# On the maximum number of $C_6's$ in a quadrilateral-free bipartite graph

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Dedicated to the memory of Professor L.A. Kalužnin.

**Abstract.** Let $G = G(n, n)$ be a 4–cycle free, bipartite graph on $2n$ vertices with partitions of equal cardinality $n$. Let $c_6(G)$ denote the number of cycles of length 6 in $G$. We prove that for $n \geq 4$, $c_6(G) \leq \frac{1}{3}\binom{n}{2}(n - r_n)$, where $r_n = \frac{1}{2} + \frac{\sqrt{4n-3}}{2}$, with equality if and only if $G$ is the incidence point-line graph of a projective plane.

## Section 1: Introduction.

Let $\mathcal{G} = \mathcal{G}_n$ denote a family of simple graphs of order $n$. For a simple graph $H$ and $G \in \mathcal{G}$, let $(G, H)$ denote the number of subgraphs of $G$ isomorphic to $H$. Let $h(n) = h(\mathcal{G}, H, n) = \max\{(G, H) | G \in \mathcal{G}\}$ and $\mathcal{G}(H, n) = \{G \in \mathcal{G} | (G, H) = h(n)\}$. We will refer to graphs of $\mathcal{G}(H, n)$ as *extremal.* The problem of finding $h(\mathcal{G}, H, n)$ and $\mathcal{G}(H, n)$, for fixed $\mathcal{G}, H, n$, has been studied extensively and is considered as central in extremal graph theory. Though it is hopeless in whole generality, some of its instances have been solved. Often the results are concerned with bounds on $h_n$ and partial description of the extremal graphs. For example, if $K_m$ denotes the complete graph of order $m$, $H = K_2$, and $\mathcal{G}$ is the family of all graphs of order $n$ which contain no $K_m$ as a subgraph, $3 \leq m \leq n$, then the solution is given by the famous Turán Theorem. For the same $H$, if $\mathcal{G}$ is the family of all $(m,n)$–bipartite graphs, we have the, so called, Zarankiewicz problem. These and many other examples can be found in [2]. For some later results see [4,5,6].

We start with the following definitions and notation. All missing ones can be found in [2].

Let $G = G(m,n)$ be a bipartite graph on $m+n$ vertices with partition $(V_1(G), V_2(G))$ such that $V_1(G) = \{u_1, \ldots, u_m\}$, $V_2(G) = \{v_1, \ldots, v_n\}$. Denote the number of edges of a graph $G$ by $e = e(G)$, the neighborhood of a vertex $v \in V(G)$ by $N(v)$ ($v \notin N(v)$), and the degree of vertex $v$ in $G$ by $\deg_G(v)$. Let $x_i = \deg_G(u_i)$, $i = 1, \ldots, m$, and $y_i = \deg_G(v_i)$, $i = 1, \ldots, n$. A subset, $\{u_{i_1}, \ldots, u_{i_k}\}$, $2 \leq k \leq n$, of $V_1(G)$ (or $\{v_{i_1}, \ldots, v_{i_k}\}$ of $V_2(G)$) is said to be *intersecting* if $N(u_{i_1}) \cap \ldots \cap N(u_{i_k}) \neq \emptyset$ (or $N(v_{i_1}) \cap \ldots \cap N(v_{i_k}) \neq \emptyset$). For a graph $G$ containing a cycle, the *girth* of $G$ is the length of a shortest cycle in $G$.

Let $n = q^2 + q + 1$ and let $\pi$ be a finite projective plane of order $q$ with point set $P = \{p_1, \ldots, p_n\}$ and line set $L = \{l_1, \ldots, l_n\}$. A bipartite graph $G$ with partitions $(P, L)$ is said to be the *incidence point-line graph of the projective plane* $\pi$ if for all $i, j \in \{1, \ldots, n\}$, $\{p_i, l_j\}$ is an edge of $G$ if and only if $p_i \in l_j$.

Let $c_6(G)$ denote the number of 6–cycles in $G$. The main goal of this paper is to find a nontrivial upper bound for $c_6(G)$, where $G = G(m,n)$ is a bipartite 4–cycle free graph. The results are summarized below.

