New Upper Bounds for the Greatest Number of Proper Colorings of a (V, E)-Graph*

Felix Lazebnik
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE

ABSTRACT

Let $\mathcal{F}$ denote the family of simple undirected graphs on v vertices having e edges ((v, e)-graphs) and $P(G; \lambda)$ be the chromatic polynomial of a graph G. For the given integers v, e, and λ, let $f(v, e, \lambda)$ denote the greatest number of proper colorings in λ or less colors that a (v, e)-graph G can have, i.e., $f(v, e, \lambda) = \max\{P(G; \lambda)\colon G \in \mathcal{F}\}$. In this paper we determine some new upper bounds for $f(v, e, \lambda)$.

1. INTRODUCTION

The definitions in this paper are based on [3]. All graphs we consider are undirected labeled graphs without loops and multiple edges. Let $V(G)$ and $E(G)$ denote a set of vertices and edges of G, respectively. The number of elements of a finite set A is denoted by $|A|$. We write $v = v(G) = |V(G)|$ and $e = e(G) = |E(G)|$. By $c = c(G)$ we denote the number of connected components of graph G. For any positive integer λ, *a proper λ-coloring* of a labeled graph G is a mapping of $V(G)$ into the set $\{1, 2, \ldots, \lambda\}$ (the set of colors) such that no two adjacent vertices of G have the same image. The *chromatic number* of a graph G, denoted $\chi(G)$, is the least λ (number of colors) for which there exists a proper coloring of G. Let $P(\lambda) = P(G; \lambda)$ denote the number of proper λ-colorings of G. This function was introduced in [2] and turned out to be a polynomial function of λ.

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Let $\mathcal{F} = \mathcal{F}_{v,e}$ be a family of all graphs having v vertices and e edges ((v, e)-graphs). Let λ be the number of colors. Denote by $f(v, e, \lambda)$ the greatest number of proper λ-colorings that a (v, e)-graph can have, i.e., $f(v, e, \lambda) = \max\{P(H; \lambda): H \in \mathcal{F}\}$. In this paper we find some new nontrivial upper bounds for $f(v, e, \lambda)$ in the general case, i.e., in the case when the only restrictions on the integers v, e, and λ are $0 \leq e \leq v(v - 1)/2$, $\lambda \geq 2$.

The main result is the following:

Theorem 1.1. Let v, e, and λ be integers, $0 \leq e \leq v(v - 1)/2$, $\lambda \geq 2$. Let $f(v, e, \lambda)$ be equal to the greatest number of proper λ-colorings of a graph with v vertices and e edges. Then

$$f(v, e, \lambda) \leq A\lambda^v,$$

where A is the least of the following three quantities:

$$\left(1 - \frac{1}{\lambda}\right)^{\lceil 1/2(\sqrt{1+8e}-1)\rceil}, \qquad \left(1 - \frac{e}{\lambda} + \left(\binom{e}{2}\Big/\lambda^2\right)\right), \qquad \frac{\lambda - 1}{\lambda - 1 + e}.$$

The question was motivated by the analysis of the running time of the backtrack algorithm for the graph coloring problem (see Wilf [10]; Bender and Wilf [1]). Another source of related problems is a paper of Wright [11], where an asymptotic approximation to the number $M_{v,e}$ = the total number of proper λ-colorings of all (v, e)-graphs was found for a fixed λ, large v, and all e. Problems similar to ours but for different families of graphs (all graphs on v vertices whose chromatic number is equal to k) were considered by Tomescu [7,8]. Several other instances of the problem were considered by the author in [4], [5].

2. PROOF OF THEOREM 1.1.

The inequality

$$f(v, e, \lambda) \leq \left(1 - \frac{1}{\lambda}\right)^{\lceil 1/2(\sqrt{1+8e}-1)\rceil} \lambda^v \tag{2.1}$$

was proved in [4]. In order to get other upper bounds, we apply some known facts based on the famous Inclusion–Exclusion Principle, or Sieve Method, and the corresponding interpretation of the chromatic polynomial of a graph due to Whitney [9]. Here we give a brief list of the corresponding facts. All proofs can be found in Lovasz [6, III. §2]).

Proposition 2.1.

(i) (Inclusion–Exclusion Formula). Let $A_1, \ldots, A_n$ be arbitrary events of a probability space (Ω, P). For each $I \subseteq \{1, \ldots, n\}$, let

$$A_I = \prod_{i \in I} A_i; \qquad A_\phi = \Omega;$$

and let

$$\sigma_k = \sum_{|I|=k} P(A_I), \qquad \sigma_0 = 1.$$

Then

$$P(\overline{A}_1 \overline{A}_2 \ldots \overline{A}_n) = \sum_{i=0}^{n} (-1)^i \sigma_i. \tag{2.2}$$

(ii) (Bonferroni Inequalities). The partial sums of

$$P(\overline{A}_1 \ldots \overline{A}_n) - \sigma_0 + \sigma_1 - \sigma_2 + \ldots. \tag{2.3}$$

are alternating in sign.

