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ABSTRACT

Let \mathscr{F} denote the family of simple undirected graphs on v vertices having e edges ((v, e)-graphs) and $P(G; \lambda)$ be the chromatic polynomial of a graph G. For the given integers v, e, and λ , let $f(v, e, \lambda)$ denote the greatest number of proper colorings in λ or less colors that a (v, e)-graph G can have, i.e., $f(v, e, \lambda) = \max\{P(G; \lambda): G \in \mathscr{F}\}$. In this paper we determine some new upper bounds for $f(v, e, \lambda)$.

1. INTRODUCTION

The definitions in this paper are based on [3]. All graphs we consider are undirected labeled graphs without loops and multiple edges. Let V(G) and E(G) denote a set of vertices and edges of G, respectively. The number of elements of a finite set A is denoted by |A|. We write v = v(G) = |V(G)| and e = e(G) = |E(G)|. By c = c(G) we denote the number of connected components of graph G. For any positive integer λ , a proper λ -coloring of a labeled graph G is a mapping of V(G) into the set $\{1, 2, \ldots, \lambda\}$ (the set of colors) such that no two adjacent vertices of G have the same image. The chromatic number of a graph G, denoted $\chi(G)$, is the least λ (number of colors) for which there exists a proper coloring of G. Let $P(\lambda) = P(G; \lambda)$ denote the number of proper λ -colorings of G. This function was introduced in [2] and turned out to be a polynomial function of λ .

Journal of Graph Theory, Vol. 14, No. 1, 25–29 (1990) © 1990 by John Wiley & Sons, Inc. CCC 0364-9024/90/010025-05\$04.00

^{*}This paper is based on a part of a Ph.D. thesis written by the author under the supervision of Prof. H.S. Wilf at the University of Pennsylvania.

Let $\mathscr{F}=\mathscr{F}_{v,e}$ be a family of all graphs having v vertices and e edges ((v,e)-graphs). Let λ be the number of colors. Denote by $f(v,e,\lambda)$ the greatest number of proper λ -colorings that a (v,e)-graph can have, i.e., $f(v,e,\lambda)=\max\{P(H;\lambda): H\in\mathscr{F}\}$. In this paper we find some new nontrivial upper bounds for $f(v,e,\lambda)$ in the general case, i.e., in the case when the only restrictions on the integers v,e, and λ are $0 \le e \le v(v-1)/2$, $\lambda \ge 2$.

The main result is the following:

Theorem 1.1. Let v, e, and λ be integers, $0 \le e \le v(v-1)/2$, $\lambda \ge 2$. Let $f(v, e, \lambda)$ be equal to the greatest number of proper λ -colorings of a graph with v vertices and e edges. Then

$$f(v, e, \lambda) \leq A\lambda^{v}$$

where A is the least of the following three quantities:

$$\left(1-\frac{1}{\lambda}\right)^{\lceil 1/2(\sqrt{1+8e-1})\rceil}, \quad \left(1-\frac{e}{\lambda}+\left(\binom{e}{2}\right)/\lambda^2\right), \quad \frac{\lambda-1}{\lambda-1+e}.$$

The question was motivated by the analysis of the running time of the backtrack algorithm for the graph coloring problem (see Wilf [10]; Bender and Wilf [1]). Another source of related problems is a paper of Wright [11], where an asymptotic approximation to the number $M_{v,e}$ = the total number of proper λ -colorings of all (v,e)-graphs was found for a fixed λ , large v, and all e. Problems similar to ours but for different families of graphs (all graphs on v vertices whose chromatic number is equal to k) were considered by Tomescu [7,8]. Several other instances of the problem were considered by the author in [4], [5].

2. PROOF OF THEOREM 1.1.

The inequality

$$f(v,e,\lambda) \le \left(1 - \frac{1}{\lambda}\right)^{\lceil 1/2(\sqrt{1+8e}-1)\rceil} \lambda^{v} \tag{2.1}$$

was proved in [4]. In order to get other upper bounds, we apply some known facts based on the famous Inclusion-Exclusion Principle, or Sieve Method, and the corresponding interpretation of the chromatic polynomial of a graph due to Whitney [9]. Here we give a brief list of the corresponding facts. All proofs can be found in Lovasz [6, III. §2]).

Proposition 2.1.

(i) (Inclusion-Exclusion Formula). Let A_1, \ldots, A_n be arbitrary events of a probability space (Ω, P) . For each $I \subseteq \{1, \ldots, n\}$, let

$$A_I = \prod_{i \in I} A_i; \qquad A_\phi = \Omega;$$

and let

$$\sigma_k = \sum_{|I|=k} P(A_I), \qquad \sigma_0 = 1.$$

Then

$$P(\overline{A}_1 \overline{A}_2 \dots \overline{A}_n) = \sum_{i=0}^n (-1)^i \sigma_i.$$
 (2.2)

(ii) (Bonferroni Inequalities). The partial sums of

$$P(\overline{A}_1 \dots \overline{A}_n) - \sigma_0 + \sigma_1 - \sigma_2 + \dots$$
 (2.3)

are alternating in sign.

