 + x, p_2, p_3, p_4 + p_2x, p_5 + 2p_3x) \\
[l]^{t_0(x)} &= [l_1, l_2 + l_1x, l_3, l_4 + 2l_2x + l_1x^2, l_5 + l_3x] \\
(p)^{t_1(x)} &= (p_1, p_2 - p_1x, p_3 - 2p_2x + p_1x^2, p_4, p_5 - p_4x) \\
[l]^{t_1(x)} &= [l_1 + x, l_2, l_3 - l_2x, l_4, l_5 + l_4x] \\
(p)^{t_2(x)} &= (p_1, p_2 + x, p_3, p_4 - p_1x, p_5 - 3p_2x) \\
[l]^{t_2(x)} &= [l_1, l_2 + x, l_3 + l_1x, l_4, l_5 - 3l_2x] \\
(p)^{t_3(x)} &= (p_1, p_2, p_3 + x, p_4, p_5 + p_1x) \\
[l]^{t_3(x)} &= [l_1, l_2, l_3 + x, l_4, l_5] \\
(p)^{t_4(x)} &= (p_1, p_2, p_3, p_4 + x, p_5) \\
[l]^{t_4(x)} &= [l_1, l_2, l_3, l_4 + x, l_5 + 2l_1x] \\
(p)^{t_5(x)} &= (p_1, p_2, p_3, p_4, p_5 + x) \\
[l]^{t_5(x)} &= [l_1, l_2, l_3, l_4, l_5 + x]
\end{aligned}
$$

Lemma 3.2.

(i) For every $x \in I\!\!F_q$ and every $i \in \{0, 1, \ldots, 5\}$, the mapping $t_i(x)$ is an automorphism of the graph B and $t_i^{-1}(x) = t_i(-x)$.
(ii) For every edge $\{[l], (p)\}$ of B there exist automorphisms α and β of B such that $[l]^\alpha = [0, 0, 0, 0, 0]$, $(p)^\alpha = (a, 0, 0, 0, 0)$, and $[l]^\beta = [b, 0, 0, 0, 0], (p)^\beta = (0, 0, 0, 0, 0)$, for some $a, b \in I\!\!F_q$. The automorphism group $Aut(B)$ acts transitively on the set of points and on the set of lines, and B is edge–transitive.

Proof. We shall prove part (i) of the lemma for $t_0(x)$ only. For other $t_i(x)$ it can be done similarly (and actually faster). Let $\{(p), [l]\}$ be an arbitrary edge of B, where $(p) = (p_1, \ldots, p_5), [l] = [l_1, \ldots, l_5]$. In terms of components their incidence is represented by the system (3.1). The condition that $\{(p)^{t_0(x)}, [l]^{t_0(x)}\}$ is an edge of B is given by the following system

$$\begin{cases} (l_2 + l_1 x) - p_2 = l_1(p_1 + x) \\ l_3 - p_3 = l_1 p_2 \\ (l_4 + 2l_2 x + l_1 x^2) - (p_4 + p_2 x) = (p_1 + x)(l_2 + l_1 x) \\ (l_5 + l_3 x) - (p_5 + 2p_3 x) = 2l_1(p_4 + p_2 x) - (p_1 + x)l_3 \end{cases} \tag{3.2}$$

It is a straightforward verification that systems (3.1) and (3.2) are equivalent for all x. It is obvious that $t_0(x)$ is a well–defined bijection and that $t_0(x)t_0(-x) = t_0(-x)t_0(x)$ is the identity mapping on $V(B)$. Therefore, part (i) of the lemma is proven.

