On a bound for the maximum number of C_8's in a 4–cycle free bipartite graph

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Abstract. Let G be a 4–cycle free bipartite graph on $2n$ vertices with partitions of equal cardinality n having e edges. Let $c_8(G)$ denote the number of cycles of length 8 in G. We prove that for $n \geq 4$, $c_8(G) \leq 3\left[\binom{n}{4} - \frac{n}{4!}f(\frac{e}{n})\right]$, where $f(t) = t(t-1)(t-2)(4n-3-3t)$. If G is extremal with respect to the number of 8–cycles, then $r_n - 2 < \frac{e}{n} \leq r_n$, where $r_n = \frac{1}{2} + \frac{\sqrt{4n-3}}{2}$. This implies that $c_8(G) \leq 3\left[\binom{n}{4} - \frac{n}{4!}f(r_n - 2)\right]$. Furthermore, if G_q is the incidence point–line graph of a finite projective plane of order q, and $n_q = q^2 + q + 1$, then $c_8(G_q) = \binom{n_q}{2}\binom{q}{2}^2 = 3\left[\binom{n_q}{4} - \frac{n_q}{4!}f(r_{n_q})\right]$, and G_q is "close" to being extremal in this sense.

Section 1: Introduction.

Let $\mathcal{G} = \mathcal{G}_n$ denote a family of simple graphs of order n. For a simple graph H and $G \in \mathcal{G}$, let (G, H) denote the number of subgraphs of G isomorphic to H. Let $h(n) = h(\mathcal{G}, H, n) = \max\{(G, H) | G \in \mathcal{G}\}$ and $\mathcal{G}(H, n) = \{G \in \mathcal{G} | (G, H) = h(n)\}$. We will refer to graphs of $\mathcal{G}(H, n)$ as *extremal*. The problem of finding $h(\mathcal{G}, H, n)$ and $\mathcal{G}(H, n)$, for fixed $\mathcal{G}, H, n$, has been studied extensively and is considered as central in extremal graph theory. Though it is hopeless in whole generality, some of its instances have been solved. Often the results are concerned with bounds on h_n and partial description of the extremal graphs. For example, if K_m denotes the complete graph of order m, $H = K_2$, and $\mathcal{G}$ is the family of all graphs of order n which contain no K_m as a subgraph, $3 \leq m \leq n$, then the solution is given by the famous Turán Theorem. For the same H, if $K_{s,t}$ denotes the complete bipartite graph with partition class sizes s, t and $\mathcal{G}$ is the family of all (m, n)–bipartite graphs with no $K_{s,t}$, we have the, so called, Zarankiewicz problem. These and many other examples can be found in [2]. For some later results see [4,5,6].

All missing definitions can be found in [2]. Let $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, $e = e(G) = |E(G)|$. The neighborhood of a vertex $v \in V(G)$ is denoted by $N(v)$ $(v \notin N(v))$, and the degree of vertex v in G by $\deg_G(v)$. For $S \subseteq V(G)$, define $N(S)$ by $N(S) = \bigcup_{v \in S} N(v)$. If G contains a cycle, the *girth* of G is the length of a shortest cycle in G. For a positive integer n, $n \geq 2$, let $\mathcal{G}_6(n,n)$ be the class of bipartite graph on $2n$ vertices with partitions of equal cardinality n and girth at least 6 (i.e. 4–cycle free). Let $G \in \mathcal{G}_6(n,n)$ have partition $(V_1(G), V_2(G))$ such that $V_1(G) = \{u_1, \ldots, u_n\}$, $V_2(G) = \{v_1, \ldots, v_n\}$. Let $x_i = \deg_G(u_i)$, $i = 1, \ldots, n$, and $y_i = \deg_G(v_i)$, $i = 1, \ldots, n$. A subset, $\{u_{i_1}, \ldots, u_{i_k}\}$, $2 \leq k \leq n$, of $V_1(G)$ (or $\{v_{i_1}, \ldots, v_{i_k}\}$ of $V_2(G)$) is said to be *intersecting* if $N(u_{i_1}) \cap \ldots \cap N(u_{i_k}) \neq \emptyset$ (or $N(v_{i_1}) \cap \ldots \cap N(v_{i_k}) \neq \emptyset$). Let a projective plane π_q of order q exist and $n_q = q^2 + q + 1$. Let $P = \{p_1, \ldots, p_n\}$ and $L = \{l_1, \ldots, l_n\}$ be the point set and the line set of π_q, respectively. A bipartite graph G_q with partition (P, L) is said to be the *incidence point-line graph of the projective plane* π_q if for all $i, j \in \{1, \ldots, n\}$, $\{p_i, l_j\}$ is an edge of G if and only if $p_i \in l_j$.