**Theorem 1.** *Let $G$ be a 4–cycle free bipartite graph on $m+n$ vertices with partition classes of size $m$ and $n$. Then for $m \geq n \geq 4$, $c_6(G) \leq \frac{m}{3}\binom{r_{m,n}}{2}(n - r_{m,n})$, where $r_{m,n} = \frac{1}{2m}[m + (m^2 + 4mn(n-1))^{\frac{1}{2}}]$.*

In the case when $m = n$ we can actually say much more.

**Theorem 2.** *Let $G$ be a 4–cycle free bipartite graph on $2n$ vertices with partitions of size $n$. Then for $n \geq 4$, $c_6(G) \leq \frac{1}{3}\binom{n}{2}(n - r_n)$, where $r_n = \frac{1}{2} + \frac{1}{2}\sqrt{4n-3}$, with equality if and only if $G$ is the incidence point-line graph of a projective plane.*

Using the terminology of finite geometries (see [1]), Theorems 1 and 2 provide an upper bound for the number of triangles in near–linear spaces with $m$ points and $n$ lines.

The paper is organized in the following way. In Section 2 we prove Theorem 1. In Section 3 we present the original proof of Theorem 2 as a corollary of Theorem 1, and sketch another proof, independent of Theorem 1, which is based on a recent result in [7].

## Section 2: General case

Let $G = G(m, n)$ be a bipartite, 4–cycle free graph on $m + n$ vertices with partition $(V_1(G), V_2(G))$ such that $V_1(G) = \{u_1, \ldots, u_m\}$, $V_2(G) = \{v_1, \ldots, v_n\}$. Let $x_i = \deg_G(u_i)$ and $y_i = \deg_G(v_i)$, $i = 1, \ldots, n$. It is clear that $\sum_{i=1}^{m}\binom{x_i}{2} \leq \binom{n}{2}$ (see [2, ch. VI.2] for a more general result).

For positive integers $m$ and $n$, define $r_{m,n} = \frac{1}{2m}[m + (m^2 + 4mn(n-1))^{\frac{1}{2}}]$. Obviously, we may assume that $\deg_G(v) \geq 1$ for all $v \in V(G)$.

To obtain a bound on the number of cycles of length six in $G$ we introduce a real–valued function in $I\!R^m$ which will provide an upper estimate of $c_6(G)$ and we show that this function attains its maximum on the hypersphere defined by $\sum_{i=1}^{m} \binom{x_i}{2} = \binom{n}{2}$. Then we prove that this maximum occurs at the point $\mathbf{r} = (r_{m,n}, \cdots, r_{m,n})$, where $r_{m,n} = \frac{1}{2m}[m + (m^2 + 4mn(n-1))^{\frac{1}{2}}]$.

Let $\mathbf{I} = [1, n] \subset I\!R$ and $\mathbf{I}^m$ be the Cartesian Product of $m$ copies of $\mathbf{I}$. Let $S_{m,n}$ be the region in $I\!R^m$ defined by

$$S_{m,n} = \left\{ \mathbf{x} \in \mathbf{I}^m \;\middle|\; \sum_{i=1}^{m} \binom{x_i}{2} \leq \binom{n}{2} \right\} .$$

Define the function $F$ on $S_{m,n}$ by $F(\mathbf{x}) = \sum_{i=1}^{m} f(x_i)$ , where $f$ is defined by $f(x) = \frac{1}{3}\binom{x}{2}(n-x)$.

**Lemma 2.1.** *Let $G = G(m, n)$, $m \geq n \geq 4$, be a 4–cycle free bipartite graph. Then $c_6(G) \leq \frac{1}{3} \sum_{i=1}^{m} \binom{x_i}{2}(n - x_i)$.*

**Proof:** Since $G$ is 4–cycle free, each cycle of length six in $G$ uniquely determines a nonintersecting set of three vertices of $V_2(G)$ each 2–element subset of which is intersecting. Conversely, each nonintersecting set of three vertices at least one pair of which is intersecting determines at most one cycle of length six in $G$. We estimate the number of cycles of length six in $G$ by determining in two ways the cardinality of the set of triples

$$T = \{(u_i, v_j, C) \mid u_i \text{ and } v_j \text{ are antipodal vertices in } 6 - \text{cycle } C\} .$$

First, since each cycle $C$ in $G$ contains exactly three distinct pairs of antipodal vertices, there exist $3c_6(G)$ such triples, that is, $|T| = 3c_6(G)$. Note that since