Let $[l] = [l_1, \ldots, l_5]$ and $(p) = (p_1, \ldots, p_5)$ be any two adjacent vertices of B. The diagram below represents the result of a sequence of applications of some automorphisms of B to the line $[l]$ and its images:

$$[l] \stackrel{t_1(-l_1)}{\longrightarrow} [0, a_2, a_3, a_4, a_5] \stackrel{t_2(-a_2)}{\longrightarrow} [0, 0, b_3, b_4, b_5] \stackrel{t_3(-b_3)}{\longrightarrow}$$

$$[0, 0, 0, c_4, c_5] \stackrel{t_4(-c_4)}{\longrightarrow} [0, 0, 0, 0, d_5] \stackrel{t_5(-d_5)}{\longrightarrow} [0, 0, 0, 0, 0]$$

Let $\alpha = t_1(-l_1)t_2(-a_2)t_3(-b_3)t_4(-c_4)t_5(-d_5)$. Then α is an automorphism of B and $[l]^\alpha = [0, 0, 0, 0, 0]$. Suppose $(p)^\alpha = (p') = (p'_1, \ldots, p'_5)$. Since α is an automorphism of B, and $[l]I(p)$, then $[0, 0, 0, 0, 0]I(p'_1, p'_2, p'_3, p'_4, p'_5)$, and conditions (3.1) give $p'_2 = p'_3 = p'_4 = p'_5 = 0$. Since α doesn't change the first coordinate of a point, then $p'_1 = p_1 = a$. Let $\alpha' = \alpha t_0(-a)$. Then $[l]^{\alpha'} = [0, 0, 0, 0, 0]$ and $(p)^{\alpha'} = (0, 0, 0, 0, 0)$. Therefore B is edge–transitive. Similarly we can show the existence of β (in this case we start with $t_0(-p_1)$). Therefore $Aut(B)$ acts transitively on each partition set of the bipartite graph B. ■

Now we show that any pair of vertices of B is connected by at most one simple path of length 4. We need not distinguish between the two cases where both vertices are lines or points as the proofs are absolutely similar. So we assume that the two vertices are lines (it is also sufficient to consider this case only, if we want to show the absence of C_8 in B). Call the vertices $[l^1]$ and $[l^3]$. Let $[l^1]I(p^1)I[l^2]I(p^2)I[l^3]$ be our path. Due to Lemma 3.2 (ii), without loss of generality, we may assume $[l^1] = [0, 0, 0, 0, 0]$ and $(p^1) = (x, 0, 0, 0, 0)$. We denote the first components of $[l^2]$ and (p^2) by y and z correspondingly, and we write $[l^3]$ as $[a_1, a_2, a_3, a_4, a_5]$ (to avoid double indices). The conditions of adjacency of subsequent vertices of the path written in terms of their components (formula (3.1)) allow us to express all the components in terms of x, y, z, a_i: $(p^1)I[l^2]$ gives $[l^2] = [y, xy, 0, x^2y, 0]$ and $[l^2]I(p^2)$ gives $(p^2) = (z, y(x-z), -y^2(x-z), xy(x-z), -2xy^2(x-z))$. The last adjacency $(p^2)I[l^3]$, written in terms of components, gives

$$\begin{cases} a_2 - y(x-z) = a_1 z \\ a_3 + y^2(x-z) = a_1 y(x-z) \\ a_4 - xy(x-z) = a_2 z \\ a_5 + 2xy^2(x-z) = 2a_1 xy(x-z) - a_3 z \end{cases} \tag{3.3}$$

We view (3.3) as a system of equations with unknown x, y, z and parameters a_i. The condition of existence of at most one simple path of length 4 between $[l^1]$ and $[l^3]$ is equivalent to the requirement that (3.3) has at most one solution which satisfies the following inequalities:

$$\begin{cases} [l^1] \neq [l^3], [l^2] \neq [l^3], [l^1] \neq [l^2] \\ (p^1) \neq (p^2) \end{cases} \tag{3.4}$$

Simplifying, we get an equivalent system

$$\begin{cases} [l^1] \neq [l^3], [l^2] \neq [l^3] \\ y \neq 0, x \neq z \end{cases} \tag{3.5}$$

Thus our goal is to prove that the combined system (3.3) and (3.5) has at most one solution for every $[l^3] = [a_1, a_2, a_3, a_4, a_5]$.