(iii) (Seiberg's Sieve, particular case). If the events A_i, $1 \leq i \leq n$, are pairwise independent and $P(A_i) = p$ for all i, $1 \leq i \leq n$, then

$$P(\overline{A}_1 \ldots \overline{A}_n) \leq \frac{1}{1 + np/(1-p)} \tag{2.4}$$

Given a labeled graph G with v vertices and e edges and an integer $\lambda \geq 1$, we associate with G the sample space Ω as follows:

$$\Omega = \{\text{all (proper and improper) colorings of } G \text{ in } \lambda \text{ colors}\}.$$

Let $V(G) = \{1, 2, \ldots, v\}$ and $E(G) = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_e, j_e\}\}$. For each k, $1 \leq k \leq e$, we define the event A_k as

$$A_k = \{w \in \Omega \mid \text{vertices } i_k \text{ and } j_k \text{ are colored in the same color}\}.$$

Obviously, $|\Omega| = \lambda^v$ and for each k, $1 \leq k \leq e$, $|A_k| = \lambda \cdot \lambda^{v-2} = \lambda^{v-1}$ (a color for i_k and j_k can be chosen in λ different ways and each of the remaining $v - 2$ vertices can be colored in λ colors independently of each other).

We define a probability function P on Ω in the usual way, assuming that the probability of each elementary event is equal to $1/|\Omega|$. Then

$$P(A_k) = \lambda^{v-1}/\lambda^v = 1/\lambda, \qquad 1 \leq k \leq e. \tag{2.5}$$

A coloring of G is proper if and only if none of the events A_k, $1 \leq k \leq e$, happens. Therefore

$$P(G; \lambda) = P(\overline{A}_1 \ldots \overline{A}_e) \cdot \lambda^v$$

and the results of Proposition 2.1 can be used in order to get upper bounds for $P(G;\lambda)$. But the following questions should be answered first: what is the probability of the intersection of two events A_k and A_s for $k \neq s$, and are they pairwise independent?

The event $A_k \cdot A_s$ takes place if and only if vertices i_k, j_k are colored in the same color and vertices i_s, j_s are colored in the same color. Therefore

$$|A_k \cdot A_s| = \begin{cases} \lambda \cdot \lambda \cdot \lambda^{v-4} = \lambda^{v-2}, & \text{if } \{i_k;j_k\} \cap \{i_s;j_s\} = \phi \\ \lambda \cdot \lambda^{v-3} = \lambda^{v-2}, & \text{if } |\{i_k;j_k\} \cap \{i_s;j_s\}| = 1 . \end{cases}$$

Thus

$$|A_k \cdot A_s| = \lambda^{v-2} \quad \text{and} \quad P(A_k \cdot A_s) = \lambda^{v-2}/\lambda^v = 1/\lambda^2 \tag{2.6}$$

for all $k, s,\ 1 \leq k < s \leq v$. Using (2.5) and (2.6) we get

$$P(A_k \cdot A_s) = P(A_k) \cdot P(A_s) ,$$

which implies the *independence* of A_k and A_s for all $k, s,\ 1 \leq k < s \leq v$.

For our model,

$$P(A_i) = \frac{1}{\lambda}, \qquad \sigma_1 = \sum_{i=1}^{e} P(A_i) = \frac{e}{\lambda},$$

$$\sigma_2 = \sum_{1 \leq i < j \leq e} P(A_i \cdot A_j) = \frac{e(e-1)}{2\lambda^2}.$$

Using this and multiplying both sides of (2.3) and (2.4) by $|\Omega|$, we obtain two upper bounds for $P(G;\lambda)$ and $f(v,e,\lambda)$:
from (2.3):

$$P(G;\lambda) \leq f(v,e,\lambda) < \left(1 - \frac{e}{\lambda} + \left(\binom{e}{2}\Big/\lambda^2\right)\right)\lambda^v \tag{2.7}$$

from (2.5):

$$P(G;\lambda) \leq f(v,e,\lambda) < \frac{\lambda - 1}{\lambda - 1 + e}\lambda^v. \tag{2.8}$$

Combining (2.1), (2.7), and (2.8), we obtain a proof of Theorem 1.1. ∎

For some ranges of parameters, the comparison of (2.1), (2.7), and (2.8) is simple, and it gives

$$\text{for} \quad e = 0, 1, \lambda + 1 ,$$

the right sides of (2.7) and (2.8) are equal;

$$\text{for} \quad e > \max\left\{\lambda + 1, (\lambda - 1)\left[\left(\frac{\lambda}{\lambda - 1}\right)^{v-1} - 1\right]\right\},$$

the bound given by (2.8) is better than those given by (2.1) and (2.7);

$$\text{for} \quad v - 1 \leq e \leq \lambda + 1 \,,$$

the bound given by (2.1) is better than ones in (2.7) and (2.8).

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