$u_i$ and $v_j$ are antipodal vertices, $d(u_i, v_j) = 3$ for each such $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ and so they are in different partition classes. However, for each $i \in \{1, \ldots, m\}$ there exist at most $n - x_i$ choices for $v_j$ at a distance 3 from $u_i$. Also for each $i \in \{1, \ldots, m\}$ and for each $v_j \in V_2(G) \setminus N(u_i)$ there exist at most $\binom{x_i}{2}$ cycles of length 6 with $u_i$ and $v_j$ antipodal on $C$. Thus $|T| \le \sum_{i=1}^{m} \binom{x_i}{2}(n - x_i)$ and so $c_6(G) \le \frac{1}{3} \sum_{i=1}^{m} \binom{x_i}{2}(n - x_i)$. ■

**Lemma 2.2.** *For $m \ge n \ge 3$, suppose $0 < b < \binom{n}{2}$, and let $\mathbf{x} \in S_{m,n}$ such that $\sum_{i=1}^{m} \binom{x_i}{2} = b$. There exists a vector $\mathbf{x}^* = (x_1^*, \ldots, x_m^*) \in S_{m,n}$ such that $\sum_{i=1}^{m} \binom{x_i^*}{2} > b$ and $F(\mathbf{x}) < F(\mathbf{x}^*)$.*

**Proof:** It is easy to show that $\mathbf{x}$ has at most two coordinates greater than or equal to $\frac{2n}{3}$. Since $n \ge 3$, we may assume $x_1 < \frac{2n}{3}$ (otherwise, relabel). Let $s = \binom{n}{2} - b > 0$. It is possible to choose $t \in (0, s]$ so that $t + \sum_{i=1}^{m} \binom{x_i}{2} \le \binom{n}{2}$ and $a \le \frac{2n}{3}$, where $a$ is defined by $\binom{a}{2} = \binom{x_1}{2} + t$. Define the vector $\mathbf{x}^* = (x_1^*, \ldots, x_m^*)$ by $x_1^* = a$ and $x_i^* = x_i$ for all $i = 2, \ldots, m$. Clearly, $\mathbf{x}^* \in S_{m,n}$. Furthermore, $f(x)$ is monotonic increasing on $[1, \frac{2n}{3}]$. Since $x_1 < x_1^* \le \frac{2n}{3}$, we have $f(x_1) < f(x_1^*)$. Therefore

$$\sum_{i=1}^{m} \binom{x_i}{2}(n - x_i) < \sum_{i=1}^{m} \binom{x_i^*}{2}(n - x_i^*) ,$$

which implies $F(\mathbf{x}) < F(\mathbf{x}^*)$. This proves the result. ■

Lemma 2.2 implies that $F$ attains a maximum value on the boundary of $S_{m,n}$ where $\sum_{i=1}^{m} \binom{x_i}{2} = \binom{n}{2}$. Thus, we may restate the problem as one where we wish to maximize $F(\mathbf{x}) = \sum_{i=1}^{m} f(x_i)$ subject to the constraint

$$\sum_{i=1}^{m} \binom{x_i}{2} = \binom{n}{2} \; ; \;\; 1 \le x_i \le n, \;\; i = 1, \ldots, m \; . \tag{2.1}$$

Define $\partial S_{m,n} = \{\mathbf{x} \in S_{m,n} | \sum_{i=1}^{m} \binom{x_i}{2} = \binom{n}{2}\}$.

**Proof of Theorem 1.** We are going to show that the maximum value of $F$ in the region $S_{m,n}$ is attained at the point $\mathbf{r} = (r_{m,n}, \cdots, r_{m,n})$, and this will prove Theorem 1.

To maximize $F$ one would naturally try a concavity argument. Unfortunately $F$ does not maintain its concavity over the intervals we required it to. Therefore, we will maximize $3F(\mathbf{x}) = \sum_{i=1}^{m} \binom{x_i}{2}(n - x_i)$ in the region $\partial S_{m,n}$ by the method of Lagrange Multipliers. For $\mathbf{x} \in \partial S_{m,n}$ let

$$G(\mathbf{x}) = 3F(\mathbf{x}) - \lambda \left[ \left( \sum_{i=1}^{m} \binom{x_i}{2} \right) - \binom{n}{2} \right] \ .$$