It will be convenient to consider two cases: $a_1 = 0$, and $a_1 \neq 0$, $a_2 = 0$. In order to see that these cases suffice, we notice that for $a_1 \neq 0$ the automorphism $t = t_0(-a_2/a_1)$ of B has the properties:

$$[l^1]^t = [l^1] = [0,0,0,0,0], (p^1)^t = (x - a_2/a_1, 0,0,0,0), [l^3]^t = [a_1, 0, a_3', a_4', a_5'].$$

Case 1. $a_1 = 0$. The combined system of (3.3) and (3.5) has the form (we shall suppress the "primes" on a_i, $i = 3,4,5$):

$$\begin{cases} y(x-z) = a_2 \\ y^2(x-z) = -a_3 \\ xy(x-z) = a_4 - a_2 z \\ 2xy^2(x-z) = -a_5 - a_3 z \\ y \neq 0, x \neq z, [l^1] \neq [l^3], [l^2] \neq [l^3] \end{cases} \tag{3.6}$$

This implies that $a_2 \neq 0, a_3 \neq 0, y = -a_3/a_2$ and

$$\begin{cases} x + z = a_4/a_2 \\ 2x - z = a_5/a_3 \end{cases}$$

The last system has at most one solution with respect to x, z. Therefore (3.6) has at most one solution.

Case 2. $a_1 \neq 0, a_2 = 0$. The combined system of (3.3) and (3.5) has the form

$$\begin{cases} y(x-z) = -a_1 z \\ y^2(x-z) = a_1 y(x-z) - a_3 \\ xy(x-z) = a_4 \\ 2xy^2(x-z) = 2a_1 xy(x-z) - a_3 z - a_5 \\ y \neq 0, x \neq z, [l^2] \neq [l^3], a_1 \neq 0 \end{cases} \tag{3.7}$$

We consider two subcases, depending on a_4 being zero or not.

Subcase 2.1. $a_4 = 0$. Then (3.7) is equivalent to

$$\begin{cases} x = 0 \\ yz = a_1 z \\ y^2 z = a_1 yz - a_3 \\ a_3 z = -a_5 \\ y \neq 0, z \neq 0, [l^2] \neq [l^3], a_1 \neq 0 \end{cases} \tag{3.8}$$

If $a_3 = 0$, then $a_5 = 0, y = a_1$ and $[l^2] = [l^3]$, i.e., (3.8) has no solution. If $a_3 \neq 0$, then $y = a_1$ and $z = -a_5/a_3$ and (3.8) has at most one solution.

Subcase 2.2. $a_4 \neq 0$. We rewrite (3.7) in the equivalent form

$$\begin{cases} a_1 xy = a_3, a_1 xz = -a_4, y(x-z) = a_1 z\ , \\ 2a_4 y = 2a_1 a_4 - a_3 z - a_5 \\ x \neq 0, y \neq 0, z \neq 0, x \neq z, a_1 \neq 0 \end{cases} \tag{3.9}$$

which implies $-a_3 z = a_4 y$ and $a_4 y = 2a_1 a_4 - a_5$. Since $a_4 \neq 0$, the last equation can be solved for y uniquely. If $y = 0$, then (3.9) has no solution. If $y \neq 0$, then x and z are determined uniquely in terms of y, and (3.9) has at most one solution. ■

Why does B contain no C_4 and no C_6? Due to Lemma 3.2, the existence of a cycle of length 4 in B would imply the existence of two interior vertex disjoint simple paths of length 2 between a pair of distinct lines $[l^1] = [0, 0, 0, 0, 0]$ and $[l^2] = [a_1, a_2, a_3, a_4, a_5]$. Let $(p) = (p_1, p_2, p_3, p_4, p_5)$ and $[l^1]I(p)I[l^2]$. Rewriting these adjacencies in terms of components, using (3.1), we get $p_2 = p_3 = p_4 = p_5 = 0$ and $a_2 = a_1 p_1$. If $a_1 \neq 0$, then p_1 is determined uniquely and, therefore, there exists only one path of length 2 between $[l^1]$ and $[l^2]$. If $a_1 = 0$, then $(p)I[l^2]$ implies $a_2 = a_3 = a_4 = a_5 = 0$, and $[l^1] = [l^2]$. Hence B contains no C_4.