Let $c_8(G)$ denote the number of 8–cycles in G. The main goal of this paper is to find a nontrivial upper bound for $c_8(G)$, where $G \in \mathcal{G}_6(n,n)$. The results are summarized below.

Theorem 1. *Let $G \in \mathcal{G}_6(n,n)$ be a 4–cycle free bipartite graph on $2n$ vertices with partition classes of size n. Then*

(i) $c_8(G) \leq 3\left[\binom{n}{4} - \frac{n}{4!} f(r_n - 2)\right]$, *where* $f(t) = t(t-1)(t-2)(4n-3-3t)$, $r_n = \frac{1}{2} + \frac{\sqrt{4n-3}}{2}$, *and* $n \geq 4$.

(ii) If G has e edges and is extremal with respect to the number of 8–cycles, then $r_n - 2 < \frac{e}{n} \leq r_n$, with r_n as in (i) above.

Using the terminology of finite geometries (see [1]), Theorem 1 provides an upper bound for the number of quadrilaterals in near–linear spaces with n points and n lines. To prove Theorem 1 we will rely on the following additional facts.

Lemma A. ([7]) *Let G be a connected graph with $\delta(G) \geq 2$ and H be a subgraph of G. Then $|V(G) \setminus V(H)| \leq |E(G) \setminus E(H)|$, with equality if and only if $H = G$.* ■

Theorem B.([2]) *Suppose G is a 4–cycle free bipartite graph with partition classes of cardinality n, $n \geq 2$. Let $e = e(G)$ be the size of G. Then $e \leq \frac{n}{2} + \frac{n}{2}\sqrt{4n-3}$ with equality if and only if $n = n_q = q^2 + q + 1$ and $G = G_q$ – the point–line incidence graph of a finite projective plane of order q.* ■

Thus G_q has the greatest number of edges among all graphs in $\mathcal{G}_6(n_q, n_q)$. In [3] we also showed that G_q has greatest number of 6–cycles among all graphs in $\mathcal{G}_6(n_q, n_q)$. Theorem 1 above grew out of our attempts to prove that G_q also has the greatest number of 8–cycles among all graphs in $\mathcal{G}_6(n_q, n_q)$.

Section 2: Proof of Theorem 1.

Let $G \in \mathcal{G}_6(n,n)$ with partition classes $V_1(G) = \{u_1, \ldots u_n\}$, $V_2(G) = \{v_1, \ldots, v_n\}$. To count the number of 8–cycles in G note that each 8–cycle in G determines a 4–set of vertices $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\} \subseteq V_2(G)$ such that no three of the vertices form an intersecting 3–set. Furthermore, each 8–cycle identifies two distinct partitions of $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$ into pairs of intersecting 2–sets. That is, if an 8–cycle is identified by the sequence $v_{i_1}u_{i_1}v_{i_2}u_{i_2}v_{i_3}u_{i_3}v_{i_4}u_{i_4}v_{i_1}$, then $\{\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\}\}$ and $\{\{v_{i_1}, v_{i_4}\}, \{v_{i_2}, v_{i_3}\}\}$ are two distinct partitions of $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$ into pairs of intersecting 2–sets of vertices.

Conversely, it is obvious that each 4–set of vertices $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\} \subseteq V_2(G)$ can be partitioned in at most three ways into pairs of intersecting 2–sets. This implies that the set $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\} \subseteq V_2(G)$ can be contained in the vertex sets of at most $\binom{3}{2} = 3$ cycles of length eight in G.