For $i = 1, \ldots, m$,

$$\frac{\partial G}{\partial x_i} = \frac{1}{2}(-3x_i^2 + 2(n+1)x_i - n - \lambda(2x_i - 1)) = 0 \ , \tag{2.2}$$

which implies there are two cases to be considered. For $i, j \in \{1, \ldots, m\}$, $i \neq j$, the equation (2.2) is equivalent to $(x_i - x_j)[-3(x_i + x_j) + 2n + 1 - 2\lambda] = 0$ . Therefore, if $\mathbf{x} = (x_1, \ldots, x_m)$ maximizes $G(\mathbf{x})$ and is not on the boundary of the hypersurface $\partial S_{m,n}$ then either:

(i) $x_i = x_j$ for all $1 \leq i < j \leq n$, or

(ii) there exist some $i, j \in \{1, \ldots, m\}$ such that $x_i + x_j = \frac{2(n+1)-2\lambda}{3}$, $1 \leq i < j \leq m$.

We will consider (ii) first. Without loss of generality, we may assume there exist $i, j \in \{1, \ldots, m\}, i < j$, such that $x_i < x_j$ and $x_i + x_j = \frac{2(n+1)-2\lambda}{3}$. Let

$$g(x) = \frac{1}{2}(-3x^2 + 2(n+1)x - n - \lambda(2x - 1)) \ .$$

Then $g(x_k) = \frac{\partial G}{\partial x_k}$ and $g'(x_k) = \frac{\partial^2 G}{\partial x_k^2}$ for $k = 1, \ldots, m$. Solving $g(x) = 0$ for $\lambda$, we have $\lambda = \frac{-3x^2+2(n+1)x-n}{2x-1}$, therefore, $\lambda$ is strictly decreasing with respect to $x$ on the interval $[1, \infty)$. This implies that $\lambda(x_i) < \lambda(x_j)$ which contradicts the fact that $\lambda$ is independent of $x_i$'s. Thus (ii) cannot occur and we must have that $x_i = x_j$ for all $1 \leq i < j \leq m$.

Therefore, let $x_i = a$ for all $i = 1, \ldots, m$. From equation (2.1) we get $a = \frac{1}{2m}[m + (m^2 + 4mn(n-1))^{\frac{1}{2}}] = r_{m,n}$.

Note that $3r_{m,n} + \lambda(r_{m,n}) = (n+1) + \frac{m+3n(n-1)}{\sqrt{m^2+4mn(n-1)}}$. Therefore, $g'(r_{m,n}) = -(3r_{m,n} + \lambda) + n + 1 = -\frac{m+3n(n-1)}{\sqrt{m^2+4mn(n-1)}} < 0$, which implies $\left.\frac{\partial^2 G}{\partial x_i^2}\right|_{x_i = r_{m,n}} < 0$ for all $i = 1, \ldots, m$. Hence, $\mathbf{r} = (r_{m,n}, \ldots, r_{m,n})$ maximizes $F$ on $\partial S_{m,n}$ except possibly on the boundary of $\partial S_{m,n}$ and $F(\mathbf{r}) = \sum_{i=1}^{m} f(r_{m,n}) = \frac{m}{3}\binom{r_{m,n}}{2}(n - r_{m,n})$.

To show $\mathbf{r}$ maximizes $F$ on $\partial S_{m,n}$ we need only prove that if $\mathbf{x}$ is on the boundary of the hypersurface $\partial S_{m,n}$ then $F(\mathbf{x}) \leq F(\mathbf{r}) = \frac{m}{3}\binom{r_{m,n}}{2}(n - r_{m,n})$. Let $\mathbf{x} = (x_1, \ldots, x_m)$ be on the boundary of $\partial S_{m,n}$. This implies that there exists at least one $i \in \{1, \ldots, m\}$ such that $x_i = 1$. To show that $F(\mathbf{x}) \leq F(\mathbf{r})$ we use induction on the number $k$ of coordinates in $\mathbf{x} = (x_1, \ldots, x_m)$ greater than one.

