An Extremal Characterization of the Incidence Graphs of Projective Planes

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Dedicated to the memory of Professor L.A. Kalužnin.

Abstract

Let G be a 4–cycle free, bipartite graph on $2n$ vertices with partitions of equal cardinality n. Let $c_6(G)$ denote the number of cycles of length 6 in G. We prove that for $n \geq 3$, $c_6(G) \leq \frac{1}{3}\binom{n}{2}(n - r_n)$, where $r_n = \frac{1}{2} + \frac{\sqrt{4n-3}}{2}$, with equality if and only if G is the incidence point-line graph of a projective plane.

Let $\mathcal{G}$ denote a family of simple graphs of order n. For simple graphs H and G, let (G, H) denote the number of subgraphs of G isomorphic to H. Let $f(\mathcal{G}, H) = \max\{(G, H) | G \in \mathcal{G}\}$ and $F(\mathcal{G}, H) = \{G \in \mathcal{G} | (G, H) = f(\mathcal{G}, H))\}$. We will refer to graphs of $F(\mathcal{G}, H)$ as *extremal.* The problem of finding $f(\mathcal{G}, H)$ and $F(\mathcal{G}, H)$, for fixed $\mathcal{G}, H$, has been studied extensively and is considered as central in extremal graph theory. Though it is hopeless in whole generality, some of its instances have been solved. Often the results are concerned with bounds on $f(\mathcal{G}, H)$ and partial description of the extremal graphs. For example, if K_m denotes the complete graph of order m, $H = K_2$, and $\mathcal{G}$ is the family of all graphs of order n which contain no K_m as a subgraph, $3 \le m \le n$, then the solution is given by the celebrated Turán Theorem. For the same H, if $\mathcal{G}$ is the family of all (m, n)–bipartite graphs with no $K_{t,s}$ (a complete bipartite graph with partition class sizes t and s) for some t, s, $1 \le t \le m, 1 \le s \le n$, we have the, so called, Zarankiewicz problem. These and many other examples can be found in [2]. For some later results see [4,5,6]. The goal of this note is to study $f(\mathcal{G}, H)$ and $F(\mathcal{G}, H)$, where $\mathcal{G}$ is the family of all 4–cycle free bipartite graphs with partition classes each of size n and H is a 6–cycle. The motivation comes from finite geometries and will be indicated below.

For $n = q^2 + q + 1$, let π be a finite projective plane of order q with point set $P = \{p_1, \ldots, p_n\}$ and line set $L = \{l_1, \ldots, l_n\}$. A bipartite graph G with partitions (P, L) is said to be the *incidence point-line graph of the projective plane* π if for all $i, j \in \{1, \ldots, n\}$, $\{p_i, l_j\}$ is an edge of G if and only if $p_i \in l_j$. Let $c_6(G)$ denote the number of 6–cycles in G. The main result of this note is the following

Theorem 1. *Let G be a 4–cycle free bipartite graph on $2n$ vertices with partitions of size n. Then for $n \ge 3$, $c_6(G) \le \frac{1}{3}\binom{n}{2}(n - r_n)$, where $r_n = \frac{1}{2} + \frac{1}{2}\sqrt{4n-3}$, with equality if and only if G is the incidence point-line graph of a projective plane.*

Using the terminology of finite geometries (see [1]), Theorem 1 provides an upper bound for the number of triangles in near–linear spaces with n points and n lines. It also states that the bound is achieved if and only if the near–linear space is a projective plane.

Our proof for Theorem 1 is based on a result from [7]. The following Theorem A is a particular case of a more general result proved in [7], but it will suffice for our needs.