Due to Lemma 3.2, the existence of a cycle of length 6 in B would imply the existence of two interior vertex disjoint simple paths of length 3 between a line $[l^1] = [0, 0, 0, 0, 0]$ and a point $(s) = (s_1, s_2, s_3, s_4, s_5)$. Let (p) and $[l^2]$ be two intermediate vertices on such a path, i.e., $[l^1]I(p)I[l^2]I(s), (p) \neq (s), [l^1] \neq [l^2]$. Rewriting the first two adjacencies in terms of components, we obtain $(p) = (x, 0, 0, 0, 0), [l^2] = [y, xy, 0, x^2y, 0]$ for some $x, y \in I\!\!F_q$. Using $[l^2]I(s)$, we obtain the following system:

$$\begin{cases} xy - s_2 = s_1 y, 0 - s_3 = s_2 y, x^2 y - s_4 = s_1 xy \\ 0 - s_5 = 2 s_4 y \\ (s) \neq (p), y \neq 0 \end{cases} \tag{3.10}$$

If $s_2 = 0$, then $s_3 = s_4 = s_5 = 0$ and $x = s_1$. This makes $(s) = (p)$, which is not the case. If $s_2 \neq 0$, then $s_3 \neq 0, y = -s_3/s_2$ and $x = s_1 - s_2^2/s_3$. Therefore (3.10) has no more than one solution with respect to x and y. Hence B contains no C_6, and $g(B) \geq 10$. To prove that B contains C_{10}, it is enough to show that there are two simple interior vertex–disjoint paths of length 5 between line $[l] = [0, 0, 0, 0, 0]$ and point $(p) = (0, 1, 1, 1, 1)$. This can be reduced to determining when the quadratic equation $3t^2 + 2t - 4 = 0$ has two distinct solutions which satisfy certain restrictions. It can be shown that for all sufficiently large values of q, which are neither divisible by 2 nor 3, and such that 13 is a quadratic residue in $I\!\!F_q$, such two solutions exist; the proof is straightforward and we omit it. We believe that $g(B) = 10$ for most other values of q (point (p) has to be chosen differently), but we leave this investigation out of the paper. This finishes the proof of part (b) of Theorem 3.1. Part (c) follows immediately from (b) since a generalized 6–gon has girth 12. ■

SECTION 4. Construction of graphs of order $2q^9$, size q^{10}, and girth ≥ 14.

The construction we present in this section is quite similar to the one we performed in Section 3. The same can be said about the logic of the proofs, though in this case some of them are more elegant.

Let $P = \{(p) = (p_1, \ldots p_9) | p_i \in \mathbb{F}_q, i = 1, \ldots 9\}$ be the set of points and $L = \{[l] = [l_1, \ldots l_9] | l_i \in \mathbb{F}_q, i = 1, \ldots, 9\}$ be the set of lines. A point $(p) = (p_1, \ldots, p_9)$ and a line $[l] = [l_1, \ldots, l_9]$ are said to be incident (and denoted $(p)I[l]$) if the following conditions are satisfied:

$$\begin{cases} l_2 - p_2 = l_1 p_1 \\ l_3 - p_3 = p_1 l_2 \\ l_4 - p_4 = l_1 p_2 \\ l_5 - p_5 = l_1 p_3 \\ l_6 - p_6 = p_1 l_4 \\ l_7 - p_7 = p_1 l_5 \\ l_8 - p_8 = l_1 p_6 \\ l_9 - p_9 = l_1 p_7 \end{cases} \tag{4.1}$$

This incidence defines a bipartite graph $G = G(q)$ whose vertex partition sets are P and L. Then G has $2q^9$ vertices, q^{10} edges, and is q–regular (the proof is the same as of Theorem 3.1 (a)).