For $k = 1$ we may assume without loss of generality that $x_1 > 1$ and $x_i = 1$ for $i = 2, \ldots, n$. Then $(x_1, 1, \ldots, 1) \in \partial S_{m,n}$ implies $\binom{n}{2} = \sum_{i=1}^{m} \binom{x_i}{2} = \binom{x_1}{2} + \sum_{i=2}^{n} \binom{1}{2} = \binom{x_1}{2}$ or $x_1 = n$. Then $F[(n, 1, \ldots, 1)] = f(x_1) = \frac{1}{3}\binom{n}{2}(n - n) = 0$. Clearly, $F[(n, 1, \ldots, 1)] \leq F(\mathbf{r})$. Next we assume that $F(\mathbf{x}) \leq F(\mathbf{r})$ holds for all $\mathbf{x} \in \partial S_{m,n}$ which have fewer than $k$ coordinates greater than 1, $1 < k \leq m$. So, suppose there exists $k$, $1 < k < m$ such that $\mathbf{x} = (x_1, \ldots, x_m)$ has $k$ coordinates greater than 1. We may assume without loss of generality that for any such $\mathbf{x}$, $x_i > 1$ for $i = 1, \ldots, k$

and $x_i = 1$ for $i = k+1, \ldots, m$. Since $f(1) = 0$, $F(\mathbf{x}) = \sum_{i=1}^{m} f(x_i) = \sum_{i=1}^{k} f(x_i)$. Also note that $\sum_{i=1}^{m} \binom{x_i}{2} = \sum_{i=1}^{k} \binom{x_i}{2} = \binom{n}{2}$. Therefore, due to Lemma 2.2 we want to maximize $F(\mathbf{x}) = \sum_{i=1}^{k} f(x_i)$ subject to the constraint

$$\sum_{i=1}^{k} \binom{x_i}{2} = \binom{n}{2} \tag{.3}$$

for $\mathbf{x} = (x_1, \ldots, x_m)$ on the boundary of $\partial S_{m,n}$. This again can be done by using the Lagrange Multipliers, and the procedure is absolutely similar to the one outlined above. It leads us to the point $\mathbf{a_k} = (x_1, \ldots, x_m) \in \partial S_{m,n}$ such that $x_i = a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{n(n-1)}{k}}$ for $i = 1, \ldots, k$, and $x_i = 1$ for $i = k+1, \ldots, m$. Then $\mathbf{a_k}$ maximizes $F$ over the region $\sum_{i=1}^{k} \binom{x_i}{2} = \binom{n}{2}$, $x_i \geq 1$, $i = 1, \ldots, k$, with the possible exception of some points on its boundary. But these boundary points have fewer than $k$ coordinates greater than one. Therefore, by the induction hypothesis, the values of $F(\mathbf{x})$ at these points are less than $F(\mathbf{r})$. So, it remains only to compare $F(\mathbf{a_k})$ with $F(\mathbf{r})$. We have

$$F(\mathbf{r}) - F(\mathbf{a_k}) = \frac{n(n-1)}{12}\left[\sqrt{1 + 4\frac{n(n-1)}{k}} - \sqrt{1 + 4\frac{n(n-1)}{m}}\right] > 0$$

for $1 \leq k < m$, and therefore $F(\mathbf{r}) > F(\mathbf{a_k})$. Thus $\mathbf{r} = (r_{m,n}, \ldots, r_{m,n})$ maximizes $F$ on $\partial S_{m,n}$, and due to Lemma 2.2 $\mathbf{r}$ maximizes $F$ on $S_{m,n}$. This completes the proof of Theorem 1. ■

**Section 3: Case of equal partition class sizes**

We are now ready to give a bound on the number of cycles of length six in a 4–cycle free, bipartite graph $G$ with partition classes of equal cardinality. We prove Theorem 2 as a corollary of Theorem 1 where $m = n$.

**Proof of Theorem 2:** Let $G$ have partition classes $V_1(G) = \{u_1, \ldots, u_n\}$, $V_2(G) = \{v_1, \ldots, v_n\}$ and let $\mathbf{r} = (r_n, \ldots, r_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$, where $r_n = r_{n,n} = \frac{1}{2} + \frac{1}{2}\sqrt{4n-3}$ and $x_i = \deg_G(u_i)$, $i = 1, \ldots, n$. Let $F(\mathbf{x}) : S_{n,n} \to I\!R$ be defined as above. Theorem 1 implies

$$c_6(G) \leq F(\mathbf{x}) \leq F(\mathbf{r}) = \frac{n}{3}\binom{r_n}{2}(n - r_n) \ . \tag{3.1}$$

Since $\binom{r_n}{2} = \frac{n-1}{2}$, we may write statement (3.1) as $c_6(G) \leq \frac{n}{3} = \frac{1}{3}\binom{n}{2}(n - r_n)$.