(iii) (Seiberg's Sieve, particular case). If the events A_i , $1 \le i \le n$, are pairwise independent and $P(A_i) = p$ for all i, $1 \le i \le n$, then

$$P(\overline{A}_1 \dots \overline{A}_n) \le \frac{1}{1 + np/(1 - p)} \tag{2.4}$$

Given a labeled graph G with v vertices and e edges and an integer $\lambda \ge 1$, we associate with G the sample space Ω as follows:

 $\Omega = \{\text{all (proper and improper) colorings of } G \text{ in } \lambda \text{ colors}\}.$

Let $V(G) = \{1, 2, ..., v\}$ and $E(G) = \{\{i_1, j_1\}, \{i_2, j_2\}, ..., \{i_e, j_e\}\}$. For each k, $1 \le k \le e$, we define the event A_k as

 $A_k = \{ w \in \Omega \mid \text{vertices } i_k \text{ and } j_k \text{ are colored in the same color} \}.$

Obviously, $|\Omega| = \lambda^{\nu}$ and for each k, $1 \le k \le e$, $|A_k| = \lambda \cdot \lambda^{\nu-2} = \lambda^{\nu-1}$ (a color for i_k and j_k can be chosen in λ different ways and each of the remaining $\nu - 2$ vertices can be colored in λ colors independently of each other).

We define a probability function P on Ω in the usual way, assuming that the probability of each elementary event is equal to $1/|\Omega|$. Then

$$P(A_k) = \lambda^{\nu-1}/\lambda^{\nu} = 1/\lambda, \qquad 1 \le k \le e.$$
 (2.5)

A coloring of G is proper if and only if none of the events A_k , $1 \le k \le e$, happens. Therefore

$$P(G;\lambda) = P(\overline{A}_1 \dots \overline{A}_e) \cdot \lambda^{\nu}$$

and the results of Proposition 2.1 can be used in order to get upper bounds for $P(G; \lambda)$. But the following questions should be answered first: what is the probability of the intersection of two events A_k and A_s for $k \neq s$, and are they pairwise independent?

The event $A_k \cdot A_s$ takes place if and only if vertices i_k, j_k are colored in the same color and vertices i_s, j_s are colored in the same color. Therefore

$$|A_k \cdot A_s| = \begin{cases} \lambda \cdot \lambda \cdot \lambda^{\nu-4} = \lambda^{\nu-2}, & \text{if } \{i_k; j_k\} \cap \{i_s; j_s\} = \phi \\ \lambda \cdot \lambda^{\nu-3} = \lambda^{\nu-2}, & \text{if } \{i_k; j_k\} \cap \{i_s; j_s\} = 1. \end{cases}$$

Thus

$$|A_k \cdot A_s| = \lambda^{\nu-2}$$
 and $P(A_k \cdot A_s) = \lambda^{\nu-2}/\lambda^{\nu} = 1/\lambda^2$ (2.6)

for all $k, s, 1 \le k < s \le v$. Using (2.5) and (2.6) we get

$$P(A_k \cdot A_s) = P(A_k) \cdot P(A_s),$$

which implies the *independence* of A_k and A_s for all $k, s, 1 \le k < s \le v$. For our model,

$$P(A_i) = \frac{1}{\lambda}, \qquad \sigma_1 = \sum_{i=1}^e P(A_i) = \frac{e}{\lambda},$$

$$\sigma_2 = \sum_{1 \le i \le j \le e} P(A_i \cdot A_j) = \frac{e(e-1)}{2\lambda^2}.$$

Using this and multiplying both sides of (2.3) and (2.4) by $|\Omega|$, we obtain two upper bounds for $P(G; \lambda)$ and $f(v, e, \lambda)$: from (2.3):

$$P(G; \lambda) \le f(v, e, \lambda) < \left(1 - \frac{e}{\lambda} + \left(\binom{e}{2} / \lambda^2\right)\right) \lambda^{v}$$
 (2.7)

from (2.5):

$$P(G;\lambda) \le f(v,e,\lambda) < \frac{\lambda-1}{\lambda-1+e}\lambda^{\nu}. \tag{2.8}$$

Combining (2.1), (2.7), and (2.8), we obtain a proof of Theorem 1.1. For some ranges of parameters, the comparison of (2.1), (2.7), and (2.8) is simple, and it gives

for
$$e = 0, 1, \lambda + 1$$
,

the right sides of (2.7) and (2.8) are equal;

for
$$e > \max \left\{ \lambda + 1, (\lambda - 1) \left[\left(\frac{\lambda}{\lambda - 1} \right)^{\nu - 1} - 1 \right] \right\}$$
,

the bound given by (2.8) is better than those given by (2.1) and (2.7);

for
$$v-1 \le e \le \lambda + 1$$
,

the bound given by (2.1) is better than ones in (2.7) and (2.8).

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