Theorem A. *Let G be a 2–connected graph on v vertices, with e edges, and girth 6. Then $c_6(G) \leq \frac{e}{6}(e - v + 1)$, and the equality holds if and only if every two edges of G are contained in a common 6–cycle.* ■

We say that a bipartite, 4–cycle free graph G is *edge–maximal* if the addition of any edge to G with end vertices in different partition classes of $V(G)$ creates a 4–cycle in G. For $v \in V(G)$, define $G - v$ to be the graph obtained from G by deleting the vertex v and all edges incident to it. We prove the following lemma.

Lemma 1. *Suppose G is a bipartite, 1–connected, edge–maximal graph with $\delta(G) \geq 2$ and girth at least 6. Then G is 2–connected.*

Proof: Let G have partition classes $V_1(G)$, $V_2(G)$. By way of contradiction, suppose G is not 2–connected. Then G has a cut vertex v which we may assume is in $V_1(G)$. Let G_1 and G_2 be two components of the graph $G - v$. Note that G_1 and G_2 are both bipartite; denote their respective partition classes by $V_1(G_1)$, $V_2(G_1)$ and $V_1(G_2)$, $V_2(G_2)$, where $V_i(G_j) = V_i(G) \cap V(G_j)$.

Since v is a cut vertex of G and $\delta = \delta(G) \geq 2$, it is easily shown that there exists $u \in V_1(G_1)$ and $w \in V_2(G_2)$ such that $dist(u, w) = 5$ and that every (u, w)–path contains vertex v. Consider the (u, v)–path $uv'vv''v'''w$ of length 5 in G. Since G is edge–maximal the graph $G + \{u, w\}$ must have girth less than 6, and since $G + \{u, w\}$ is bipartite it must contain a 4–cycle which uses the new edge $\{u, w\}$.

Denote it by $uwxyu$. Since v is a cut vertex it must appear in the 4–cycle; so, in fact, $v = x$. But then $vv''v'''wv$ is a 4–cycle in G, contradiction. ■

Proof of Theorem 1. For n a positive integer, define $r_n = \frac{1}{2} + \frac{1}{2}\sqrt{4n-3}$. It is known (see [2, ch. VI.2]) that for a bipartite 4–cycle free graph G with partition classes of equal cardinality n, $e(G) \leq nr_n$, and $e(G) = nr_n$ if and only if G is the point–line incidence graph of a projective plane. Let G be a 2–connected, bipartite, 4–cycle free graph on $v = 2n$ vertices with both partitions of equal size n. Then by Theorem A,

$$c_6(G) \leq \frac{e}{6}(e - v + 1) \leq \frac{nr_n}{6}(nr_n - 2n + 1) = \frac{1}{3}\binom{n}{2}(n - r_n).$$

The equality in the second inequality holds if and only if $e(G) = nr_n$, and therefore, G must be the incidence point–line graph of a projective plane. But it is obvious that such a graph has any pair of edges belonging to a 6–cycle. Therefore the equality in the first inequality holds and Theorem 1 is proven for 2–connected graphs.

Suppose G is not 2–connected. Then it is either disconnected or 1-connected. Though the inequality of Therorem A does fail for some of these graphs, the inequality of Theorem 1 never does.

First we observe that it suffices to prove Theorem 1 for 1–connected graphs G only, since G may be assumed to be edge–maximal, and therefore connected. Thus, what is left is to prove Theorem 1 for 1–connected, edge–maximal graphs. Due to Lemma 1, for such graphs G, $\delta(G) = 1$.

We proceed by induction on n. Suppose $n = 3$. Clearly $c_6(G) \leq 1$. However, $r_3 = \frac{1}{2} + \frac{3}{2} = 2$ and so $\frac{1}{3}\binom{3}{2}(3 - 2) = 1$. Thus, the result is true for $n = 3$.