For equality suppose $G_q$ is an incidence point-line graph of a finite projective plane of order $q$. Then $r_n = q + 1$ is the degree of each vertex in $V(G_q)$. To count the number of cycles of length six in $G_q$ note that every 6–cycle in $G_q$ determines a nonintersecting 3–set of vertices from $V_2(G_q)$ such that any pair of vertices from this set is intersecting. Conversely, by definition of $G_q$, every distinct pair of vertices in $V_2(G_q)$ form an intersecting pair, thus every nonintersecting 3–set of vertices from $V_2(G_q)$ determines a unique cycle of length six in $G$. Therefore, for each $i = 1, \ldots, n$, the number of cycles of length six in $G_q$ that pass through $u_i \in V_1(G)$ is $\binom{q+1}{2} \cdot q^2$. Summing over all $i$, the quantity $n\binom{q+1}{2} \cdot q^2$ counts each 6–cycle in $G_q$ exactly three times. Therefore $c_6(G_q) = \frac{q^2}{3}\binom{n}{2} = \frac{1}{3}\binom{n}{2}(n - r_n)$.

Next, suppose

$$c_6(G) = F(\mathbf{x}) = F(\mathbf{r}) = \frac{1}{3}\binom{n}{2}(n - r_n) \tag{3.2}$$

for $G$. Theorem 1 implies that $\mathbf{x} = \mathbf{r}$. Hence, the degree of every vertex in $V_1(G)$ is $r_n$, and $r_n$ is an integer. Therefore, equation (3.2) implies that every nonintersecting 3–set of $V_2(G)$ determines a unique 6–cycle. Hence, for all $1 \leq i < j \leq n$, the set $\{v_i, v_j\} \subseteq V_2(G)$ is intersecting. Similarly, $c_6(G) \leq \frac{1}{3}\sum_{i=1}^{n}\binom{y_i}{2}(n - y_i) \leq \frac{1}{3}\binom{n}{2}(n - r_n)$

implies that for all $1 \leq i < j \leq n$, the set $\{u_i, u_j\} \subseteq V_1(G)$ is intersecting. Therefore, $G$ is $r_n$–regular and $\text{diam}(G) \leq 3$, which implies there exists exactly $n - r_n$ vertices in $V_2(G)$ of distance 3 from $u_i \in V_1(G)$ for each $i = 1, \ldots, n$. Therefore, if $q$ is an integer such that $r_n = q + 1$, then $G$ is an incidence point-line graph of a finite projective plane of order $q$. ■

Below we sketch another proof of Theorem 2 which is independent of Theorem 1 and which is based on a recent article [7]. The following Theorem A is a particular case of a more general result proved in [7], but it will suffice for our needs.

**Theorem A.** *Let $G$ be a 2–connected graph on $v$ vertices, with $e$ edges, and girth 6. Then $c_6(G) \leq \frac{e}{6}(e - v + 1)$, and the equality holds if and only if every two edges of $G$ are contained in a common 6–cycle.* ■

**Alternate Proof of Theorem 2.** Let $G$ be 2–connected and let $v = 2n$. It is known (see [2, ch. VI.2]) that for a bipartite 4–cycle free graph $G$, $e(G) \leq nr_n$, and $e(G) = nr_n$ if and only if $G$ is the point–line incidence graph of a projective plane. Therefore, by Theorem A,

$$c_6(G) \leq \frac{e}{6}(e - v + 1) \leq \frac{nr_n}{6}(nr_n - 2n + 1) = \frac{1}{3}\binom{n}{2}(n - r_n).$$

The equality in the second inequality holds if and only if $e(G) = nr_n$, and therefore, $G$ must be the incidence point–line graph of a projective plane. But it is obvious that such a graph has any pair of edges belonging to a 6–cycle. This proves Theorem 2 for 2–connected graphs.

Suppose $G$ is not 2–connected. Then it is either disconnected or 1-connected, and the inequality of Theorem 2 can be extended to such graphs. The proof we have is straightforward but rather long and we refer the reader to [3]. For such graphs the equality is never achieved if $n \geq 4$. ■

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