For every $x \in \mathbb{F}_q$, we introduce the following mappings $t_i : V(G) \to V(G), i = 0, \ldots, 9$:

$$(p)^{t_0(x)} = (p_1 + x, p_2, p_3 + p_2x, p_4, p_5 + p_4x, p_6, p_7 + p_6x, p_8, p_9 + p_8x)$$

$$[l]^{t_0(x)} = [l_1, l_2 + l_1x, l_3 + 2l_2x + l_1x^2, l_4, l_5 + l_4x, l_6 + l_4x, l_7 + (l_5 + l_6)x + l_4x^2, l_8, l_9 + l_8x]$$

$$(p)^{t_1(x)} = (p_1, p_2 - p_1x, p_3, p_4 - 2p_2x + p_1x^2, p_5 - p_3x, p_6 - p_3x, p_7, p_8 - (p_5 + p_6)x + p_3x^2, p_9 - p_7x)$$

$$[l]^{t_1(x)} = [l_1 + x, l_2, l_3, l_4 - l_2x, l_5, l_6 - l_3x, l_7, l_8 - l_5x, l_9]$$

$$(p)^{t_2(x)} = (p_1, p_2 + x, p_3 - p_1x, p_4, p_5 - p_2x, p_6 + p_2x, p_7 - p_3x, p_8 + p_4x, p_9 - p_5x)$$

$$[l]^{t_2(x)} = [l_1, l_2 + x, l_3, l_4 + l_1x, l_5 - l_2x, l_6 + l_2x, l_7 - l_3x, l_8 + l_4x, l_9 - l_5x]$$

$$(p)^{t_3(x)} = (p_1, p_2, p_3 + x, p_4, p_5, p_6, p_7 + p_2x, p_8, p_9 + p_4x)$$

$$[l]^{t_3(x)} = [l_1, l_2, l_3 + x, l_4, l_5 + l_1x, l_6, l_7 + l_2x, l_8, l_9 + l_4x]$$

$$(p)^{t_4(x)} = (p_1, p_2, p_3, p_4 + x, p_5, p_6 - p_1x, p_7, p_8 - p_2x, p_9)$$

$$[l]^{t_4(x)} = [l_1, l_2, l_3, l_4 + x, l_5, l_6, l_7, l_8 - l_2x, l_9]$$

$$(p)^{t_5(x)} = (p_1, p_2, p_3, p_4, p_5 + x, p_6, p_7 - p_1x, p_8, p_9 - p_2x)$$

$$[l]^{t_5(x)} = [l_1, l_2, l_3, l_4, l_5 + x, l_6, l_7, l_8, l_9 - l_2x]$$

$$(p)^{t_6(x)} = (p_1, p_2, p_3, p_4, p_5, p_6 + x, p_7, p_8, p_9)$$

$$[l]^{t_6(x)} = [l_1, l_2, l_3, l_4, l_5, l_6 + x, l_7, l_8 + l_1x, l_9]$$

$$(p)^{t_7(x)} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7 + x, p_8, p_9)$$

$$[l]^{t_7(x)} = [l_1, l_2, l_3, l_4, l_5, l_6, l_7 + x, l_8, l_9 + l_1x]$$

$$(p)^{t_8(x)} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8 + x, p_9)$$

$$[l]^{t_8(x)} = [l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8 + x, l_9]$$

$$(p)^{t_9(x)} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9 + x)$$

$$[l]^{t_9(x)} = [l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9 + x]$$

The proof of the following lemma is absolutely similar to the one given for Lemma 3.2 and we omit it.

Lemma 4.1.

(i) For every $x \in I\!F_q$ *and every* $i \in \{1, \ldots 9\}$*, the mapping* $t_i(x)$ *is an automorphism of the graph* G *and* $t_i^{-1}(x) = t_i(-x)$*.*

(ii) For every edge $\{[l], (p)\}$ *of* G *there exist automorphisms* α *and* β *of* G *such that* $[l]^\alpha = [0, \ldots, 0], (p)^\alpha = (a, 0, \ldots, 0)$*, and* $[l]^\beta = [b, 0, \ldots, 0], (p)^\beta = (0, \ldots, 0)$*, for some* $a, b \in I\!F_q$*. The automorphism group* $Aut(G)$ *acts transitively on each of the sets* P *and* L*, and* G *is edge–transitive.* ■