Clearly, the number of intersecting 4–sets in $V_2(G)$ is $\sum_{i=1}^{n} \binom{\deg_G(u_i)}{4} = \sum_{i=1}^{n} \binom{x_i}{4}$. Also, the number of 4–sets in $V_2(G)$ such that exactly three of the four vertices form an intersecting 3–set of vertices is $\sum_{i=1}^{n} \binom{\deg_G(u_i)}{3}(n - \deg_G(u_i)) = \sum_{i=1}^{n} \binom{x_i}{3}(n - x_i)$. Therefore, the number of 4–sets in $V_2(G)$ such that no three vertices form an intersecting 3–set is

$$\binom{n}{4} - \sum_{i=1}^{n} \left[\binom{x_i}{4} + \binom{x_i}{3}(n - x_i) \right] = \binom{n}{4} - \frac{1}{4!} \sum_{i=1}^{n} x_i(x_i-1)(x_i-2)(4n-3-3x_i) .$$

Hence, we have proved the following

Lemma 2.1. *For $n \geq 2$, let $G \in \mathcal{G}_6(n,n)$ with $V_1(G) = \{u_1, \ldots, u_n\}$ and $x_i = \deg_G(u_i)$, $i = 1, \ldots, n$. Then*

$$c_8(G) \leq 3 \left[\binom{n}{4} - \frac{1}{4!} \sum_{i=1}^{n} f(x_i) \right]$$

where $f(t) = t(t-1)(t-2)(4n-3-3t)$. Moreover, equality holds if every 4–set of $V_2(G)$ for which no subset of three vertices is intersecting determines exactly three 8–cycles of G. ■

Since every pair of vertices in $V_2(G_q)$ is an intersecting pair of vertices, it is easy to see that every 4–set of vertices in $V_2(G_q)$ no three vertices of which form an intersecting set determines exactly three 8–cycles of G_q. Therefore, Lemma 2.1 implies that

$$c_8(G_q) = 3\left[\binom{n_q}{4} - \frac{n_q}{4} f(q+1)\right] = \binom{n_q}{2}\binom{q}{2}^2 = \binom{q^2+q+1}{2}\binom{q}{2}^2 . \qquad (2.2)$$

Equation (2.2) provides us with a lower estimate of the upper bound on the max $c_8(G)$, $G \in \mathcal{G}_6(n_q, n_q)$.

In our next step, we find an upper bound on $c_8(G)$ in terms of $e = e(G)$ and n. Define the function $F(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_n) \in I\!R^n$, by $F(\mathbf{x}) = \sum_{i=1}^{n} f(x_i)$, where $f(t) = t(t-1)(t-2)(4n-3-3t), t \in [1, n]$.

Theorem 2.2. *Let $G \in \mathcal{G}_6(n,n), n \geq 4$. Then $c_8(G) \leq 3\left[\binom{n}{4} - \frac{n}{4!} f\left(\frac{e}{n}\right)\right]$, where $\overline{\mathbf{x}} = (\frac{e}{n}, \ldots, \frac{e}{n})$.*

Proof: Let $G \in \mathcal{G}_6(n)$ have partition classes $V_1(G) = \{u_1, \ldots, u_n\}$, $V_2(G) = \{v_1, \ldots, v_n\}$. It is easy to show that for all $n \geq 2$, there exist real values $\alpha = \alpha(n)$, $\beta = \beta(n)$, $0 < \alpha < 1$, $\frac{2n}{3} < \beta < \frac{2n+1}{3}$, such that $f''(t) = 6(-6t^2+(4n+6)t+1-4n) > 0$ for $t \in (\alpha, \beta)$. This implies $f(t)$ is concave up on the interval (α, β). We consider two cases:

Case 1: $\alpha < x_i < \beta$ for all $i = 1, \dots, n$.

Since f is concave up on (α, β) and $\alpha < x_i < \beta$, $i = 1, \dots, n$, Jensen's inequality implies $F(\mathbf{x}) = \sum_{i=1}^{n} f(x_i) \geq n \cdot f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = nf(\frac{e}{n}) = F(\overline{\mathbf{x}})$. Therefore

$$c_8(G) \leq 3\left[\binom{n}{4} - \frac{1}{4!}F(\overline{\mathbf{x}})\right] = 3\left[\binom{n}{4} - \frac{n}{4!}f(\frac{e}{n})\right] \quad .$$

Case 2: $x_i > \beta$ for some $i \in \{1, \dots, n\}$.