Let $\mathcal{G}(n)$ denote the family of all bipartite, 1–connected, edge–maximal, 4–cycle free graphs G with partition classes of equal cardinality n, $n \geq 3$, and $\delta(G) = 1$. For $t \geq 1$, we define $f(t) = \binom{t}{2}(t - r_t)$, where $r_t = \frac{1}{2} + \frac{\sqrt{4t-3}}{2}$. Thus, we want to prove that $c_6(G) \leq \frac{1}{3}f(n)$. Next, suppose that $c_6(G) \leq \frac{1}{3}f(n-1)$ for all $G \in \mathcal{G}(n-1)$. Starting with $G \in \mathcal{G}(n)$, construct the bipartite graphs $G^{(1)}$ and $G^{(2)}$ in the following manner:

Choose $v^{(1)} \in V_1(G)$ so that $\deg_G\,(v^{(1)}) = 1$ (if no vertex in $V_1(G)$ is of degree 1 interchange $V_1(G)$ and $V_2(G)$). Let $G^{(1)} = G - v^{(1)}$. Note that $G^{(1)}$ is a bipartite graph of girth at least six with partition class sizes $n, n-1$. Since $\deg_G(v^{(1)}) = 1$, it is obvious that $c_6(G^{(1)}) = c_6(G)$. Next, choose $v^{(2)} \in V_2(G^{(1)})$ so that $v^{(2)}$ is of minimum degree in the partition class $V_2(G^{(1)})$, that is, $\deg_{G^{(1)}}\,(v^{(2)}) = \min\limits_{u \in V_2(G^{(1)})} \{\deg_{G^{(1)}}\,(u)\}$. Since $e(G) \leq r_n n$, this implies that the minimum degree from each partition class is at most r_n. Therefore $\deg_{G^{(1)}}(v^{(2)}) \leq \deg_G(v^{(2)}) \leq r_n$. Let $G^{(2)} = G^{(1)} - v^{(2)}$.

We claim that $c_6(G^{(2)}) \leq \frac{1}{3}f(n-1)$. If $G^{(2)}$ is 2–connected, we are done. If $G^{(2)}$ is not edge–maximal, let $\widetilde{G}^{(2)}$ be an edge–maximal graph containing $G^{(2)}$. If $\widetilde{G}^{(2)}$ is 1–connected, then $c_6(G^{(2)}) \leq c_6(\widetilde{G}^{(2)}) \leq \frac{1}{3}f(n-1)$ due to the induction hypothesis. If $\widetilde{G}^{(2)}$ is 2–connected, then $c_6(G^{(2)}) \leq c_6(\widetilde{G}^{(2)}) \leq \frac{1}{3}f(n-1)$, as was shown at the beginning of the proof of the theorem. It is easy to see that $c_6(G) = c_6(G^{(1)}) = c_6(G^{(2)}) + c_6(v^{(2)})$, where $c_6(v^{(2)})$ is the number of cycles of length 6 in $G^{(1)}$ that pass through $v^{(2)}$. By a simple counting argument, we have that $c_6(v^{(2)}) \leq \binom{\deg_{G^{(1)}}(v^{(2)})}{2}(n - \deg_{G^{(1)}}(v^{(2)}))$. Note that $\deg_{G^{(1)}}(v^{(2)}) \leq r_n \leq$

$\frac{n+1+\sqrt{n^2-n+1}}{3}$. Therefore, $c_6(v^{(2)}) \le \binom{r_n}{2}(n-1-r_n)$, and so

$$c_6(G) \le c_6(G^{(2)}) + c_6(v^{(2)}) < \frac{1}{3}(f(n-1) + \binom{r_n}{2}(n-1-r_n)) \ .$$

However, $f(n-1) + \binom{r_n}{2}(n-1-r_n) < f(n)$ for all $n \ge 3$. Therefore $c_6(G) < \frac{1}{3}f(n)$ for $G \in \mathcal{G}(n)$, and this completes the proof of Theorem 1. ■

This proof also shows that the equality in Theorem 1 is not achieved by graphs which are not 2–connected. Another proof of Theorem 1, based on a completely different approach, can be found in [3]. It is much longer, but it establishes a slightly more general result.

References

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