In order to prove that $g(G) \geq 14$ we choose any pair of distinct lines, $[\tilde{l}^1]$ and $[\tilde{l}^4]$ and show that in any simple path $[\tilde{l}^1]I(\tilde{p}^1)I[\tilde{p}^2]I(\tilde{p}^2)I[\tilde{l}^3]I(\tilde{p}^3)I[\tilde{l}^4]$ of length six, the first point $(\tilde{p}^1)$ is determined uniquely by $[\tilde{l}^1]$ and $[\tilde{l}^4]$. This implies that there are no two paths of length six between $[\tilde{l}^1]$ and $[\tilde{l}^4]$ having disjoint sets of interior vertices, and therefore that G contains no cycle of length 12. Then, by similar arguments, we show that G contains no C_4, C_6, C_8, C_{10}. Since G is bipartite, this will imply that $g(G) \geq 14$.

Thus let $[\tilde{l}^1]I(\tilde{p}^1)I \ldots I(\tilde{p}^3)I[\tilde{l}^4]$ be a path of length six joining $[\tilde{l}^1]$ and$[\tilde{l}^4]$, and having all its vertices distinct. By Lemma 4.1, there exists $\alpha \in Aut(G)$ such that $[\tilde{l}^1]^\alpha = [0, \ldots, 0]$ and $(\tilde{p}^1)^\alpha = (t, 0, \ldots, 0)$ for some $t \in I\!F_q$. Let $\beta = t_0(-t)$, and let $[l^i] = [\tilde{l}^i]^{\alpha\beta}, (p^i) = (\tilde{p}^i)^{\alpha\beta}, i = 1, 2, 3, 4$. Then we have $[l^1] = [0, \ldots, 0]$ and $(p^1) = (0, \ldots, 0)$. Suppose $[\tilde{l}^4]^\alpha = [a_1, \ldots, a_9]$. Then $[l^4] = [a_1, \ldots, a_9]^\beta = [a_1, a_2 - a_1 t, a_3 - a_2 t + a_1 t^2, a_4, a_5 - a_4 t, a_6 - a_4 t, a_7 - (a_5 + a_6)t + a_4 t^2, a_8, a_9 - a_8 t]$. Let x, y, z, v be the first components of $[l^2], (p^2), [l^3], (p^3)$ respectively. Then the incidence condition

(4.1) allows us to express all other components of $[l^2], (p^2), [l^3], (p^3)$ in terms of x, y, z, v only and we get:

$$[l^2] = [x, 0, \dots, 0]$$

$$(p^2) = (y, -xy, 0, x^2y, 0, \dots, 0)$$

$$[l^3] = [z, y(z-x), y^2(z-x), xy(x-z), 0, y^2x(x-z), 0, 0, 0]$$

$$(p^3) = (v, -xy+yz-vz, y(z-x)(y-v), x^2y+z^2v-z^2y, yz(z-x)(v-y), xy(x-z)(y-v), 0, xyz(z-x)(y-v), 0)\ .$$

It is important to notice that if two lines (points) are adjacent in G to the same point (line) and their first components are equal, then they must coincide. This follows immediately from (4.1). Therefore, having all vertices of our path distinct, we get

$$x \neq 0, y \neq 0, x \neq z, y \neq v, z \neq a_1 \tag{4.2}$$

Conditions (4.2) can be met if and only if $q \geq 3$. Since $(p^3)I(l^4)$, the last two equations of (4.1) give:

$$\begin{cases} a_9 - a_8 t = 0 \\ a_8 - xyz(z-x)(y-v) = a_1 \cdot xy(x-z)(y-v) \end{cases}$$

or

$$\begin{cases} a_8 t = a_9 \\ a_8 = xy(x-z)(y-v)(a_1 - z)\ . \end{cases}$$

Due to (4.2), $a_8 \neq 0$ and t is determined from the first equation uniquely. It means that $(\tilde{p}^1)^\alpha = (t, 0, \dots, 0)$ is determined by $[\tilde{l}^1]^\alpha$ and $[\tilde{l}^4]^\alpha$ uniquely, and therefore every simple path between $[\tilde{l}^1]^\alpha$ and $[\tilde{l}^4]^\alpha$ of length six passes through it. So every path between $[\tilde{l}^1]$ and $[\tilde{l}^4]$ must pass through $(\tilde{p}^1)$ and there are no two such paths between $[\tilde{l}^1]$ and $[\tilde{l}^4]$ with disjoint sets of interior vertices. This implies that G has no C_{12} as a subgraph.