Suppose G has a vertex u of degree at least β and, relabeling if necessary, assume $u = u_n$. If there exists $u_i \in V_1(G)$, $i \neq n$, such that $\deg_G(u_i) \geq \beta$, then $|N(u_n) \cap N(u_i)| > 1$, which means that G contains a 4–cycle, contradiction. Therefore, Case 2 is equivalent to $x_n > \beta$ and $\alpha < x_i < \beta$ for $i = 1, \dots, n-1$. We will consider two subcases: $x_n = n$ and $x_n \leq n - 1$.

First, suppose $x_n = n$. Since G is 4–cycle free, $\deg_G(u_n) = n$ implies $\deg_G(u_i) \leq 1$ for all $u_i \in V_1(G)$, $i = 1, \dots, n-1$. Therefore $c_8(G) = 0$ and, trivially, we have that $c_8(G) = 0 \leq 3\left[\binom{n}{4} - \frac{n}{4!}f(\frac{e}{n})\right]$ and the result is proved.

So, suppose $\beta \leq x_n \leq n - 1$. For all $n \geq 2$, there exist real values $\eta = \eta(n)$ and $\zeta = \zeta(n)$, $1 < \eta < 2$, $n-1 < \zeta < n$, such that $f'(t) = 2(-6t^3 + (6n+9)t^2 + (3 - 12n)t + 4n - 3) > 0$ for $t \in (\eta, \zeta)$. This implies f is strictly increasing on the interval (η, ζ). Hence, $\frac{2n}{3} < \beta \leq x_n < n$ implies $f(x_n) > f\left(\frac{2n}{3}\right)$, therefore $F(\mathbf{x}) = \sum_{i=1}^{n} f(x_i) > \sum_{i=1}^{n-1} f(x_i) + f\left(\frac{2n}{3}\right)$. However, $f(1) = f(2) = 0$ and $f(t) > 0$ for $2 < t < \beta$. Therefore $f(x_i) \geq 0$ for $i = 1, \dots, n-1$ and so $\sum_{i=1}^{n-1} f(x_i) \geq 0$. Hence $F(\mathbf{x}) > f\left(\frac{2n}{3}\right)$ and so we have,

$$c_8(G) \leq 3\left[\binom{n}{4} - \frac{1}{4!}F(\mathbf{x})\right] < 3\left[\binom{n}{4} - \frac{1}{4!}f\left(\frac{2n}{3}\right)\right] \quad . \tag{2.3}$$

Let $r_n = \frac{1}{2} + \frac{1}{2}\sqrt{4n-3}$. Theorem B implies $\frac{e}{n} \le r_n$; thus $f(\frac{e}{n}) \le f(r_n)$ and so $nf(\frac{e}{n}) = F(\overline{\mathbf{x}}) \le F(\mathbf{r}) = nf(r_n)$ where $\mathbf{r} = (r_n, \ldots, r_n)$. However, $f\left(\frac{2n}{3}\right) - nf(r_n) = \frac{n}{27}(54n(n-1)\sqrt{4n-3} - 16n^3 - 147n^2 + 306n - 135) > 0$, which implies $f\left(\frac{2n}{3}\right) > F(\overline{\mathbf{x}}) = nf(\frac{e}{n})$. Therefore, inequality (2.3) becomes

$$c_8(G) < 3\left[\binom{n}{4} - \frac{1}{4!}nf\left(\frac{e}{n}\right)\right]$$

and the result is proved. ■

Clearly, $F(\overline{\mathbf{x}}) = nf(\frac{e}{n})$ depends on $e = e(G)$. Since f is strictly increasing on the interval (η, ζ), this implies that the bound in Theorem 2.2 holds for small values of e which would produce a trivial upper bound for $c_8(G)$. This problem could be resolved by giving a lower bound on e in terms of n. Indeed, we would like to determine a bound on $c_8(G)$ that depends only on n, the number of vertices in a partition class of G. When $n_q = q^2 + q + 1$, equation (2.2) and inequality (2.3) imply that any bound on $\max\limits_{G \in \mathcal{G}_6(n_q, n_q)} c_8(G)$ must be at least $\binom{n_q}{2}\binom{r_{n_q}-1}{2}^2$. With this in mind we can determine an appropriate range for e.

Lemma 2.3. *For $n \ge 2$, suppose $G \in \mathcal{G}_6(n,n)$ satisfies $c_8(G) \ge \binom{n}{2}\binom{r_n-1}{2}^2$, where $r_n = \frac{1}{2} + \frac{1}{2}\frac{\sqrt{4n-3}}{2}$. Then $r_n - 2 < \frac{e}{n} \le r_n$.*

Proof: Theorem B implies $\frac{e}{n} \le r_n$. Here we show that $\frac{e}{n} > r_n - 2$.

Let $\mathcal{P}(G)$ represent the set of all paths of length three in G. Define the function g by $g(x) = x(n-x)$. Since G is a bipartite graph, the distance between $v \in N(u_i)$ and $v' \in V_2(G) \setminus N(u_i)$ is at least 2. If the distance is exactly 2, since G has girth at least six, the path of length 2 joining v to v' is unique. Therefore, the path from v' to u_i through v is unique and of length 3. This implies there are $x_i(n-x_i)$ paths of length 3 with u_i as one end vertex of those paths. Therefore $|\mathcal{P}(G)| \leq \sum_{i=1}^{n} g(x_i)$, and since g is convex on the interval $(-\infty, \infty)$, by Jensen's inequality, we have

$$|\mathcal{P}(G)| \leq \sum_{i=1}^{n} g(x_i) \leq ng(\frac{e}{n}) \ , \tag{2.4}$$

where $e = \sum_{i=1}^{n} x_i$. Since g is increasing on $\left[1, \frac{n}{2}\right]$ and $\frac{e}{n} \leq r_n \leq \frac{n}{2}$ for $n \geq 2$, inequality (2.4) implies

$$|\mathcal{P}(G)| \leq ngr_n = 2(r_n - 1)\binom{n}{2} \ . \tag{2.5}$$

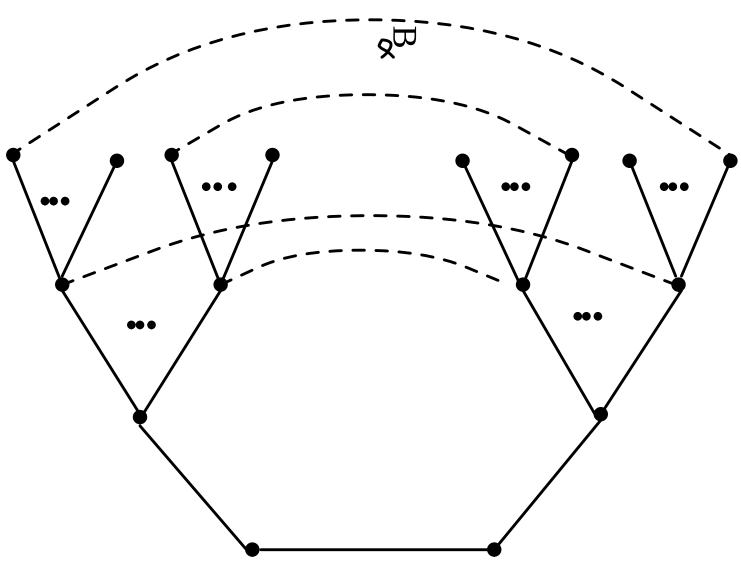

Figure 2.1

Let $\alpha = u'vuv' \in \mathcal{P}(G)$. Let $S = N(u') \setminus \{v\}$, $T = N(v') \setminus \{u\}$. At this point we make use of a construction (see Figure 2.1) similar to one which appears in the proof of Lemma A in [7] and which allows us to estimate the number of 8–cycles passing through α. We describe it presently.

Let $V_\alpha = N(u') \cup N(v') \cup N(S) \cup N(T)$. Define $G_\alpha = G[V_\alpha]$ to be the subgraph of G induced by the vertex set V_α. Let $q_\alpha = |E(G) \setminus E(G_\alpha)|$ and $p_\alpha = |V(G) \setminus V(G_\alpha)|$. Lemma A implies $q_\alpha - p_\alpha \geq 0$.