The absence of shorter cycles of even length can be proven in a similar way. Let $[\tilde{l}^1]I(\tilde{p}^1)I\ldots$ be a simple path of length $r, 2 \le r \le 5$. Let $\alpha \in Aut(G)$ such that $[\tilde{l}^1]^\alpha = [0,\ldots,0]$ and $(\tilde{p}^1)^\alpha = (t,0,\ldots,0)$. The image of the last vertex of the path is denoted by $[a_1,\ldots,a_9]$ if the vertex is a line, and by $(b_1,\ldots,b_9)$ if it is a point. Then we apply $\beta = t_0(-t)$. We get $[l^1] = [\tilde{l}^1]^{\alpha\beta} = [0,\ldots,0], (l^1) = (\tilde{p}^1)^{\alpha\beta} = (0,\ldots,0)$. Let $[l^2] = [x,\ldots], (p^2) = (y,\ldots), [l^3] = [z,\ldots]$.

For $r = 5$, the 6^{th} and 7^{th} equations of (4.1) give:

$$\begin{cases} b_7 - b_6 t = 0 \\ b_6 = xy(x-z)(b_1 - y) \end{cases}$$

Since $x \ne 0, y \ne 0, x \ne z, b_1 \ne y$, then $b_6 \ne 0$ and t is determined from the first equation uniquely. Thus for any pair point–line there is at most one simple path of length 5 joining them. So G contains no C_{10}.

For $r = 4$, the 4^{th} and 5^{th} equations of (4.1) give:

$$\begin{cases} a_5 - a_4 t = 0 \\ a_4 = xy(x - a_1) \end{cases}$$

Since $x \ne 0, y \ne 0, x \ne a_1$, then $a_4 \ne 0$ and t is determined uniquely from the first equation. This implies that G contains no C_8.

For $r = 3$, the 2^{nd} and 3^{rd} equations of (4.1) give:

$$\begin{cases} b_3 - b_2 t = 0 \\ b_2 = -xy \end{cases}$$

Since $x \ne 0, y \ne 0, t$ is determined uniquely from the first equation. Therefore G contains no C_6. In a similar way we show that G contains no C_4. The proof is left for the reader.

We combine all the results of this section in

Theorem 4.2. *Let q be a prime power, $q \geq 3$. Then graph $G(q)$ is a q–regular bipartite graph of order $2q^9$ and girth ≥ 14. The automorphism group $Aut(G(q))$ is transitive on each of the sets of points and lines, and $G(q)$ is edge–transitive.* ■

Let $UB(q)$ be a group generated by the automorphisms $t_i(x)$, $i = 0, 1, \ldots, 5$, of the graph $B(q)$, $UB_1(q) = \langle t_0(x) \mid x \in \mathbb{F}_q \rangle$, $UB_2(q) = \langle t_1(x) \mid x \in \mathbb{F}_q \rangle$. Let $UG(q)$ be a group generated by the automorphisms $t_i(x)$, $i = 0, 1, \ldots, 9$, of the graph $G(q)$, $UG_1(q) = \langle t_0(x) \mid x \in \mathbb{F}_q \rangle$, $UG_2(q) = \langle t_1(x) \mid x \in \mathbb{F}_q \rangle$. Analogously with the case of graphs $S_n(q)$, $n = 3, 4, 6$, we obtain

Proposition 4.3. *The incidence structure $B(q)$ and $G(q)$ are isomorphic to the group incidence structure $\gamma(UB(q), \{UB_1(q), UB_2(q)\})$ and $\gamma(UG(q), \{UG_1(q), UG_2(q)\})$, respectively.* ■

Remark. All graphs in this paper, but $G(q)$, are connected (for odd q). Graphs $B(q)$ and $G(q)$ are blow–ups of $K_{1,1}$.

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