Let $A_\alpha = \{\{u_i, v_j\} \in E(G) \mid u_i \in T \text{ and } v_j \in S\}$ and $B_\alpha = \{\{u_i, v_j\} \in E(G) \mid u_i \in N(T) \text{ and } v_j \in N(S)\}$. Define $G_\alpha - (A_\alpha \cup B_\alpha)$ as the graph obtained from G_α by removing the edges of $A_\alpha \cup B_\alpha$. Note that the graph $G_\alpha - (A_\alpha \cup B_\alpha)$ is a tree with $2n - p_\alpha$ vertices, where $|A_\alpha|$ is the number of 6–cycles through α and $|B_\alpha|$ is the number of 8–cycles through α. Therefore $e = |E(G) \setminus E(G_\alpha)| + |E(G_\alpha)| = q_\alpha + 2n - p_\alpha - 1 + |A_\alpha| + |B_\alpha|$, which implies for each $\alpha \in \mathcal{P}(G)$,

$$e - 2n + 1 \geq |B_\alpha| \ . \tag{2.6}$$

Now, summing both sides of (2.6) over all paths of length three, the left hand side expression becomes

$$\sum_{\alpha \in \mathcal{P}(G)} (e - 2n + 1) = |\mathcal{P}(G)|(e - 2n + 1) \leq 2(e - 2n + 1)(r_n - 1)\binom{n}{2} \ , \tag{2.7a}$$

and the right hand side expression becomes

$$\sum_{\alpha \in \mathcal{P}(G)} |B_\alpha| = 8c_8(G) \geq 8\binom{n}{2}\binom{r_n - 1}{2}^2 \ . \tag{2.7b}$$

Therefore, combining both expressions we have $2(e-2n+1)(r_n-1)\binom{n}{2} \geq 8\binom{n}{2}\binom{r_n-1}{2}^2$. Solving this inequality for e we see that $e \geq r_n^3 - 3r_n^2 + 6r_n - 3 > r_n^3 - 3r_n^2 + 3r_n - 2 = (r_n - 2)(r^2 - r + 1)$, which implies $\frac{e}{n} > r_n - 2$. ■

We are now ready to prove the first statement of Theorem 1.

Proof of Theorem 1, part (i): If $c_8(G) \leq \binom{n}{2}\binom{r_n-1}{2}^2$, then clearly $\binom{n}{2}\binom{r_n-1}{2}^2 < 3\left[\binom{n}{4} - \frac{n}{4!}f(r_n-2)\right]$ and the result is proved. If $c_8(G) > \binom{n}{2}\binom{r_n-1}{2}^2$, Theorem 2.2 implies that $c_8(G) \leq 3\left[\binom{n}{4} - \frac{n}{4!}f(\frac{e}{n})\right]$. Since f is strictly increasing on the interval (η, ζ), where $1 < \eta < 2$ and $n-1 < \zeta < n$, Lemma 2.3 implies $f(\frac{e}{n}) > f(r_n - 2)$ and so $c_8(G) < 3\left[\binom{n}{4} - \frac{n}{4!}f(r_n-2)\right]$ and the result is proved. ∎

We make a final comment on the lower bound on $\frac{e}{n}$ appearing in Lemma 2.3. Central to the proof of Lemma 2.3 is the calculation of $|\mathcal{P}(G)|$, the number of paths of length three in $G \in \mathcal{G}_6(n)$. Clearly, there exists $\alpha \in \mathcal{P}(G)$ for which $q_\alpha - p_\alpha > 0$ and $|A_\alpha| > 0$. Hence, it would seem that if we could find functions ϕ and ψ such that $\sum_{\alpha\in\mathcal{P}(G)} (q_\alpha - p_\alpha) > \phi(n)$ and $\sum_{\alpha\in\mathcal{P}(G)} |A_\alpha| > \psi(n)$, then $\phi(n)$ and $\psi(n)$ could be inserted into the inequalities (2.6), (2.7a) and (2.7b) to produce a tighter bound on $c_8(G)$. However, we have been unable to find such functions so far. This would imply that the incidence point–line graph of order q indeed is the best candidate for the extremal graphs for those values of n for which it exists. However, such a proof would require another technique.

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